Existence of Maximal and Minimal Solutions of Differential Inclusion of Closed Interval-Valued Maps

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Abstract

Let $F(x)$ be a bounded closed interval-valued map. Then the differential inclusion, $Dx(t) \in F(x(t)), x(0) = x_0, t \in [0, T]$ has maximal and minimal solutions, where F is upper semi-continuous, D is one of four Dini derivatives and $T > 0$. Also this paper shows that general differential inclusion solvable and solutions depend continuously on the initial condition.

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0. INTRODUCTION

Suppose u and v are bounded real valued functions. Let $F(x) = [v(x), u(x)]$ be closed interval-valued map. We consider differential inclusion,

$$(1) \quad Dx(t) \in F(x(t)), \quad x(0) = x_0, \quad t \in [0, T],$$

where D (throughout this paper) is one of four Dini derivatives as follow,

$$D^+x(t) = \limsup_{s \rightarrow t_+} (x(s) - x(t))/(s - t), \quad D^-x(t) = \limsup_{s \rightarrow t_-} (x(s) - x(t))/(s - t)$$

and $D_+x(t)$ and $D_-x(t)$ are defined as $D^+x(t)$ and $D^-x(t)$ respectively, when we use $\liminf$ instead of $\limsup$, also $F : R \rightarrow 2^R$ is upper semi-continuous map that 2^R is the space of compact subsets of real numbers R , with the Vietoris topology. In this topology R with usual topology, and a net (A_i) converges to A if and only if, given an open U in R with $A \subseteq U$ and a finite number of open sets $V_1, \dots, V_n$ in R with $A \cap V_j \neq \emptyset$ for $1 \leq j \leq n$, then

eventually both $A_i \subseteq U$ and $A_i \cap V_j \neq \emptyset$ for $1 \leq j \leq n$. See [5] for more details.

A mapping $F : R \rightarrow 2^R$ is called upper semi-continuous (abbreviated to *usc*) if whenever $a_i \rightarrow a$ in R , then for each open set U in R with $F(a) \subseteq U$, eventually $F(a_i) \subseteq U$. If $F(x)$ is singleton, for each x , this means that F is an ordinary continuous function. See [3].

In this paper we show that (1) has maximal (and minimal) solution so, it is solvable. To prove this we look at the maximal solution ϕ_M (and minimal solution ϕ_m) of the differential inequalities,

$$(2) \quad Dx(t) \leq u(x(t)) \quad (\text{and} \quad Dx(t) \geq v(x(t)) \quad), \quad x(0) = x_0, \quad t \in [0, T],$$

where u is bounded above upper semi-continuous, briefly *usc* (and v is bounded below lower semi-continuous, briefly *lsc*) function that satisfies,

$$v(\phi_M(t)) \leq D\phi_M(t) \leq u(\phi_M(t)), \quad \phi_M(o) = x_o, \quad t \in [0, T],$$

$$(\text{and} \quad v(\phi_m(t)) \leq D\phi_m(t) \leq u(\phi_m(t)), \quad \phi_m(o) = x_o, \quad t \in [0, T]).$$

So, ϕ_M and ϕ_m are solutions of (1). In [1], he has shown that (2) has maximal (and minimal) solutions when u and v are continuous maps. In [4] has proved that if $v(x) \leq u(x)$ for all $x \in R$, then the set of solutions of $dx/dt = v(x(t))$, $t \in [o, T]$ is bounded by the maximal and minimal solutions of $dx/dt = u(x(t))$, $t \in [o, T]$, where $u(x)$ is continuous.

1. PRELIMINARY

The purpose of this section is to prove some results which we need later. By the definition of Dini derivatives,

1.1 COROLLARY. *If $\phi : [a, b] \rightarrow R$ is a continuous map then,*

$$(i) \quad D_+\phi \leq D^+\phi \quad \text{and} \quad D_-\phi \leq D^-\phi,$$

$$(ii) \quad D^+(-\phi) = -D_+\phi, \quad D_+(-\phi) = -D^+\phi, \quad D^-(-\phi) = -D_-(\phi)$$

$$\text{and} \quad D_-(-\phi) = -D^-\phi.$$

1.2 LEMMA. *Let $\phi : [a, b] \rightarrow R$ be a continuous map and $D^+\phi(t) \geq 0$ nearly every where (i.e. for all t , except countable subset) in $[a, b]$, then ϕ is increasing.*

Proof. It is obviously enough to prove that $\phi(a) \leq \phi(b)$. First we show that if $\phi(a) > \phi(b)$ and $A = \{t \in [a, b] ; D^+\phi(t) \leq 0\}$, then $(\phi(b), \phi(a)) \subseteq \phi(A)$. Let $x \in (\phi(b), \phi(a))$. By the intermediate value theorem, there is a greatest element $t_x \in (a, b)$ such that $\phi(t_x) = x$ and for $t_x < t \leq b$ we have

$\phi(t) < x$ and so, $\phi(t) - \phi(t_x) < x - x = 0$. Hence $D^+\phi(t_x) \leq 0$, i.e. $t_x \in A$. Therefore $x = \phi(t_x) \in \phi(A)$.

Now let $\epsilon > 0$, and let $\psi(t) = \phi(t) + \epsilon t$. So, nearly every where in $[a, b]$, $D^+\psi(t) = D^+\phi(t) + \epsilon > 0$, hence the set t , where $D^+\psi(t) \leq 0$ is countable. Then the image of this set by ψ is countable, and therefore contains no non-empty open interval. Now by the above argument we conclude that $\psi(a) \leq \psi(b)$. Since ϵ is arbitrary, this gives the result.

1.3 COROLLARY. *If $\phi : [a, b] \rightarrow R$ is continuous and $D\phi(t) \geq 0$ nearly every where in $(a, b]$, then ϕ is increasing.*

Proof. For $D = D^-$, this follows by applying (1.2) to the function $t \rightarrow \phi(a + b - t)$ on $[a, b]$. For $D = D_+$ and $D = D_-$ we use the facts that $D_+(-\phi) = -D^+\phi(t)$ and $D_-(-\phi) = -D^-\phi$.

1.4 LEMMA. *Let $M \in R$ and $\phi : [a, b] \rightarrow R$ be continuous function such that $D\phi(t) \leq M$ nearly every where in $[a, b]$. Then $\phi(b) - \phi(a) \leq M(b - a)$.*

Proof. Let us define $\psi(t) = \phi(a) - \phi(t) + M(t - a)$ for all $t \in [a, b]$. Obviously we have $D^+\psi(t) = M - D_+\phi(t)$, $D^-\psi(t) = M - D_-\phi(t)$, $D_-\psi(t) = M - D^-\phi(t)$ and $D_+\psi(t) = M - D^+\phi(t)$. Then by (1.2) and (1.3), ψ is increasing on $[a, b]$. But $\psi(a) = 0$, therefor, $\psi(b) \geq 0$ and hence the inequality follows.

1.5 COROLLARY. *Let $M \in R$ and $\phi : [a, b] \rightarrow R$ be a continuous map such that $D\phi(t) \geq M$ for nearly every where in $[a, b]$. Then $\phi(b) - \phi(a) \geq M(b - a)$.*

Proof. We prove it for $D = D^+$, in other the cases the proof will be similar. We have $D_+(-\phi)(t) = -D^+\phi(t) \leq -M$, so by (1.4), $(-\phi)(b) - (-\phi)(a) \leq -M(b - a)$. Then the inequality follows.

The next result is a generalizing of ([1], 1.4.5). He has proved it for continuous functions but we prove it for *lsc* functions.

1.6 LEMMA. *Let $\psi : R \rightarrow R$ be a *lsc* function and let $\phi : [a, b] \rightarrow R$ be a continuous map. If $D\phi(t) \geq \psi(t)$ nearly every where in $[a, b]$, then $D'\phi(t) \geq \psi(t)$ for all $t \in (a, b)$, where D' denotes a fixed one of the four Dini derivatives.*

Proof. Let $\varepsilon > 0$ be given, and $t \in (a, b)$. Since ψ is *lsc* at t , we can find $\delta > 0$ such that $\psi(s) \geq \psi(t) - \varepsilon$ for all $s \in (t - \delta, t + \delta)$. Then $D\phi(s) \geq \psi(t) - \varepsilon$ nearly every where in $(t - \delta, t + \delta)$. Then by (1.5), $\phi(s) - \phi(t) \geq (\psi(t) - \varepsilon)(s - t)$, so we conclude that $D'\phi(t) \geq \psi(t) - \varepsilon$. Since ε is arbitrary, this gives the result.

1.7 COROLLARY. *Let in (1.6), $\psi : R \rightarrow R$ be an *usc* function. If $D\phi(t) \leq \psi(t)$ nearly every where in $[a, b]$. Then $D'\phi(t) \leq \psi(t)$ for all $t \in [a, b]$.*

Proof. The proof is the same as the proof of (1.6). Only we use *usc*

property instead of the *lsc* and obviously, the inequalities will be changed.

2. DIFFERENTIAL INEQUALITIES

Let $u : R \rightarrow R$ be an *usc* function, $k_1, k_2 \in R$ and $T > 0$. This section is devoted to existence of the maximal solution of the differential inequalities,

$$Dx(t) \leq u(x(t)) , \quad k_1 \leq Dx(t) \leq k_2 , \quad x(0) = x_0, \quad t \in [0, T],$$

which we shall prove it is monotone. The topology on each set of functions will be the norm topology.

2.1 LEMMA. *Let T be a positive real number and $k_1, k_2 \in R$. Then*

$$S = \{\phi : [0, T] \rightarrow R \mid \phi \text{ continuous, } k_1 \leq D\phi(t) \leq k_2 \text{ and } \phi(0) = x_0\}$$

is closed compact equicontinuous set.

Proof. Let $\phi \in S$, so $D\phi(t) \leq \max\{|k_1|, |k_2|\} = k$. Then by (1.4), $\phi(t) - \phi(s) \leq k(t - s)$ for all $0 \leq s \leq t \leq T$. Hence, obviously, S is equicontinuous and bounded, so by the Arzela-Ascoli theorem $\text{cls}(S)$ is compact. Suppose (ϕ_n) is a sequence in S which converges to ϕ . Since the topology is norm topology, so ϕ is continuous on $[0, T]$. Also for all n , $k_1 \leq D\phi_n(t) \leq k_2$, then by (1.4) and (1.5), for $0 \leq s < t \leq T$, $k_1(t - s) \leq \phi_n(t) - \phi_n(s) \leq k_2(t - s)$. Hence $k_1 \leq (\phi_n(t) - \phi_n(s))/(t - s) \leq k_2$ for all n and by taking the limit as $n \rightarrow \infty$, we obtain, $k_1 \leq (\phi(t) - \phi(s))/(t - s) \leq k_2$. If $D = D^-$ we take $\limsup$ or $\liminf$ as $s \rightarrow t_-$, we have $k_1 \leq D\phi(t) \leq k_2$. If $D = D^+$ or $D = D_+$ we consider $0 \leq t < s \leq T$, and as above, we have, $k_1 \leq (\phi(s) - \phi(t))/(s - t) \leq k_2$. By taking $\limsup$ or $\liminf$ as $s \rightarrow t_+$, $k_1 \leq D\phi(t) \leq k_2$ as desired.

Now we discuss “some compact subsets” of S in (2.1).

2.2 LEMMA. *Let $u : R \rightarrow R$ be a usc map and bounded above by k_2 . Then the set $\Omega = \{\phi \in S \mid D\phi(t) \leq u(\phi(t)) , \phi(0) = x_0\}$ is compact.*

Proof. Since S is compact, it is enough to show that Ω is closed. Let (ϕ_n) be a sequence in Ω such that $\phi_n \rightarrow \phi$. We have to show that $\phi \in \Omega$. Let $\varepsilon > 0$ be given. By the *usc* property of u , there is a nbhd V of $\phi(t_0)$ such that

$$(1) \quad u(y) \leq u(\phi(t_0)) + \varepsilon, \quad (y \in V).$$

Equicontinuity of Ω gives us that the map $(\psi, t) \rightarrow \psi(t) : \Omega \times [0, T] \rightarrow R$ is continuous. Since V is a nbhd of $\phi(t_0)$, then there are $\delta(t_0) > 0$ and N (not depending on $\delta(t_0)$) such that $\phi_n(t) \in V$ for all $|t - t_0| \leq \delta$ & $n \geq N$. Hence by (1), $u(\phi_n(t)) \leq u(\phi(t_0)) + \varepsilon$ for all $|t - t_0| \leq \delta$ & $n \geq N$. So by the definition of Ω , $D\phi_n(t) \leq u(\phi(t_0)) + \varepsilon$ for $|t - t_0| \leq \delta$ and $n \geq N$. Then by (1.4), $\phi_n(t) - \phi_n(t_0) < (u(\phi(t_0)) + \varepsilon)(t - t_0)$ for all $t \in (t_0, t_0 + \delta)$ & $n \geq N$.

So, $(\phi_n(t) - \phi_n(t_0))/(t - t_0) \leq u(\phi(t_0)) + \varepsilon$. By letting $n \rightarrow \infty$ we obtain, $(\phi(t) - \phi(t_0))/(t - t_0) \leq u(\phi(t_0)) + \varepsilon$ for all $t \in (t_0, t_0 + \delta)$. Also, similarly we have $(\phi(t_0) - \phi(t))/(t_0 - t) \leq u(\phi(t_0)) + \varepsilon$ for all $t \in (t_0 - \delta, t_0)$. Now by taking limit as desired, we conclude that, $D\phi(t_0) \leq u(\phi(t_0)) + \varepsilon$. Since ε is arbitrary, $D\phi(t_0) \leq u(\phi(t_0))$. By (2.1), S is closed, so $\phi \in S$ and it follows that $\phi \in \Omega$, as required.

In lemma (2.2) we can change us.continuity to ls.continuity in the following way.

2.3 COROLLARY. *Let $v : R \rightarrow R$ be a lsc function which is bounded below by k_1 . Then the set $\Omega' = \{\phi \in S \mid D\phi(t) \geq v(\phi(t)), \phi(0) = x_0\}$ is compact.*

Proof. The proof is as for lemma (2.2). We use ls.continuity instead of us.continuity and obviously, the inequalities are reversed.

In the following lemma we study the existence of a maximal member of Ω .

2.4 THEOREM. *The set Ω in (2.2) has a unique maximal element.*

Proof. Let first, D be a fixed one of D^+ or D^- . Suppose $\phi_1, \phi_2 \in \Omega$. We observe that if $\phi_1(t) > \phi_2(t)$, then $D(\phi_1 \vee \phi_2)(t) = D\phi_1(t)$. Also if $\phi_1(t) = \phi_2(t)$, $D(\phi_1 \vee \phi_2)(t) = \max\{D\phi_1(t), D\phi_2(t)\}$, so it follows that $\phi_1 \vee \phi_2 \in \Omega$.

Now for each finite set $F = \{\phi_1, \dots, \phi_n\}$ of elements of Ω , we consider $U_F = \phi_1 \vee \phi_2 \vee \dots \vee \phi_n$. Then, $\{U_F : F \text{ any finite subset of } \Omega\}$ is an increasing net which is indexed by the collection of all finite subsets of Ω . Now let us take any convergent subnet $U_{F'} \rightarrow \phi \in \Omega$ (this is allowed by the lemma (2.2)). We claim that, $\phi = \sup_{f \in \Omega} f$. Let $f \in \Omega$. Then if $f \in F$, so, $U_f \leq U_F$. Since $(U_{F'})$ is a subnet, there is F'_0 such that whenever $F'_0 \subseteq F'$, we have $F \subseteq F'$, and so $U_f \leq U_F \leq U_{F'}$. It follows that $f \leq \lim_{F'} U_{F'} = \phi$. Hence ϕ is an upper bound of Ω . Assume ϕ' is any upper bound for Ω . Then $\phi' \geq f_i$, $1 \leq i \leq n$ for any n elements of Ω . We obtain that $\phi' \geq f_1 \vee f_2 \vee \dots \vee f_n = U_{\{f_1, f_2, \dots, f_n\}}$ and so, $\phi' \geq U_G$ for all finite subset G of Ω . We deduce that $\phi' \geq \lim U_{F'} = \phi$ and thus $\sup \Omega = \phi$. Therefore ϕ is a maximal member of Ω .

Let now, $D = D_+$ or $D = D_-$. Since $D_+x(t) \leq D^+x(t)$, then each solution of, (1) $D^+x(t) \leq u(x(t)), x(0) = x_0$, is a solution of (2) $D_+x(t) \leq u(x(t)), x(0) = x_0$. Suppose ϕ be a solution of (2). Since $u \circ \phi$ is usc, by (1.7), we have $D^+\phi(t) \leq u(\phi(t))$, $\phi(0) = x_0$. Thus the differential inequalities (1) and (2) have the same solutions. So (2) has unique maximal solution which is equal to the unique maximal solution of (1). Similarly for $D^-x(t) \leq u(x(t))$ and $D_-x(t) \leq u(x(t))$, $x(0) = x_0$. Therefore Ω has a maximal member. Since the maximal member of Ω is $\phi = \sup \Omega$, then it is unique.

2.5 LEMMA. *Let $u : R \rightarrow R$ be usc map such that $u(x) \leq k_2$ for all $x \in R$, then the differential inequalities, (1) $D^+x(t) \leq u(x(t))$, (2) $D_+x(t) \leq u(x(t))$, (3) $D^-x(t) \leq u(x(t))$, (4) $D_-x(t) \leq u(x(t))$, with $k_1 \leq Dx(t) \leq k_2$, $x(0) = x_0$*

and $t \in [0, T]$ have the same unique maximal solution.

Proof. We prove only (1) and (4) have the same unique maximal solution the proof for other cases will be similar. Let ϕ be a solution of (1). Since $u \circ \phi$ is *usc* so by (1.7), $D_-\phi(t) \leq u(\phi(t))$. Then every solution of (1) is a solution of (4). Similarly, every solution of (4) is a solution of (1). Therefore (1) and (4) have the same maximal solution.

The parallel results to (2.4) and (2.5) hold when we use a *lsc* map v instead of a *usc* map u . The proof will be similar.

2.6 COROLLARY. *Suppose $v : R \rightarrow R$ is a *lsc* function with $v(x) \geq k_1$, for all $x \in R$. Then the set Ω has a unique minimal continuous solution.*

Proof. We use *lsc* property instead of *usc* property in (2.4) and obviously the inequalities are reversed.

2.7 COROLLARY. *Let $v : R \rightarrow R$ be the *lsc* function, then the differential inequalities $D^+x(t) \geq v(x(t))$, $D_+x(t) \geq v(x(t))$, $D^-x(t) \geq v(x(t))$, $D_-x(t) \geq v(x(t))$, with $k_2 \geq Dx(t) \geq k_1$, $x(0) = x_0$, $t \in [0, T]$ have the same unique minimal solution.*

Proof. The proof is exactly the same as the proof of (2.5).

In general, the minimal solution ϕ of the inequalities in (2.7) does not satisfy

$$D\phi(t) = v(\phi(t)), \quad \phi(0) = x_0, \quad t \in [0, T].$$

Let $D = D^+$ and $a, x_0 \in R$ with $0 \leq x_0 < a$. We define $v(x) = 2$ for all $0 \leq x < a$, $v(x) = 1/4$ for $x = a$ and $v(x) = 1/2$ for all $x > a$. Obviously, v is a *lsc* real valued function. The minimal solution of $D^+x(t) \geq v(t(t))$, $x(0) = x_0$, ϕ say, is a continuous function whose graph has slope 2 on $[0, (a - x_0)/2]$ and slope 1/2 on the rest (in fact ϕ is the unique continuous solution). But we have

$$D^+\phi((a - x_0)/2) = 1/2 > v(\phi((a - x_0)/2)) = v(a) = 1/4.$$

See the following figures.

Figure 1: Graph of v . Figure 2: Graph of ϕ .

Also, the same argument for the maximal solution of the inequalities (2.5) holds.

2.8 THEOREM. *The maximal element of the set Ω in (2.2) is monotone.*

Proof. Let ϕ be the maximal solution of the inequalities,

$$k_1 \leq D\phi(t) \leq k_2, \quad D\phi(t) \leq u(\phi(t)), \quad \phi(0) = x, \quad t \in [0, T]$$

and ϕ is not monotone. Then there are $0 \leq t_1 < t_2 < t_3 \leq T$ such that,

$$\phi(t_1) > \phi(t_3) > \phi(t_2) \quad (\text{or } \phi(t_1) < \phi(t_3) < \phi(t_2)).$$

Suppose, now, $M = (\phi(t_2) + \phi(t_3))/2$ so, $\phi(t_2) < M$ and $\phi(t_3) > M$. We define $s_1 = \inf\{t \geq t_1; \phi(t) = M\}$ and $s_2 = \sup\{t \leq t_3; \phi(t) = M\}$. Now there are two cases for $\phi(t)$ when $t > s_2$.

i) $\phi(t) > M$ for $t > s_2$.

We define the continuous map $\psi(t)$ to be equal to $\phi(t)$ on $[0, s_1]$ and $\phi(t + s_2 - s_1)$ for all $t \geq s_1$. Obviously $D\psi(t) \leq u(\psi(t))$ for all $t \geq 0$, but, $\psi(t_2) = \phi(t_2 + s_2 - s_1) > M > \phi(t_2)$, because $t_2 + s_2 - s_1 > s_2$. Thus, this is a contradiction to maximality of ϕ .

ii) $\phi(s) = M$ for some $s > s_2$.

Assume $s_0 = \inf\{s > t_3; \phi(s) = M\}$. Then, $s_0 > s_2$, $\phi(t) \geq M$ for $t \in [s_2, s_0]$. Put $l = s_0 - s_2$. Define $\psi(t)$ to be equal to $\phi(t)$ on $[0, s_1]$ and to be periodic of period l from s_1 . i.e. $\psi(t) = \phi(t)$ for $t \in [0, s_1]$, $\psi(t) = \phi(t + s_2 - s_1 - kl)$ for all t , with $s_1 + kl \leq t \leq s_1 + (k+1)l \leq T$, $k = 0, 1, 2, \dots, N$, for some N . Since $\phi(t + s_2 - s_1) > M$ for $s_1 \leq t \leq s_1 + l$ then $\phi(s) \geq M$ for all $s \geq s_1$. Let $s_1 \leq t \leq T$, then,

$$D\psi(t) = D\phi(t + s_2 - s_1 - kl) \leq u(\phi(t + s_2 - s_1 - kl)) = u(\psi(t))$$

for some $k = 0, \dots, N$; so ψ is a solution of the inequality. But by the definition of ψ , $\psi(t_2) = \phi(t_2 + s_2 - s_1 - kl) \geq M > \phi(t_2)$ and this is a contradiction to maximality of ϕ .

In the case of $\phi(t_1) < \phi(t_3) < \phi(t_2)$ the proof will be parallel. Hence ϕ is monotone.

2.9 COROLLARY. *Let $v : R \rightarrow R$ be lsc map with $v(x) \geq k_1$ for all $x \in R$, and $k_1, k_2 \in R$. Then the minimal continuous solution ϕ of the inequalities, $k_1 \leq D\phi(t) \leq k_2$, $D\phi(t) \geq v(\phi(t))$, $\phi(0) = x_0$, $t \in [0, T]$ is monotone.*

Proof. The proof will be the same as the proof of (2.8), we use only the minimality of the solution instead of the maximality of the solution.

3. DIFFERENTIAL INCLUSION

Assume u and v are real functions and let $F(x) = [v(x), u(x)]$ be bounded closed interval-valued map, then there are $k_1, k_2 \in R$, such that $F(x) \subseteq [k_1, k_2]$, for all x . In this section we show that maximal and minimal solutions of the differential inclusion

$$Dx(t) \in F(x(t)), \quad x(0) = x_0, \quad t \in [0, T],$$

are exist (so, it is solvable); where F is usc map. The maximal solution is equal to the maximal solution of $Dx(t) \leq u(x(t))$ and the minimal solution is equal to the minimal solution of $Dx(t) \geq v(x(t))$ that $x(0) = x_0$ and $t \in [0, T]$.

In the following lemma, we study the relation between closed interval-valued map F and real functions u and v .

3.1 LEMMA. *Let $F(x) = [v(x), u(x)]$ be a set valued map as above. Then u and v are usc and lsc as real functions respectively if and only if F is a usc set-valued map.*

Proof. Let u and v be usc and lsc respectively and $\varepsilon > 0$ be given. Then, for $x_0 \in R$, by us.continuity and ls.continuity there are nbhds $N_1(x_0)$ and $N_2(x_0)$ of x_0 such that, $u(x) \leq u(x_0) + \varepsilon/2$ ($x \in N_1(x_0)$), $v(x) \geq v(x_0) - \varepsilon/2$ ($x \in N_2(x_0)$). Hence, for all $x \in N_1(x_0) \cap N_2(x_0)$, $[v(x), u(x)] \subseteq (v(x_0) - \varepsilon, u(x_0) + \varepsilon)$. Therefore F is usc.

Conversely, let $(v(x_0) - \varepsilon, u(x_0) + \varepsilon)$ be an open nbhd of $F(x_0) = [v(x_0), u(x_0)]$, so by us.continuity of F , there is a nbhd $N(x_0)$ of x_0 such that $F(x) \subset (v(x_0) - \varepsilon, u(x_0) + \varepsilon)$ for all $x \in N(x_0)$. Then $v(x) > v(x_0) - \varepsilon$, $u(x) < u(x_0) + \varepsilon$ for all $x \in N(x_0)$ as required.

Now we combine the lemmas (2.2), (2.3), (2.5), (2.7) and (2.9).

3.2 LEMMA. *Let $F(x) = [v(x), u(x)]$ be bounded usc closed interval-valued map. Then the set,*

$$\Lambda = \{\phi : [0, T] \rightarrow R \mid \phi \text{ is continuous, } D\phi(t) \in F(\phi(t)), \phi(0) = x_0\},$$

is compact.

Proof. Since F is bounded, then there are $k_1, k_2 \in R$, such that $F(x) = [v(x), u(x)] \subseteq [k_1, k_2]$, for all x . Then $\Lambda = \Omega \cap \Omega'$, and since Ω and Ω' are compact, then, Λ is compact.

3.3 LEMMA. *Let $F(x) = [v(x), u(x)]$ be bounded usc closed interval-valued map and ϕ_M be the maximal element of Ω . Then $D\phi_M(t) \geq v(\phi_M(t))$, $\phi_M(0) = x_0$ for all $t \in [0, T]$. Hence $D\phi_M(t) \in F(\phi_M(t))$, for all $t \in [0, T]$.*

Proof. Assume $D = D^+$ or $D = D_+$ and $t \in [0, T]$. Let $\varepsilon > 0$ be given. By ls.continuity of $v\phi_M$ at t_0 there is $\delta > 0$ such that, $u(\phi_M(t)) \geq v(\phi_M(t)) > v(\phi_M(t_0)) - \varepsilon$ for all $t \in (t_0 - \delta, t_0 + \delta)$. We define now, ψ equals to ϕ_M on $[0, t_0]$ and equals to $(t - t_0)(v(\phi_M(t_0)) - \varepsilon) + \phi_M(t_0)$ on $[t_0, t_0 + \delta]$ and whose graph on the rest of $[0, T]$ is a line segment of slope k_1 . Clearly, $D\psi(t) = k_1 \leq u(\psi(t))$ for all $T > t \geq t_0 + \delta$, because we assumed F is bounded so, $k_1 \leq v(x) \leq u(x)$ for all x . Then we conclude that $D\psi(t) \leq u(\psi(t))$ for all $t \in [0, T]$, so by maximality of ϕ_M , we have $\phi_M(t) \geq \psi(t)$ for all $t \in [0, T]$. Since $\phi_M(t_0) = \psi(t_0)$, $(\phi_M(t) - \phi_M(t_0))/(t - t_0) \geq (\psi(t) - \psi(t_0))/(t - t_0)$ for all $t \in (t_0, t_0 + \delta)$. But the right hand side of the inequality is the slope of ψ on $[t_0, t_0 + \delta]$, therefore $D\phi_M(t_0) \geq v(\phi_M(t_0)) - \varepsilon$. Since ε is arbitrary. $D\phi_M(t_0) \geq v(\phi_M(t_0))$. For $D = D^-$ or $D = D_-$, we define ψ equals to ϕ_M on $[0, t_0 - \delta]$, and equals to $(t - t_0)(v(\phi_M(t_0)) - \varepsilon) + \phi_M(t_0)$ on $[t_0 - \delta, t_0]$ and whose graph on the rest of $[0, T]$ is a line segment of slope k_1 . As in the above argument, $D\phi_M(t_0) \geq v(\phi_M(t_0))$. Since t_0 is arbitrary in $[0, T]$, then $D\phi_M(t) \in [v(\phi_M(t)), u(\phi_M(t))] = F(\phi_M(t))$ for all $t \in [0, T]$.

3.4 COROLLARY. *Let $F(x) = [v(x), u(x)]$ be bounded usc set-valued map and ϕ_m be the minimal element of Ω . Then $D\phi_m(t) \leq u(\phi_m(t))$, $\phi_m(0) = x_0$ for all $t \in [0, T]$. Hence $D\phi_m(t) \in F(\phi_m(t))$ for all $t \in [0, T]$.*

Proof. Proof will be parallel to (3.3). We use lsc property instead of usc property and obviously, the inequalities are reversed.

3.5 THEOREM. *Let $F(x) = [v(x), u(x)]$ be a usc bounded closed interval-valued map. Then, $Dx(t) \in F(x(t))$, $x(0) = x_0$, $t \in [0, T]$ has maximal and minimal solutions. the maximal solution is equal to the maximal solution of $Dx(t) \leq u(x(t))$, $x(0) = x$, $t \in [0, T]$, and the minimal solution is equal to the minimal solution of $Dx(t) \geq v(x)$, $x(0) = x_0$, $t \in [0, T]$.*

Proof. Let ϕ_m be the minimal solution of (1) $Dx(t) \geq v(x(t))$, $x(0) = x_0$, $t \in [0, T]$. By (3.4), ϕ_m is a solution of (2) $Dx(t) \in F(x(t))$, $x(0) = x_0$, $t \in [0, T]$. And obviously if ψ is a solution of (2), then $D\psi(t) \geq v(\psi(t))$. So, by minimality of ϕ_m we have $\psi \geq \phi_m$. Therefore ϕ_m is a minimal solution of (2). For maximal solution is parallel. Converse by (3.3) and (3.4) trivial.

3.6 COROLLARY. *Let $f : [0, T] \rightarrow R$ be a continuous function. Then*

$Dx(t) = f(x(t))$, $x(0) = x_0$ is solvable.

Proof. Since f is continuous on $[0, T]$, then f is bounded. Now, by (3.5) the assertion follows.

4. GENERAL DIFFERENTIAL INCLUSION

In this section we define right and left (set-valued) derivatives of a continuous function in general. Then we prove the existence of the right and the left derivatives. Also we show that the sets of the right and left derivatives as set-value maps are not *usc* not *lsc*.

4.1 Definition. Let $x : [0, T] \rightarrow R$ be a continuous map. We define $(S_D x)(t)$, set-valued derivative of x at t , as follows,

$$(S_D^+ x)(t) = \{\text{all limit points of } (x(s) - x(t))/(s - t) \text{ as } s \rightarrow t_+\},$$

$$(S_D^- x)(t) = \{\text{all limit points of } (x(s) - x(t))/(s - t) \text{ as } s \rightarrow t_-\},$$

$$(S_D x)(t) = (S_D^+ x)(t) \cup (S_D^- x)(t).$$

We call $(S_D^+ x)(t)$ and $(S_D^- x)(t)$, right and left set-valued derivatives of x at t respectively. Obviously, $(S_D^+ x)(t)$ on $(0, T]$ and $(S_D^- x)(t)$ on $[0, T)$ are defined.

4.2 THEOREM. Let $F(x) = [v(x), u(x)]$ be a *usc* closed interval-valued which $k_1 \leq F(x) \leq k_2$ for all $x \in R$. Then $(S_D x)(t) \in F(x(t))$, $t \in [0, T]$, $x(0) = x_0$ is solvable.

This theorem can be obtained by the theorem (3.5). Because, if ϕ_m is the minimal solution of $Dx(t) \in F(x(t))$, $x(0) = x_0$, $t \in [0, T]$, we conclude that,

$$u(\phi_m(t)) \geq D^+ \phi_m(t) \geq D_+ \phi_m(t) \geq v(\phi_m(t)).$$

Hence by definition of D^+ and D_+ ,

$$(S_D^+ \phi_m)(t) \subseteq [v(\phi_m(t)), u(\phi_m(t))], \quad t \in [0, T].$$

Similarly, $(S_D^- \phi_m)(t) \subseteq F(\phi_m(t))$, then $(S_D \phi_m)(t) \subseteq F(\phi_m(t))$ for all $t \in [0, T]$.

The following proof is just an alternative way of obtaining the results.

Proof. Let T be a large positive real number. Let $\phi_1 : [0, T] \rightarrow R$ be any continuous function with $\phi_1(0) = x_0$, for which its graph consists of a finite number of straight line segments with vertices at the points of any partition of $[0, T]$,

$$P_1 = \{0 = t_0, t_1, \dots, t_N = T\}.$$

Construct ϕ_1 inductively by,

$$\phi_1(0) = x_0, \quad \phi_1(t) = m_i(t - t_{i-1}) + \phi_1(t_{i-1}) \quad (t_{i-1} \leq t \leq t_i),$$

where m_i is arbitrarily chosen in $F(\phi_1(t_{i-1}))$ for $(1 \leq i \leq N)$. Suppose now, $0 \leq s \leq t \leq T$, so there exist $t_i, \dots, t_j$ such that $t_i \leq s < t_{i+1} < \dots < t \leq t_j$. Since F is bounded by $k = \max\{|k_1|, |k_2|\}$, then we obtain that,

$$|\phi_1(t) - \phi_1(s)| \leq |m_{i+1}|(t_{i+1} - s) + |m_{i+2}|(t_{i+2} - t_{i+1}) + \dots + |m_j|(t - t_{j-1}) \leq k(t - s).$$

Thus ϕ_1 is bounded and continuous. Assume ϕ_2 is made on $[0, T]$ by a similar method as ϕ_1 . Its graph consists of straight line segments with successive vertices at points of the partition,

$$P_2 = \{0, t_0, (t_0 + t_1)/2, t_1, \dots, t_{N-1}, (t_{N-1} + t_N)/2, t_N\}.$$

Let us continue this process, then we obtain a continuous function ϕ_{2^n} with its graph consisting of straight line segments with successive vertices at points of $P_{2^n} = \{0, t'_0, t'_1, \dots, t'_p\}$ in which each subinterval is made by dividing each subinterval of $P_{2^{n-1}}$ at its mid-point. Since F is bounded by k , for $0 \leq s \leq t \leq T$ we have,

$$|\phi_{2^n}(t) - \phi_{2^n}(s)| \leq k(t - s),$$

so ϕ_{2^n} is bounded and continuous.

The sequence $(\phi_{2^n})_{n=0}$ which is established in this method is bounded and obviously equicontinuous. By the Arzela-Ascoli theorem it has a convergent subsequence $(\phi_{i(2^n)})$ which uniformly converges to ϕ say. Hence ϕ is continuous.

Let $s \in [0, T]$ and U be a closed nbhd of $F(\phi(s))$. By us. continuity of F there is a nbhd W of $\phi(s)$ such that $F(y) \subset U$ for all $y \in W$. By equicontinuity there are M and $\delta > 0$ such that $\phi_{i(2^n)}(t) \in W(\phi(s))$ for $i(2^n) \geq M$ and $t \in (s - \delta, s + \delta)$ (δ not depending on M). So we deduce that,

$$F(\phi_{i(2^n)}(t)) \subseteq U \text{ for } i(2^n) \geq M \text{ and all } t \in (s - \delta, s + \delta).$$

Assume now, $i(2^n) \geq M$, $t \in (s - \delta, s + \delta)$. Since $D^+\phi_{i(2^n)}(t)$ is the slope of one segment of the graph of $\phi_{i(2^n)}$ in $(s - \delta, s + \delta)$, thus there exists $t_j \in (s - \delta, s + \delta)$ such that

$$D^+\phi_{i(2^n)}(t) \in F(\phi_{i(2^n)}(t_j)) \subseteq U(F(\phi(s))).$$

Hence by (1.4) and (1.5) we conclude that

$$(\phi_{i(2^n)}(t) - \phi_{i(2^n)}(s))/(t - s) \in U(F(\phi(s))),$$

for all $t \in (s - \delta, s + \delta)$, $t \neq s$ and $i(2^n) \geq M$. Now, let $i(2^n) \rightarrow \infty$ then $(\phi(t) - \phi(s))/(t - s) \in U(F(\phi(s)))$ for all $t \in (s - \delta, s + \delta)$, $t \neq s$ because U is closed. Then $(S_D\phi)(s) \subset U(F(\phi(s)))$. Since U is arbitrary closed nbhd of $F(\phi(s))$ then we conclude that $(S_D\phi)(s) \subseteq F(\phi(s))$.

If $s = 0$ or $s = T$ we consider the right differential $(S_D^+\phi)(0)$ or the left differential $(S_D^-\phi)(T)$ of ϕ at 0 or T .

4.3 Example. Let ϕ be a continuous map from $[0, T]$ to R . Then $t \rightarrow S_D^+ \phi(t)$ and $t \rightarrow S_D^- \phi(t)$ are not necessarily *usc* or *lsc*, nor is $t \rightarrow S_D \phi(t)$.

Proof. Let us consider coordinate axes. Choose the interval $[0, T]$ on the x -axis and on the y -axis too. Now we draw two lines through the origin, Δ and Δ' with slopes m and m' respectively with $0 < m < m'$. Let $0 < a < m$, $m' < b < \infty$. Let ϕ be the continuous function with its graph consisting of the straight line segments, $\Delta_1, \Delta_2, \Delta_3, \dots$ as follows: Δ_1 with the slope a passing through the point (T, T) ; meets Δ' at $A_1 = (t_1, x_1)$ (since slope of Δ_1 is less than m). Δ_2 with slope b passing through A_1 and intersects Δ at $A_2 = (t_2, x_2)$ (since slope of Δ_2 is bigger than m'). By continuing this process we will find continuous map ϕ whose graph consists of $\{\Delta_i : i = 1, 2, \dots\}$. For each number α with $m \leq \alpha \leq m'$, let the points of intersection of $y = \alpha x$ with $y = \phi(x)$ be $(s_n, \phi(s_n))$ with $s_n \searrow 0$. Therefore

$$(\phi(s_n) - \phi(0))/(s_n - 0) = (\phi(s_n))/(s_n) \rightarrow \alpha.$$

So, we have $S_D^+ \phi(0) = [m, m']$. Now, let $(r_n, \phi(r_n))$ be the points of intersection of $y = mx$ (*i.e.* Δ' with $y = \phi(x)$) where $r_n \searrow 0$. Then $(S_D^+ \phi)(r_n) = \{b\}$ and $(S_D^- \phi)(r_n) = \{a\}$ for all n . The assertion now follows.

Figure 3: Graph of ϕ .

5. INITIAL CONDITION

Let $f : R \rightarrow R$ be a given function and $x' = f(x)$ be an ordinary differential equation ($x'(t) = dx/dt$). An initial condition for a solution $x : R^+ \rightarrow R$ is a condition of the form $x(t_0) = x_0$ where $t_0 \in R^+$ and $x_0 \in R$. For simplicity usually we take $t_0 = 0$, so $x(0) = x_0$.

Suppose (x_n) is a sequence in R such that $x_n \rightarrow x_0$. Let ϕ_n be a solution of $x' = f(x)$, with initial condition $x(0) = x_n$, for all n and ϕ be a solution of $x' = f(x)$ with $x(0) = x_0$. Can we say, $\phi_n \rightarrow \phi$ as $n \rightarrow \infty$ with respect to the sup norm? The answer is no, except under some conditions. For example let $f(x) = 3x^{2/3}$ with $x(0) = x_n$, such that $x_n \rightarrow 0$. Then $\phi_n(t) = (t + \sqrt[3]{x_n})^3$ is a solution of the differential equation for all n . Also the map $\psi(t) = 0$ for all t , is a solution of the differential equation with initial condition $x(0) = 0$. But obviously, (ϕ_n) doesn't tend to ψ with respect to sup norm.

Let $v : R \rightarrow R$ be an lsc function with $k_1 \leq v(x) \leq k_2$ for all $x \in R$. Consider the differential inequality $Dx(t) \geq v(x(t))$ with $x(0) = x_n$, $x_n \in R$ and $t \in [0, T]$. Suppose ϕ_n to be the minimal solution of the differential inequality. In this section we investigate the question whether, if $x_n \rightarrow x_0$, then $\phi_n \rightarrow \phi$ uniformly on $[0, T]$, where ϕ is the minimal solution of $D\phi(t) \geq v(\phi(t))$, $\phi(0) = x_0$. Also we have shown that if $k_1 > 0$ or $k_2 < 0$, and $\phi(t, x)$ is a minimal solution of $Dx(t) \geq v(x(t))$, $x(0) = x$ for $t \in [0, T]$. Then $(t, x) \rightarrow \phi(t, x)$ is continuous.

5.1 LEMMA. *Let $v : R \rightarrow R$ be an lsc map. Suppose $\phi_n \rightarrow \phi$ uniformly, that $D\phi_n(t) \geq v(\phi_n(t))$ for $t \in [0, T]$, $\phi_n(0) = x_n$ for all n and $x_n \rightarrow x_0$. Then $D\phi(t) \geq v(\phi(t))$ for $t \in [0, T]$, $\phi(0) = x_0$.*

Proof. Since $\phi_n \rightarrow \phi$, with respect to the sup norm and $F = \{\phi_n; n = 1, 2, \dots\} \cup \{\phi\}$ is compact, then the set F is equicontinuous. Let $\varepsilon > 0$ be given and $t \in R$. By the *lsc* property of v , there exists a nbhd W of $\phi(t)$ such that $v(y) \geq v(\phi(t)) - \varepsilon$, for all $y \in W(\phi(t))$. By equicontinuity there are $\delta > 0$ and N (each of them not depending on the other) such that, $D\phi_n(s) \geq v(\phi_n(s)) \geq v(\phi(t)) - \varepsilon$ for all $s \in (t - \delta, t + \delta)$ and $n \geq N$. Hence by (1.4), $\phi_n(s) - \phi_n(t) \geq (v(\phi(t)) - \varepsilon)(s - t)$, $s \in (t, t + \delta)$ and also, $\phi_n(t) - \phi_n(s) \geq (v(\phi(t)) - \varepsilon)(t - s)$, $s \in (t - \delta, t)$. By taking the limit as $n \rightarrow \infty$, we have $\phi(s) - \phi(t) \geq (v(\phi(t)) - \varepsilon)(s - t)$, $s \in (t, t + \delta)$, $\phi(t) - \phi(s) \geq (v(\phi(t)) - \varepsilon)(t - s)$, $s \in (t - \delta, t)$. Then we conclude that $D\phi(t) \geq v(\phi(t)) - \varepsilon$. Since ε is arbitrary, $D\phi(t) \geq v(\phi(t))$ and $\phi(0) = x_0$ because $\phi_n(0) \rightarrow \phi(0)$.

5.2 COROLLARY. *Let $u : R \rightarrow R$ be usc map. Suppose $\phi_n \rightarrow \phi$ uniformly on $[0, T]$ that $D\phi_n(t) \leq u(\phi_n(t))$ for $t \in [0, T]$, $\phi_n(0) = x_n$ for all n and $x_n \rightarrow x_0$. Then, $D\phi(t) \leq u(\phi(t))$, $\phi(0) = x_0$.*

Proof. The proof is exactly the same as (5.1). Just we use *usc* instead of *lsc* and reverse the inequalities.

5.3 COROLLARY. *Let $F(x) = [v(x), u(x)]$ be usc closed interval-valued map. Assume $\phi_n \rightarrow \phi$ uniformly on $[0, T]$, that $D\phi_n(t) \in F(\phi_n(t))$ for $t \in [0, T]$, $\phi_n(0) = x_n$ for all n and $x_n \rightarrow x_0$. Then $D\phi(t) \in F(\phi(t))$, $\phi(0) = x_0$ for all $t \in [0, T]$.*

Proof. The proof by (5.1) and (5.2) is obvious.

5.4 LEMMA. *Let $v : R \rightarrow R$ be lsc map such that $0 < k_1 \leq v(x) \leq k_2$ for all x . Let ϕ_n be the minimal solution of $Dx(t) \geq v(x(t))$ for $t \in [0, T]$, $x(0) = x_n$ for all n , where (x_n) is a sequence such that $x_n \nearrow x_0$. Let ϕ be the minimal solution of $Dx(t) \geq v(x(t))$ for $t \in [0, t]$, $x(0) = x_0$, then $\phi_n \rightarrow \phi$ uniformly on $[0, T]$.*

Proof. By (2.7), it is enough to prove it for $D = D_+$. Consider the sequence (ϕ_n) . If $i \leq j$ then $\phi_i \leq \phi_j$, otherwise there exists s , such that $\phi_i(s) > \phi_j(s)$. We write $\phi(t) = (\phi_i \wedge \phi_j)(t)$, obviously $D_+\phi(t) = D_+\phi_i(t)$ if $\phi_i(t) < \phi_j(t)$ and $D_+\phi(t) = \min\{D_+\phi_i(t), D_+\phi_j(t)\}$ if $\phi_i(t) = \phi_j(t)$. Then ϕ is a solution of $D_+x(t) \geq v(x(t))$, for $t \in [0, T]$, $x(0) = x_i$. But $\phi(s) = \phi_j(s) < \phi_i(s)$ so, this is contradiction to the minimality of ϕ_i . Thus $\phi_i(t) \leq \phi_j(t)$ for all $t \in [0, T]$.

Clearly, $D_+\phi_n(t) \geq k_1$ for all $t \in [0, T]$, so by (1.5), $\phi_n(t) \geq k_1 t + x_n$ for all $t \in [0, T]$. Then by the intermediate value theorem there exists $\varepsilon_n > 0$ that $\phi_n(\varepsilon_n) = x_0 = \phi(0)$, because $x_n < x_0$. Therefore $x_0 \geq k_1 \varepsilon_n + x_n$ so, $(x_0 - x_n)/k_1 \geq \varepsilon_n$ and by letting $n \rightarrow \infty$, $\varepsilon_n \rightarrow 0$. Since $\phi_n(\varepsilon_n) = x_0 = \phi(0)$, we conclude that $\phi_n(t + \varepsilon_n)$ is a solution of $D_+x(t) \geq v(x(t))$, $x(0) = x_0$. Thus, by minimality of ϕ , (1) $\phi_n(t + \varepsilon_n) \geq \phi(t)$ for all $t \in [0, T]$. We define now ψ_n agrees with ϕ_n on $[0, \varepsilon_n]$ and agrees with $\phi(t - \varepsilon_n)$ for $t \geq \varepsilon_n$. Then obviously

$D_+\psi_n(t) \geq v(\psi_n(t))$, $\psi_n(0) = x_n$. Hence $\psi_n(t) = \phi(t - \varepsilon_n) \geq \phi_n(t)$ for all $t \geq \varepsilon_n$, by minimality of ϕ_n . We replace $t - \varepsilon_n$ by t so we obtain (2) $\phi(t) \geq \phi_n(t + \varepsilon_n)$ for all $t \geq 0$. Now, by (1) and (2) we have (3) $\phi(t) = \phi_n(t + \varepsilon_n)$ for all $t \in [0, T]$. Obviously $k_1 \leq D_+\phi_n(t) \leq k_2$ for all $t \in [0, T]$, then so (ϕ_n) is equicontinuous. Then by (3), $\phi(t) = \lim_n \phi_n(t + \varepsilon_n) = \lim_n \phi_n(t)$. Hence (ϕ_n) is point wise convergent to ϕ . Since (ϕ_n) is uniformly bounded, by the Arzela-Ascoli theorem, (ϕ_n) has a subsequence which converges uniformly. But (ϕ_n) is monotone, therefore $\phi_n \rightarrow \phi$ uniformly.

5.5 LEMMA. *Suppose that the hypotheses of the lemma (5.4) are satisfied but $x_n \searrow x_0$. Then as in (5.4), $\phi_n \rightarrow \phi$ uniformly on $[0, T]$.*

Proof. Since $D\phi(t) \geq k_1$ for all $t \in [0, T]$, by (1.3), $\phi(t) \geq k_1 t + x_0$ for all $t \in [0, T]$. Then by the intermediate value theorem, for each n , $\phi(\varepsilon_n) = x_n = \phi_n(0)$ for some $\varepsilon_n > 0$. Then $\phi(\varepsilon_n) = x_n \geq k_1 \varepsilon_n + x_0$, and so $\varepsilon_n \leq (x_n - x_0)/k_1$ for all n . Hence $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Clearly $\phi(t + \varepsilon_n)$ is a solution of $Dx(t) \geq v(x(t))$, $t \in [0, T]$, $x(0) = x_n$. So by minimality of ϕ_n , (1) $\phi(t + \varepsilon_n) \geq \phi_n(t)$ for all $t \in [0, T]$. We define now, $\psi_n(t) = \phi(t)$ on $[0, \varepsilon_n]$, $\psi_n(t) = \phi_n(t - \varepsilon_n)$ for $t \in [\varepsilon_n, T]$. Obviously ψ_n is a solution of $Dx(t) \geq v(x(t))$, $t \in [0, T]$, $x(0) = x_0$, thus $\psi_n(t) = \phi_n(t - \varepsilon_n) \geq \phi(t)$ for $t \in [\varepsilon_n, T]$, by minimality of ϕ . We replace now, $t - \varepsilon_n$ by t , so we have (2) $\phi_n(t) \geq \phi(t + \varepsilon_n)$ for all $t \in [0, T]$. By (1), (2) we conclude $\phi_n(t) = \phi(t + \varepsilon_n)$ for all $t \in [0, T]$, and all n . Hence $|\phi(t) - \phi_n(t)| = |\phi(t) - \phi(t + \varepsilon_n)|$ for all $t \in [0, T]$. Since $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, $|\phi(t) - \phi(t + \varepsilon_n)| \rightarrow 0$. So, $\phi_n \rightarrow \phi$ pointwise and uniformly as in the lemma (5.4).

5.6 COROLLARY. *Let $u : R \rightarrow R$ be usc function with $k_2 \geq u(x) \geq k_1 > 0$ (or $k_2 < 0$). Assume ϕ_n is the maximal solution of $Dx(t) \leq u(x(t))$ for $t \in [0, T]$, $x_0 = x_n$, where (x_n) is a sequence such that $x_n \rightarrow x_0$. Also let ϕ be the maximal solution of $Dx(t) \leq u(x(t))$, for $t \in [0, T]$, $x(0) = x_0$. Then $\phi_n \rightarrow \phi$ uniformly on $[0, T]$.*

Proof. We consider first, $x_n \nearrow x_0$ and second $x_n \searrow x_0$. Also we use us.continuity instead of ls.continuity and obviously reverse the inequalities.

5.7 THEOREM. *Let $F(x) = [v(x), u(x)]$ be an usc interval-valued map with $0 < k_1 \leq F(x) \leq k_2$ (or $k_2 < 0$) for all x . Let ϕ_n be the maximal (resp. minimal) solution of $Dx(t) \in F(x(t))$ for $t \in [0, T]$ and $x(0) = x_n$, where (x_n) is a sequence such that $x_n \rightarrow x_0$. Also, suppose ϕ is the maximal (resp. minimal) solution of $Dx(t) \in F(x(t))$, $x(0) = x_0$. Then $\phi_n \rightarrow \phi$ uniformly on $[0, T]$.*

Proof. The proof by (5.4), (5.5) and (5.6) is obvious.

5.8 THEOREM. *Let $u : R \rightarrow R$ be a usc function with $k_2 \geq u(x) \geq k_1 > 0$ (or $k_2 < 0$). Assume $\phi(t, x)$ is the maximal solution of $Dx(t) \leq u(x(t))$, $x(0) =$*

x for $t \in [0, T]$. Then $(t, x) \rightarrow \phi(t, x)$ is continuous.

Proof. By (5.6) for $t \in [0, T]$, $\phi(t, x_n) \rightarrow \phi(t, x)$ uniformly as $x_n \rightarrow x$ and also $t \rightarrow \phi(t, x)$ is continuous for each $x \in R$, so $(t, x) \rightarrow \phi(t, x)$ is continuous. Because $[0, T]$ is compact and R is metrizable. For more details see [3, p.314].

There are the same result for *lsc* functions.

5.9 COROLLARY. Let $v : R \rightarrow R$ be an *lsc* function with $k_2 \geq v(x) \geq k_1 > 0$ (or $k_2 < 0$). Assume $\phi(t, x)$ is the minimal solution of $Dx(t) \geq v(x(t))$, $x(0) = x$ for all $t \in [0, T]$. Then $(t, x) \rightarrow \phi(t, x)$ is continuous.

Proof. The proof is the same as the proof of (5.8).

5.10 THEOREM. Let $F(x) = [v(x), u(x)]$ be as in (5.7). Suppose $\phi(t, x)$ be maximal (or minimal) solution of $Dx(t) \in F(x(t))$, $x(0) = x$ for all $t \in [0, T]$. Then $(t, x) \rightarrow \phi(t, x)$ is continuous.

Proof. The proof by (5.7), (5.8) and (5.9) is clear.

In the preceding lemmas, the conditions $v(x) \geq k > 0$ and $u(x) \geq k > 0$ (or $v(x) \leq k < 0$ and $u(x) \leq k < 0$) are essential. To illustrate this, consider the *usc* function $u(x)$ as follows.

5.11 Example. Suppose *usc* set-valued map $u : [0, 2] \rightarrow R$ is defined by:

$$u(x) = \begin{cases} 1/2^{n-1} & 1 - 1/2^{n-1} < x \leq 1 - 1/2^n, \\ 1 & 0 \leq x \leq 1/2 \text{ or } 1 \leq x \leq 2, \end{cases}$$

for all n . The solution for initial value $x_0 \geq 1$ will be $\phi(t) = t + x_0$ and for $x_0 < 1$ the solution will not be a straight line (see figure 5). Thus if $x_n \nearrow x_0 = 1$ and ϕ_n is the solution for initial value x_n , then ϕ_n doesn't tend to ϕ . The graphs of some solution curves are shown in the following figures.

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Figure 4: Graph of u .

Figure 5: Graph of some solutions.