Round Fuzzy Metric Spaces

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Abstract. The purpose of this paper to introduce the concept of roundness is fuzzy metric spaces. Also some theorems and definitions is given on round fuzzy metric spaces.

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1. Introduction

In 1965, the concept of fuzzy set was introduced by Zadeh [10]. Many authors have introduced the concept of fuzzy metric space in different ways [1-5]. George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [6] and defined a Hausdorff topology on this fuzzy metric space. Nathanson [7] introduced the concept of round metric space and gave the relationship between round metrics and equivalent metrics.

In this paper we study on the concept of round fuzzy metric space in the sense of George and Veeramani.

2. Preliminaries

Definition 2.1[8]: A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t-norm if $*$ is satisfying the following conditions:

- (i) $*$ is commutative and associative;
- (ii) $*$ is continuous;
- (iii) $a * 1 = a$ for all $a \in [0, 1]$;
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Definition 2.2[3]: A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X, s, t > 0$,

- (i) $M(x, y, t) > 0$,
- (ii) $M(x, y, t) = 1$ if and only if $x = y$,
- (iii) $M(x, y, t) = M(y, x, t)$,
- (iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
- (v) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Remark 2.3: In fuzzy metric space X , $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.

Example 2.4: Let (X, d) be a metric space. Denote $a * b = ab$ for all $a, b \in [0, 1]$ and let M_d be a fuzzy set on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}$$

for all $k, m, n \in \mathbb{R}^+$. Then $(X, M_d, *)$ is a fuzzy metric space.

Remark 2.5: Note the above example holds even with the t-norm $a * b = \min\{a, b\}$ and hence M is a fuzzy metric with respect to any continuous t-norm. In the above example by taking $k = m = n = 1$, we get

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

We call this fuzzy metric induced by a metric d the standard fuzzy metric.

Definition 2.6[3]: Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0, 1)$, $t > 0$ and $x \in X$. The set

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

is called the open ball with center x and radius r with respect to t .

Theorem 2.7[3]: Every open ball $B(x, r, t)$ is an open set.

Definition 2.8[3]: Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0, 1)$, $t > 0$ and $x \in X$. The set

$$B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}$$

is called the closed ball with center x and radius r with respect to t .

Theorem 2.9[3]: Every closed ball $B[x, r, t]$ is a closed set.

Remark 2.10: Let $(X, M, *)$ be a fuzzy metric space. Define $\tau = \{A \subset X : \text{for each } x \in X, \text{ there exist } t > 0, r \in (0, 1) \text{ such that } B(x, r, t) \subset A\}$. Then τ is a topology on X .

Remark 2.11:

- (i) Since $\{B(x, \frac{1}{n}, \frac{1}{n}) : n = 1, 2, \dots\}$ is a local base at x , the topology τ is first countable.
- (ii) Every fuzzy metric space is Hausdorff.
- (iii) Let $(X, M, *)$ be an fuzzy metric space and τ be the topology on X induced by the fuzzy metric. Then for a sequence $(x_n)_n$ in X , $x_n \rightarrow x$ if and only if $M(x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$.

(iv) In a fuzzy metric space every compact set is closed and bounded.

Definition 2.12[9]: Let $(X, M, *)$ be a fuzzy metric space, $x \in X$ and $A \subset X$. The distance between x and A is defined by

$$M(A, x, t) = \sup\{M(y, x, t) : y \in A\}$$

for all $t > 0$.

3. Main Results

Let $(X, M, *)$ be a fuzzy metric space. If $A \subset X$, let $\overline{A}$ denote the closure of A in X . Then $\overline{B(x, r, t)} \subset B[x, r, t]$ for all $x \in X$, each $r \in (0, 1)$ and each $t > 0$.

Definition 3.1: Let $(X, M, *)$ be a fuzzy metric space. We call M a round fuzzy metric for X if $\overline{B(x, r, t)} = B[x, r, t]$ for all $x \in X$, each $r \in (0, 1)$ and each $t > 0$.

Remark 3.2: Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric M is round iff $y \in \overline{B(x, 1 - M(x, y, t), t)}$ for all $x, y \in X$ with $x \neq y$ and each $t > 0$.

Definition 3.3: A fuzzy metrizable space X is a round fuzzy metric space if there exist a round fuzzy metric for X .

Example 3.4: Let $X = \mathbb{R}$. Define $a * b = ab$ and

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

Then $(X, M, *)$ is a round fuzzy metric space.

Theorem 3.5: Let $X = A \cup K$ be a fuzzy metrizable space, where A and K are nonempty, disjoint, closed sets and K is compact. Then no fuzzy metric for X is round.

Proof. Let M be a fuzzy metric for X . Choose $a \in A$. Since K is compact, there exists $k \in K$ such that $M(a, K, t) = \sup\{M(a, x, t) : x \in K\} = M(a, k, t)$. Then $B(a, 1 - M(a, k, t), t) \subset A$. Since A is closed, $\overline{B(a, 1 - M(a, k, t), t)} \subset A$, and so $k \notin \overline{B(a, 1 - M(a, k, t), t)}$. Therefore, M is not a round metric for X . ■

Conclusion 3.6: If a fuzzy metric space $(X, M, *)$ has an isolated point, then no metric for X is round.

Theorem 3.7: Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be two fuzzy metric spaces without isolated points, and let $f : X_1 \rightarrow X_2$ be a surjection such that for $x, y, z \in X_1$, if $M_1(x, z, t) > M_1(x, y, t)$, then $M_2(f(x), f(z), t) > M_2(f(x), f(y), t)$ for all $t > 0$. If M_1 is a fuzzy round metric for X_1 , then M_2 is a fuzzy round metric for X_2 .

Proof. We first show that f is continuous. Let $x \in X_1$, and let $(y_n)_n$ be a sequence in X_1 such that $M_1(y_n, x, t) < M_1(y_{n+1}, x, t)$ for all $n \geq 1$, $t > 0$ and

such that $y_n \rightarrow x$. Then

$$M_2(f(y_n), f(x), t) < M_2(f(y_{n+1}), f(x), t)$$

for all $n \geq 1$ and for all $t > 0$. Let $r = \sup\{M_2(f(y_n), f(x), t) : t > 0\}$. Suppose $r < 1$. Since $f(x)$ is not an isolated point of X_2 , there exists $f(z) \in B(f(x), 1 - r, t)$ with $z \neq x$ for all $t > 0$. Then

$$M_2(f(z), f(x), t) > r \geq M_2(f(y_n), f(x), t)$$

for all $n \geq 1$ and for all $t > 0$. But this is impossible, since $(M_1(y_n, x, t))_n$ increases to 1. Therefore, $r = 1$, the sequence $(f(y_n))_n$ converges to $f(x)$, and the function f is continuous (In fact, f is a homeomorphism of X_1 onto X_2).

Let $f(x), f(y) \in X_2$ with $x \neq y$. If M_1 is a round fuzzy metric for X_1 , then $y \in \overline{B(x, 1 - M_1(x, y, t), t)}$, and so there is a sequence $(z_n)_n$ in X_1 such that $M_1(z_n, x, t) > M_1(y, x, t)$ and $(z_n)_n$ converges to y , for all $t > 0$. Then

$$M_2(f(z_n), f(x), t) > M_2(f(y), f(x), t)$$

and so $f(z_n) \in B(f(x), 1 - M_2(f(x), f(y), t), t)$ for all $t > 0$. Since f is continuous, $(f(z_n))_n$ converges to $f(y)$, and so $f(y) \in \overline{B(f(x), 1 - M_2(f(x), f(y), t), t)}$ for all $t > 0$. Therefore $(X_2, M_2, *)$ is a round fuzzy metric space. ■

Definition 3.8: Let X be a nonempty set and M_1 and M_2 be two fuzzy metrics on X . Then M_1 and M_2 are said to be equivalent if there exist $a, b \in (0, 1]$ such that

$$a \cdot M_2(x, y, t) \leq M_1(x, y, t) \leq b \cdot M_2(x, y, t)$$

for all $x, y \in X$ and $t > 0$. We denote it by $M_1 \sim M_2$.

Corollary 3.9: The relation " $\sim$ " is defined as above is an equivalence relation.

Proof. The proof is the same as in classic metric spaces. ■

Corollary 3.10: Let M_1 be a round fuzzy metric for X . If M_2 is a fuzzy metric on X equivalent to M_1 such that $M_2(x, z, t) > M_2(x, y, t)$ whenever $M_1(x, z, t) > M_1(x, y, t)$, then M_2 is also a round fuzzy metric for X .

Proof. Let $f : (X, M_1, *) \rightarrow (X, M_2, *)$ be the identity map. Then the proof is clear from Theorem 3.7. ■

Problem: How can be characterize the roundness in fuzzy metric spaces?

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