A Taylor Polynomial Approach for Solving Volterra-Hammerstein Integral Equations of Mixed Type

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Abstract

Integral equations of mixed Volterra-Fredholm type arise in various physical and biological problems. In the present paper, we obtain the approximate solution of the nonlinear Volterra-Hammerstein integral equations of mixed type in terms of Taylor polynomial. The reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments.

Keywords: Taylor polynomials, Volterra-Hammerstein integral equations of mixed type

1 Introduction

In this paper, we will consider the approximate solution of the nonlinear Volterra-Hammerstein integral equation of mixed type

$$y(x) = f(x) + \sum_{\mu=1}^m \int_a^x K_{\mu}(x, t) g_{\mu}(t, y(t)) dt, \quad a \leq x \leq X, \quad (1)$$

where $f(x), K_\mu(x, t), g_\mu(t, y)$ ($\mu = 1, \dots, m$) are known functions with $g_\mu(t, y)$ nonlinear in y and $y(x)$ is a solution to be determined.

The main aim of the present paper is to give an approximate solution of the nonlinear Volterra-Hammerstein integral equations of mixed type by a Taylor expansion approach. This method has been presented for solution of integral equations by Kanwal and Liu in [5] and then has been extended by Sezer to Volterra integral equations in [3] and to differential equations in [4]. The results of [3] had been developed and applied to high-order linear Volterra-Fredholm integro-differential equations in [6]. Recently, Yalcinbas in [7] has been concerned with the Taylor polynomial solution of certain nonlinear Volterra-Fredholm integral equations with algebraic nonlinearity.

The object of the present article is to present developments of the operational approach of the Taylor polynomials for Volterra-Hammerstein equations of mixed type. We proceed to investigate the numerical solvability of (1), by a Taylor expansion approach. Finally, some numerical results are presented in the final section which support the theoretical results obtained in this paper. Throughout this paper, we assume that (1) has a unique solution $y(x)$ to be determined.

2 Analysis of the Method for Volterra-Hammerstein Integral Equations of mixed type

Consider the integral equation (1), where $f(x), K_\mu(x, t), g_\mu(t, y(t))$ ($\mu = 1, \dots, m$) are known functions and $y(x)$ is a solution to be determined. and the solution is expressed in the form:

$$y(x) = \sum_{m=0}^N \frac{1}{m!} y^{(m)}(c)(x - c)^m, \quad (2)$$

which is a Taylor polynomial of degree N at $x = c$, where $y^{(m)}(c), m = 0, 1, \dots, N$ are coefficients to be determined.

We define:

$$g_\mu(t, y(t)) = G_\mu(y(t)),$$

where $G_\mu (\mu = 1, \dots, M)$ are known smooth functions with $G_\mu(y(t))$ nonlinear in $y(t)$. Then by substituting the Taylor expansion of $G_\mu(y(t))$ at $y(t) = 0$, it follows:

$$g_\mu(t, y(t)) = \sum_{l=0}^N \alpha_{l\mu} [y(t)]^l,$$

where $\alpha_{l\mu} = \frac{G_\mu^{(l)}(y(0))}{l!}$.

Now, suppose that:

$$Y_l(t) = [y(t)]^l,$$

if $l = 0$, then $Y_0(t) = 1$, So the equation (1) can be written as follows:

$$y(x) = f(x) + \sum_{\mu=1}^M \sum_{l=0}^N \alpha_{l\mu} \int_a^x K_\mu(x, t) Y_l(t) dt. \quad (3)$$

To obtain the solution of equation (1), we first differentiate it n times with respect to x :

$$y^{(n)}(x) = f^{(n)}(x) + \sum_{\mu=1}^M \sum_{l=0}^N \alpha_{l\mu} V_\mu^{(n)}(x), \quad (4)$$

where

$$V_\mu^{(n)}(x) = \frac{d^n}{dx^n} \int_a^x K_\mu(x, t) Y_l(t) dt.$$

Obviously, for $n = 0$:

$$V_\mu^{(0)}(x) = V_\mu(x) = \int_a^x K_\mu(x, t) Y_l(t) dt,$$

and for $n > 0$, by applying successively n times the Leibnitz's rule to the integral $V_\mu(x)$ we have:

$$V_\mu^{(n)}(x) = \sum_{j=0}^{n-1} [h_{j\mu}(x) Y_l(x)]^{(n-j-1)} + \int_a^x \frac{\partial^{(n)} K_\mu(x, t)}{\partial x^n} Y_l(t) dt, \quad (5)$$

where

$$h_{j\mu} = \frac{\partial^{(j)} K_\mu(x, t)}{\partial x^j} \Big|_{t=x}.$$

From the Leibnitz's rule, we evaluate $[h_{j\mu}(x) Y_l(x)]^{(n-j-1)}$ as follows:

$$[h_{j\mu}(x) Y_l(x)]^{(n-j-1)} = \sum_{m=0}^{n-j-1} \binom{n-j-1}{m} h_{j\mu}^{(n-m-j-1)}(x) Y_l^{(m)}(x),$$

and by substituting the above relation in equation (5), this equation becomes:

$$V_{\mu}^{(n)}(x) = \sum_{m=0}^{n-1} \sum_{j=0}^{n-m-1} \left\{ \binom{n-j-1}{m} h_{j\mu}^{(n-m-j-1)}(x) Y_l^{(m)}(x) \right\} + \int_a^x \frac{\partial^{(n)} K_{\mu}(x, t)}{\partial x^n} Y_l(t) dt. \quad (6)$$

Following [2], we have:

$$\sum_{j=0}^{n-1} \sum_{m=0}^{n-j-1} (\dots) = \sum_{m=0}^{n-1} \sum_{j=0}^{n-m-1} (\dots).$$

Now, we put $x = c$ in relations (4) and (6):

$$y^{(n)}(c) = f^{(n)}(c) + \sum_{\mu=1}^M \sum_{l=0}^N \alpha_{l\mu} \left\{ \sum_{m=0}^{n-1} \sum_{j=0}^{n-m-1} \binom{n-j-1}{m} h_{j\mu}^{(n-m-j-1)}(c) Y_l^{(m)}(c) \right\} \\ + \sum_{\mu=1}^M \sum_{l=0}^N \alpha_{l\mu} \left\{ \int_a^c \frac{\partial^{(n)} K_{\mu}(x, t)}{\partial x^n} \Big|_{x=c} Y_l(t) dt \right\}.$$

By substituting the Taylor expansion of $Y_l(t)$ at $t = c$ for $l > 0$ in above equation we obtain:

$$y^{(n)}(c) = f^{(n)}(c) + \sum_{\mu=1}^M \sum_{l=0}^N \alpha_{l\mu} \left\{ \sum_{m=0}^{n-1} \sum_{j=0}^{n-m-1} \binom{n-j-1}{m} h_{j\mu}^{(n-m-j-1)}(c) Y_l^{(m)}(c) \right\} \\ + \sum_{\mu=1}^M \sum_{l=0}^N \alpha_{l\mu} \left\{ \int_a^c \frac{\partial^{(n)} K_{\mu}(x, t)}{\partial x^n} \Big|_{x=c} \left[\sum_{m=0}^N \frac{1}{m!} Y_l^{(m)}(c) (t-c)^m \right] dt \right\},$$

or briefly:

$$y^{(n)}(c) = f^{(n)}(c) + \sum_{l=0}^N \left\{ \sum_{m=0}^{n-1} H_{nm,l} Y_l^{(m)}(c) + \sum_{m=n}^N T_{nm,l} Y_l^{(m)}(c) \right\},$$

in other words:

$$y^{(n)}(c) = f^{(n)}(c) + \sum_{l=0}^N \left\{ \sum_{m=0}^{n-1} (H_{nm,l} + T_{nm,l}) Y_l^{(m)}(c) + \sum_{m=n}^N T_{nm,l} Y_l^{(m)}(c) \right\}, \quad (7)$$

where for $n = 0$:

$$\sum_{m=0}^{n-1} H_{nm,l} + T_{nm,l} Y_l^{(m)}(c) = 0, \quad (l = 0, 1, \dots, N)$$

for nm)

$$H_{nm,l} = \sum_{\mu=1}^M \sum_{j=0}^{n-m-1} \alpha_{l\mu} \binom{n-j-1}{m} h_{j\mu}^{(n-m-j-1)}(c),$$

also for $n, m, l = 0, 1, 2, \dots, N$:

$$T_{nm,l} = \sum_{\mu=1}^M \alpha_{l\mu} \frac{1}{m!} \int_a^c \frac{\partial^{(n)} K_{\mu}(x, t)}{\partial x^n} \Big|_{x=c} (t-c)^m dt.$$

The quantities $Y_l^{(m)}(c)$ for $m = 0, 1, \dots, N$ in equation (7) can be found from the permutation relation:

$$Y_l^{(m)}(c) = \sum_{t_1+t_2+\dots+t_l=m} \binom{m}{t_1 t_2 \dots t_l} y^{(t_1)}(c) y^{(t_2)}(c) \dots y^{(t_l)}(c), \quad (l > 0)$$

and for $m \neq 0$, $Y_0^{(m)}(c) = 0$, where $\binom{m}{t_1 t_2 \dots t_l} = \frac{m!}{t_1! t_2! \dots t_l!}$. ($t_1, t_2, \dots, t_l$ are positive integers or zero).

Note that the above relation can be obtained from the generalized Leibnitz's rule (dealing with differentiation of product of p -function). If we take $n, m = 0, 1, \dots, N$ then equation (7) becomes:

$$\begin{aligned} y^{(0)}(c) &= f^{(0)}(c) + \sum_{l=0}^N \sum_{m=0}^N T_{nm,l} Y_l^{(m)}(c), \\ y^{(n)}(c) &= f^{(n)}(c) + \sum_{l=0}^N \left\{ \sum_{m=0}^{n-1} H_{nm,l} + T_{nm,l} Y_l^{(m)}(c) + \sum_{m=n}^N T_{nm,l} Y_l^{(m)}(c) \right\}, \end{aligned} \quad (8)$$

which is a nonlinear system of $N+1$ equations for $N+1$ unknowns $y^{(n)}(c)$ ($n = 0, 1, \dots, N$), which can be solved numerically by any standard methods.

This system can be written as a matrix form:

$$Y - \sum_{l=0}^N T_l Y_l^* = F, \quad (9)$$

where Y, F, Y_l^*, T_l are matrices defined by:

$$Y = \begin{bmatrix} y^{(0)}(c) \\ y^{(1)}(c) \\ \vdots \\ y^{(N)}(c) \end{bmatrix}, \quad F = \begin{bmatrix} f^{(0)}(c) \\ f^{(1)}(c) \\ \vdots \\ f^{(N)}(c) \end{bmatrix},$$

$$T_l = \begin{bmatrix} T_{00,l} & T_{01,l} & \dots & T_{0N,l} \\ T_{10,l} + H_{10,l} & T_{11,l} & \dots & T_{1N,l} \\ \vdots & \vdots & & \vdots \\ T_{N0,l} + H_{N0,l} & T_{N1,l} + H_{N1,l} & \dots & T_{NN,l} \end{bmatrix}, \quad Y_l^* = \begin{bmatrix} Y^{(0)}(c) \\ Y^{(1)}(c) \\ \vdots \\ Y^{(N)}(c) \end{bmatrix}.$$

To make easy the calculations, we put $c = a$, then system (8) becomes:

$$\begin{aligned} y^{(0)}(a) &= f^{(0)}(a) + \sum_{l=0}^N \sum_{m=0}^N T_{0m,l} Y_l^{(m)}(a), \\ y^{(n)}(a) &= f^{(n)}(a) + \sum_{l=0}^N \left\{ \sum_{m=0}^{n-1} (H_{nm,l} + T_{nm,l}) Y_l^{(m)}(a) + \sum_{m=n}^N T_{nm,l} Y_l^{(m)}(a) \right\} \\ n &= 1, 2, \dots; \quad m = 0, 1, \dots, N \end{aligned}$$

3 Error Estimation

In this section, we perform the estimating error for the Volterra-Hammerstein integral equations. Since the truncated Taylor series or the corresponding polynomial expansion is an approximate solution of equations (1), then by substituting the solutions $y^{(n)}(x)$ ($n = 0, 1, \dots, N$), in equation (2), we have:

$$e(x) = \left| y(x) - f(x) - \sum_{\mu=1}^M \sum_{l=0}^N \alpha_{l\mu} I_{\mu}(x) \right|,$$

where $e(x)$ is defined as an error function. Also for the Volterra-Hammerstein equations, the function $I_{\mu}(x)$ is defined as follows:

$$I_{\mu}(x) = \int_a^x K_{\mu}(x, t) Y_l(t) dt, \quad a \leq t \leq x$$

When we put $x = x_r$,

$$e(x_r) = |y(x_r) - f(x_r) - \sum_{\mu=1}^M \sum_{l=0}^N \alpha_{l\mu} I_{\mu}(x_r)|,$$

then our aim is: $e(x_r) \leq 10^{-k_r}$ (k_r is any positive integer). If we prescribe $\max(10^{-k_r}) = 10^{-k}$, then we increase N as far as the following inequality holds at each points x_r :

$$e(x_r) \leq 10^{-k}.$$

In other words, by increasing N the error function $e(x_r)$ approaches to 0.

4 Numerical Results and some Comments

In this section, we report on numerical results of some test problems solved by the proposed method of this article. We consider the following test problems:

Example 1. Let us now study the following nonlinear integro-differential equation:

$$\begin{cases} y'(x) = 1 + \frac{x}{2} - e^x - \frac{xe^{-x^2}}{2} + e^{y(x)} - \int_0^x xte^{-y^2(t)} dt, & 0 \leq x \leq 1 \\ y(0) = 0, \end{cases}$$

this initial value problem may be viewed as a special case of Volterra-Hammerstein integral equation of mixed type:

$$y(x) = f(x) + \sum_{\mu=1}^2 \int_0^x K_{\mu}(x, t) g_{\mu}(t, y(t)) dt,$$

when we let:

$$\begin{aligned} f(x) &= x + \frac{x^2}{4} - e^x - \frac{e^{-x^2}}{4} + \frac{3}{4}, \\ K_1(x, t) &= 1, & K_2(x, t) &= \frac{t^3}{2} - \frac{tx^2}{2}, \\ g_1(t, y(t)) &= e^{y(t)}, & g_2(t, y(t)) &= e^{-y^2(t)}. \end{aligned}$$

Let us set: $M = 2$, $c = 0$, $a = 0$, we apply the method in the case $N = 3$. Then we evaluate the quantities $\alpha_{l\mu}$ ($l = 0, 1, 2, 3$; $\mu = 1, 2$), $f^{(n)}(0)$ ($n = 0, 1, 2, 3$) and matrices T_l ($l = 0, 1, 2, 3$) as follows:

$$\begin{aligned} f(0) &= 0, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = -1, \\ \alpha_{01} &= 1, \quad \alpha_{02} = 1, \\ \alpha_{11} &= 1, \quad \alpha_{12} = 0, \\ \alpha_{21} &= \frac{1}{2}, \quad \alpha_{22} = 1, \\ \alpha_{31} &= \frac{1}{6}, \quad \alpha_{32} = 0, \\ T_0 = T_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \end{bmatrix}. \end{aligned}$$

We substitute these values in (8), and get the nonlinear system of equations:

$$\begin{aligned} y(0) &= 0, \\ y'(0) - 1 - y(0) - \frac{1}{2}[y(0)]^2 - \frac{1}{6}[y(0)]^3 &= 0, \\ y''(0) - y'(0) - y(0)y'(0) - \frac{1}{2}[y(0)]^2y'(0) &= -1, \\ y'''(0) - y''(0) - [y'(0)]^2 - y(0)y''(0) - y(0)y'(0) - \frac{1}{2}[y(0)]^2y''(0) &= -1. \end{aligned}$$

From this system, the coefficients $y^{(n)}(0)$ ($n = 0, 1, 2, 3$) are computed as: $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 0$ and thus the solution of the nonlinear Volterra-Hammerstein integro-differential equation becomes : $y(x) = x$,

which is an exact solution.

Example 2. Our last test problem is the nonlinear Volterra-Hammerstein integral equation

$$y(x) = -x^3(-1 + e^{\sin x}) + \sin x + \int_0^{x^3} \cos t e^{y(t)} dt,$$

which has the exact solution $y(x) = \sin x$.

In this example we have :

$$\begin{aligned} f(x) &= -x^3(-1 + e^{\sin x}) + \sin x, \\ M &= 1, \quad K_1(x, t) = x^3 \cos t, \quad g_1(t, y(t)) = e^{y(t)}. \end{aligned}$$

First, we find the following sparse matrices :

$$\begin{aligned} T_0 = T_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ T_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{2} & 0 & 0 & 0 & 0 \end{bmatrix}, \\ T_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -24 \\ -59 \end{bmatrix}. \end{aligned}$$

Then, we have :

$$y^{(0)}(0) = y(0) = 0,$$

$$y^{(1)}(0) = 1,$$

$$y^{(2)}(0) = 0,$$

$$y^{(3)}(0) = -1,$$

$$y^{(4)}(0) - 24 - 24y(0) - 12[y(0)]^2 - 4[y(0)]^3 - [y(0)]^4 - \frac{1}{5}[y(0)]^5 = -24,$$

$$y^{(5)}(0) - 60y^{(1)}(0) - 60y(0)y^{(1)}(0) - 30[y(0)]^2 y^{(1)}(0) - 10[y(0)]^3 y^{(1)}(0) - \frac{5}{2}[y(0)]^4 y^{(1)}(0) = -59.$$

Thus, the coefficients are obtained as:

$$y(0) = 0, \quad y^{(1)}(0) = 1, \quad y^{(2)}(0) = 0, \quad y^{(3)}(0) = -1, \quad y^{(4)}(0) = 0, \quad y^{(5)}(0) = 1.$$

By substituting these coefficients in (2), we get the solution of the problem as :

$$y(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5,$$

which is the first three terms of the Taylor expansion of function $\sin x$ at $x = 0$.

Now let us find the solution of this problem taking $N = 3, 5, 8$ for $c = 0$. Comparing the solution given above with the exact solution $y(x) = \sin x$ of the problem, we obtain :

r	x_r	Exact solution	Approximate solution $y(x_r)$		
			$c = 0$	$c = 0$	$c = 0$
			$N = 3$	$N = 5$	$N = 8$
0	0	0	0	0	0
1	0.2	0.19866	0.19866	0.19866	0.19866
2	0.4	0.38941	0.38933	0.38941	0.38941
3	0.6	0.56460	0.56400	0.56464	0.56464
4	0.8	0.71735	0.71466	0.71739	0.71735
5	1	0.84147	0.83333	0.84166	0.84146

Table-1

To get the best approximate solution of the equation, we take more terms from the Taylor expansion of functions, that is, the truncation limit N must be chosen large enough.

5 Conclusion

In this paper, a variation of Taylor polynomial approach has been used for the approximate solution of Volterra-Hammerstein integral equations of mixed type. With the availability of this methodology, it will now be possible to investigate the approximate solution of nonlinear applied problems, particularly of the nonlinear problems in dynamic model of a chemical reactor.

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