On Multiplication Modules

Ünsal Tekir

Marmara University, Department of Mathematics
Ziverbey, Göztepe-İstanbul, Turkey
utekir@marmara.edu.tr

Abstract. Let $R$ be a commutative ring with identity and $M$ be a unital $R$-module. Then $M$ is called a multiplication module provided for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. Our objective is to investigate properties of prime and semiprime submodules of multiplication modules.

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Throughout this paper all rings will be commutative with identity and all modules will be unitary. Let $R$ be a ring and $M$ be a unital $R$-module. For any submodule $N$ of $M$, we define $(N : M) = \{ r \in R : rM \subseteq N \}$. A submodule $N$ of $M$ is called prime if $N \neq M$ and whenever $r \in R, m \in M$ and $rm \in N$, then $m \in N$ or $r \in (N : M)$. A submodule $N$ of $M$ is called semiprime if $N \neq M$ and whenever $r \in R, m \in M$, and $r^n m \in N$ for some positive integer $n$, then $rm \in N$. In recent years, prime and semiprime submodules have attracted a good deal of attention; see, for example $[2-5]$.

An $R$-module $M$ is called a multiplication module provided for each submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. We say that $I$ is a presentation ideal of $N$. Clearly, every submodule of $M$ has a presentation ideal if and only if $M$ is a multiplication module. Let $N$ and $K$ be submodules of a multiplication with $N = I_1 M$ and $K = I_2 M$ for some ideals $I_1$ and $I_2$ of $R$. The product $N$ and $K$ denoted by $NK$ is defined by $NK = I_1 I_2 M$. Then by $[6,\text{theorem 3.4}]$, the product of $N$ and $K$ is independent of presentation of $N$ and $K$. Note that this definition is different from the definition of ordinary ideal multiplication. Indeed, let $R = \mathbb{Z}$ be the ring of integers, and let $M = 2\mathbb{Z}$ and $N = K = 4\mathbb{Z}$. Then $NK$ is $16\mathbb{Z}$ by the usual definition and is $8\mathbb{Z}$ by the our definition. Moreover, for $a, b \in M$, by $ab$ we mean the product
of $Ra$ and $Rb$. Clearly, $NK$ is a submodule of $M$ and $NK \subseteq N \cap K$ see, for example,[[7] – [9]].

The purpose of this paper is to introduce interesting and useful properties of prime and semiprime submodules of multiplication modules.

**Theorem 1.** Let $M$ be a non-zero $R$-module. Then, $M$ is a faithful $R$-module (not necessarily multiplication) such that every proper submodule is prime if and only if $R$ is a field.

*Proof. $\Leftarrow$: Clear.*

$\Rightarrow$: Suppose that $R$ is not field. Note that the zero submodule of $M$ is prime. Then $R$ is a domain and $M$ is a torsion-free $R$-module. Because $R$ is not field, $M$ is not simple. Let $Rm$ be a proper non-zero submodule of $M$. Assume that $0 \neq a$ is not invertible element of $R$. Then $Ram$ is prime so that $aM$ is contained in $Ram$ (which gives contradiction $M = Rm$) or $m$ belongs to $Ram$ (which gives $Ra = R$, a contradiction).

**Corollary 2.** Let $M$ be a faithful multiplication $R$-module. Then, $M$ is simple if and only if every proper submodule of $M$ is prime.

**Proposition 3.** Let $M$ be a multiplication $R$-module and $N_1, N_2, \ldots, N_k$ be submodules of $M$. Let $N$ be prime submodule of $M$. Then the following statements are equivalent.

1. $N_j \subseteq N$ for some $j$ with $1 \leq j \leq k$.
2. $\bigcap_{i=1}^{k} N_i \subseteq N$
3. $\prod_{i=1}^{k} N_i \subseteq N$

*Proof. $(i) \Rightarrow (ii)$ : Clear.

$(ii) \Rightarrow (iii)$ : Since $\prod_{i=1}^{k} N_i \subseteq \bigcap_{i=1}^{k} N_i$, $\prod_{i=1}^{k} N_i \subseteq N$ by $(ii)$.

$(iii) \Rightarrow (i)$ : We have $N_i = I_iM$ for some ideals $I_i (1 \leq i \leq k)$ of $R$. Then $N_1N_2\ldots N_k = I_1I_2\ldots I_kM \subseteq N$ and so $I_1I_2\ldots I_k \subseteq (N : M)$. Since $(N : M)$ is a prime ideal of $R$, $I_j \subseteq (N : M)$ for some $j (1 \leq j \leq k)$. Therefore, $N_j = I_jM \subseteq N$ for some $j (1 \leq j \leq k)$.

**Definition 1.** Let $M$ be a multiplication $R$-module. A nonempty subset $S^*$ of $M$ is said to be multiplicatively closed if $mn \cap S^* \neq \emptyset$ whenever $m, n \in S^*$.

**Proposition 4.** Let $M$ be a multiplication $R$-module. Then, a proper submodule $N$ of $M$ is prime if and only if $M \setminus N$ is a multiplicatively closed.
Proof. Let $N$ be a prime submodule of $M$ and let $a, b \in M \setminus N$. Since $N$ is prime, $ab \notin N$. Then $ab \cap (M \setminus N) \neq \emptyset$. Conversely, let $a, b \notin N$. Then $a, b \in M \setminus N$. Since $M \setminus N$ is a multiplicatively set, $ab \cap (M \setminus N) \neq \emptyset$. Therefore, $ab \notin N$ [see, 6]. 

**Theorem 5.** Let $M$ be a multiplication $R$-module. Let $A$ be a submodule of $M$ and let $S^*$ be a multiplicatively closed set in $M$ such that $A \cap S^*$ is empty. Then there is a submodule $N$ of $M$ which is maximal with respect to the properties that $A \subseteq N$ and $N \cap S^*$ are empty. Furthermore, $N$ is prime submodule of $M$.

Proof. Let $H$ be the set of all submodules $B$ of $M$ such that $A \subseteq B$ and $B \cap S^*$ is empty; $H$ is not empty since $A \in H$. By Zorn’s Lemma, $H$ has a maximal element $N$. To show that $N$ is prime, suppose that it is not and let $ab \subseteq N$, $a \notin N$, $b \notin N$ where $a, b \in M$. Then $N \subseteq N \cap Ra$ and $N \subseteq N \cap Rb$ so there are elements $s, t \in S^*$ such that $s \in N + Ra$ and $t \in N + Rb$. Hence $st \in S^*$ and $st \subseteq (N + Ra) (N + Rb) \subseteq N$, which contradicts the fact that $N \cap S^*$ is empty. 

**Definition 2.** Let $M$ be a multiplication module. A zero divisor in $M$ is an element $0_M \neq a \in M$ for which there exists $b \in M$ with $b \neq 0_M$ such that $ab = RaRb = 0_M$.

**Theorem 6.** Let $M$ be a multiplication $R$-module. Let $N$ be a submodule of $M$ such that $N \neq M$. Then, $N$ is prime if and only if $M/N$ has no zero divisor.

Proof. Suppose that $N$ be a prime submodule of $M$. Since $M$ is a multiplication module, $M/N$ is a multiplication module [see 6, Theorem 3.21]. Let $a \in M$ be such that the element $0_{M/N} \neq \overline{a} = a + N$ in $M/N$ is zero divisor, so that there exists $b \in M$ such that $0_{M/N} \neq \overline{b} = b + N$ and $\overline{ab} = 0_{M/N}$. Let $I$ and $J$ be presentation ideals $\overline{a}$ and $\overline{b}$, respectively. Then $\overline{ab} = (IJ)M/N = N$ and so $ab \subseteq N$. Since $N$ is prime, $a \in N$ or $b \in N$. This is a contradiction. Conversely, let $M/N$ has no zero divisor. Let $ab \subseteq N$ where $a, b \in M$. Then $\overline{ab} = 0_{M/N}$. Since $M/N$ has no zero divisor, $\overline{a} = 0_{M/N}$ or $\overline{b} = 0_{M/N}$. Therefore, $a \in N$ or $b \in N$. 

**Definition 3.** Let $M$ be a multiplication $R$-module and let $N$ be a submodule of $M$. Then

(i) $N$ is called nilpotent if $N^k = 0$ for some positive integer $k$, where $N^k$ means the product of $N, k$ times;

(ii) An element $m$ of $M$ is called nilpotent if $m^k = 0$ for some positive integer $k$.

The set of all nilpotent elements of $M$ is denoted by $N_M$. 
Definition 4. Let $M$ be an $R$-module and $N$ be a submodule of $M$. Then, the radical of $N$ denoted by $M - \text{rad}(N)$ or $r(N)$ is defined to be intersection of all prime submodules of $M$ containing $N$. If $N$ is not contained in any prime submodule of $M$, then $M - \text{rad}(N) = M$.

Theorem 7. [6, Theorem 3.13] Let $N$ be a submodule of a multiplication $R$-module $M$. Then $M - \text{rad}(N) = \{m \in M : m^k \subseteq N \text{ for some } k > 0\}$.

Corollary 8. Let $M$ be a multiplication $R$-module. Then $N_M$ is the intersection of all prime submodules of $M$ ( $r(0) = N_M$ ).

Definition 5. A proper submodule $N$ of a module $M$ over a commutative ring $R$ is said to be weakly prime submodule if whenever $0 \neq rm \in N$, for some $r \in R, m \in M$, then $m \in N$ or $r \in (N : M)$.

Clearly, every prime submodule of a module is weakly prime submodule. However, since 0 is always weakly prime (by definition), a weakly prime submodule need not be prime.

Theorem 9. Let $N$ be a weakly prime submodule of $M$. If $(N : M)N \neq 0$, then $N$ is a prime submodule of $M$.

Proof. Suppose that $(N : M)N \neq 0$; we show that $N$ is prime. Let $rm \in N$. If $rm \neq 0$, then $N$ weakly prime gives $m \in N$ or $r \in (N : M)$. So assume that $rm = 0$. First suppose that $rN \neq 0$, say $rn_0 \neq 0$ where $n_0 \in N$. Then $0 \neq rn_0 = r(m + n_0) \in N$, so $r \in (N : M)$ or $m + n_0 \in N$. Hence $r \in (N : M)$ or $m \in N$. So we can assume that $rN = 0$. Next suppose that $m(N : M) \neq 0, mk_0 \neq 0$ where $k_0 \in (N : M)$. Then $0 \neq mk_0 = (r + k_0)m \in N$. Therefore $m \in N$ or $r + k_0 \in (N : M)$. Then $m \in N$ or $r \in (N : M)$. So we can assume that $m(N : M) = 0$.

Since $(N : M)N \neq 0$, there exists $k \in (N : M)$ and $n \in N$ with $kn \neq 0$. Then $0 \neq kn = (r + k)(m + n) \in N$; $r + k \in (N : M)$ or $m + n \in N$. Hence $r \in (N : M)$ or $m \in N$. So $N$ is prime submodule of $M$. 

Compare the following Theorem with Theorem 1 in [1].

Theorem 10. Let $M$ be a multiplication $R$-module. Let $N$ be a weakly prime submodule of $M$. If $N$ is not prime, $N^2 = 0$.


Corollary 11. Let $M$ be a multiplication $R$-module and $N$ be weakly prime submodule of $M$. Then $N \subseteq r(0)$ or $r(0) \subseteq N$. 

Proof. If $N$ is a prime submodule, $r(0) \subseteq N$ by Corollary 8. If $N$ is not prime submodule, $N \not\subseteq r(0)$ by Theorem 10. \]

Recall that an ideal $I$ in a commutative ring $R$ is called semiprime if $r^n \in I$ for some $n \in \mathbb{Z}^+$ implies that $r \in I$. It is well known that an ideal $I$ is semiprime if and only if $I = \sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^+ \}$. A submodule $N$ of $M$ is called semiprime if $r^n m \in N$ for some $n \in \mathbb{Z}^+$ implies that $rm \in N$. It is clear that $N$ is semiprime if and only if $I^n V \subseteq N$ for some $n \in \mathbb{Z}^+$ implies that $IV \subseteq N$.

**Theorem 12.** Let $N$ be a proper submodule of a multiplication module $M$. Then $N$ is semiprime if and only if $U^n \subseteq N$ implies that $U \subseteq N$ for each submodule $U$ of $M$.

**Proof.** Let $N$ be a semiprime submodule and $U^n \subseteq N$ for some submodule $U$ of $M$. Suppose that $I$ be a presentation of $U$. Then $U^n = I^n M \subseteq N$. Therefore, $I^n \subseteq (N : M)$, Since $N$ is a semiprime submodule of $M$, $(N : M)$ is a semiprime ideal of $R$. Therefore, $I \subseteq (N : M)$ and so $U = IM \subseteq N$.

Conversely, let $U^n \subseteq N$ for some $n \in \mathbb{Z}^+$ implies that $U \subseteq N$ for any submodule $U$ of $M$. Let $I^n V \subseteq N$ for some ideal $I$ of $R$ and a submodule $V$ of $M$. Suppose that $J$ is a presentation of $V$. Therefore, $I^n V = I^n JM \subseteq N$ and so $(IV)^n = (IJ)^n M \subseteq N$. Then $IV \subseteq N$. □

**Corollary 13.** Let $N$ be a proper submodule of $M$. Then $N$ is semiprime if and only if $m^n \subseteq N$ for some $n \in \mathbb{Z}^+$ implies that $m \in N$ for every $m \in M$.

**Proof.** Let $N$ be a semiprime submodule. It is clear that $m^n \subseteq N$ for some $n \in \mathbb{Z}^+$ implies that $m \in N$ for every $m \in M$. Conversely, let $U \not\subseteq N$. Thus, there is $u \in U \setminus N$. Then $u^n \not\subseteq N$. Therefore $U^n \not\subseteq N$. Thus, $N$ is semiprime. □

If $N$ is a submodule of $M$ such that $N$ is an intersection of prime submodules of $M$, then $N$ is semiprime submodule of $M$. We don’t know if the converse is true in general, but it is true but it is true in the following special case (see, 3)

**Theorem 14.** Let $M$ be a multiplication $R$-module. Then, $N$ is a semiprime submodule of $M$ if and only if $r(N) = N$.

**Proof.** It is clear that $N \subseteq r(N)$ for any submodule $N$ of $M$. Let $N$ be a semiprime submodule of $M$. Let $m \in r(N)$. Then $m^k \subseteq N$ for some $k \in \mathbb{Z}^+$. Since $N$ is semiprime, $m \in N$. Therefore, $N = r(N)$.

Conversely, let $N = r(N)$. Let $m^n \subseteq N$ for some $n \in \mathbb{Z}^+$. Since $N = r(N)$, $m \in N$. Therefore, $N$ is semiprime submodule of $M$. □
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