On a Decomposition of the Second Cohomology of an Abelian Topological Group

H. Sahleh

Department of Mathematics
Guilan University
P.O. Box 1914, Rasht, Iran
sahleh@guilan.ac.ir

Abstract

Let C be an abelian topological group, A a trivial C -module. In this paper we show that the second cohomology of C with the 2-divisible coefficient is isomorphic to the direct sum of the second cohomology of a Lie ring c , whose underlying group is C , and the group of symmetric cocycles. Also there is an isomorphism between the categories of abelian topological groups of nilpotency class two, with a continuous section, and that of Lie rings of nilpotency class two.

Mathematics Subject Classification: 22A05

Keywords: Topological extension ; Second cohomology; Lie ring

Introduction

All spaces are assumed to be Tychanov (completely regular, Hausdorff). A *topological extension* of G by N is a short exact sequence $0 \rightarrow N \xrightarrow{i} Q \xrightarrow{\pi} G \rightarrow 0$, where i is a topological embedding onto a closed subgroup and π an open continuous onto homomorphism. The extension is *central* if N is in the center of G . We consider extensions with a continuous section i.e. $u : G \rightarrow Q$ such that $\pi u = Id$. For example, if G is a connected locally compact group, then any topological extension of G by a connected simply connected Lie group has a continuous section [6, theorem 2]. Notation and definitions as in [2].

In section 1, we give some results concerning the abelian group of nilpotency class two. In section 2, an isomorphism is constructed between the second cohomology of an abelian topological group C and the direct sum of the second cohomology of a Lie ring c , with the underlying group C , and the group of symmetric cocycles with the trivial coefficient.

1 Nilpotent group of class two

In this section we define nilpotent topological group of class two and express it as an exact sequence with a continuous section. Moreover ,we define an operation on a nilpotent group G of class two and will show that it induces the same group structure on G .

Definition 1.1 Let G be an abelian topological group .Then G is called a nilpotent group of class two if there exists an extension ,(e): $0 \rightarrow A \rightarrow G \rightarrow C \rightarrow 0$,with a continuous section where A and C are abelian and A is in the center of G .

Cohomology of topological groups

Let G be a topological group and A an abelian topological group on which G acts continuously.

Let $C^n(G, A)$ be the continuous maps $\phi : G^n \rightarrow A$ with the coboundary map

$$C^n(G, A) \xrightarrow{\delta_n} C^{n+1}(G, A)$$

given by

$$\begin{aligned} \delta\phi(g_1, \dots, g_n) &= g_1 \cdot \phi(g_2, \dots, g_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i \phi(g_1, \dots, g_i g_{i+1}, \dots, g_n) + (-1)^n \phi(g_1, \dots, g_{n-1}) \end{aligned}$$

Note that this is analogous to the inhomogeneous resolution for the discrete case.

Definition1.2 .The continuous group cohomology of G with coefficient in A is

$$H^n(G, A) = Ker\delta_n / Im\delta_{n-1}$$

Let $Ext_s(C, A)$ be the set of extensions of A by C with a continuous section. It is known [1] ,by the Baer sum, that $Ext_s(C, A)$ is an abelian topological group. By [3], if C is a topological group and A a trivial C -module then there is an isomorphism between the second cohomology of C and the group of extensions of C by A with continuous sections, namely

$$H^2(C, A) \simeq Ext_s(C, A)$$

Let G be a nilpotent group of class two and (e) : $0 \rightarrow A \rightarrow G \rightarrow C \rightarrow 0$ the corresponding extension. In fact , G and $A \times C$ are homeomorphic [1]. Now we define a new operation on G and will show that this is a group structure.

By [3] , there is a 2-cocycle $[\phi] \in H^2(C, A)$ which corresponds to (e).

$$(a_1, c_1) \cdot (a_2, c_2) = (a_1 + a_2 + \phi(c_1, c_2), c_1 + c_2), \quad a_1, a_2 \in A, \quad c_1, c_2 \in C$$

By the cocycle condition it is easy to show that the operation is associative. Since $[\phi] \in H^2(C, A)$ and A is a trivial module, the associativity implies the following identity:

$$\phi(c_1, c_2) + \phi(c_1 + c_2, c_3) = \phi(c_1, c_2 + c_3) + \phi(c_2, c_3)$$

Hence $\phi(0, c) = \phi(c, 0) = \phi(0, 0)$. For such ϕ , then $(-\phi(0, 0), 0)$ is the identity of G :

$$(a, c)(-\phi(0, 0), 0) = (a - \phi(0, 0), 0) + \phi(c, 0), c = (a, c)$$

Now for the inverse element set $(a, c)^{-1} = (-a - \phi(c, -c) - \phi(0, 0), c)$.

The following lemma indicates that the group structure of a nilpotent topological group of class two is uniquely determined by 2-cocycles modulo coboundaries of 1-chains.

Lemma 1.3 *Let G be a nilpotent topological group of class two and $0 \rightarrow A \rightarrow G \rightarrow C \rightarrow 0$ the associated extension of G . Then the group structure of G is uniquely determined by the elements of $H^2(C, A)$, with A a trivial module.*

Proof: Let G be a nilpotent group of class two and $0 \rightarrow A \rightarrow G \rightarrow C \rightarrow 0$ the central extension where A and C are abelian. Let $\phi \in Z^2(C, A)$ be the associated 2-cocycle of the extension. In this case, the maps

$$A \rightarrow A \times \{0\}, a \mapsto (a - \phi(0, 0), 0)$$

and

$$C \rightarrow G/A, c \mapsto (0, c)A$$

are isomorphisms.

Now we choose other representation than $(0, c)$. For any continuous function $q : C \rightarrow A$, define $C \rightarrow G/A$

$$c \mapsto (0, c)(q(c) - \phi(0, 0), 0)A = (q(c), c)A$$

The map $\psi : G \rightarrow G, (a, c) \mapsto (a + q(c), c)$ is one-one, onto continuous homomorphism:

$$\begin{aligned} \psi((a_1, c_1) \cdot (a_2, c_2)) &= (a_1 + a_2 + \phi(c_1, c_2) + q(c_1 + c_2), c_1 + c_2) \\ &= (a_1 + 1 + a_2 + \phi(c_1, c_2) + q(c_1) + q(c_2), c_1 + c_2) \\ &= \psi((a_1, c_1))\psi((a_2, c_2)) \end{aligned}$$

Now if we consider $(a + q(c), c)$ as an element of G we conclude that the same group structure in G can be defined by 2-cocycle $\phi(c_1, c_2) + q(c_1) + q(c_2) -$

$q(c_1 + c_2)$. Note that the difference between ϕ and this new 2-cocycle is equal to the coboundaries of 1-chain q . Hence The group structure in G is uniquely determined by 2-cocycle modulo coboundaries of 1-chains, that is, by $H^2(C, A)$.

Definition 1.4 A 2-cocycle $\psi \in C^2(C, A)$ is called central iff $\psi(0, 0) = 0$.

If ψ is central then for any $c \in C$

$$\psi(0, c) = \psi(c, 0) = \psi(0, 0) = 0$$

By choosing $q(0) = -\psi(0, 0)$, we get a central cocycle.

Definition 1.5 An abelian topological group is 2-divisible if the continuous homomorphism $A \rightarrow A$, $a \mapsto a + a$ is an automorphism.

For example an abelian topological p-group is 2-divisible [4]. Let $a/2$ be the image of a under the inverse map.

Definition 1.6 A cocycle $\psi \in C^2(C, A)$ is normalized iff $\psi(c, -c) = 0$ for all $c \in C$.

If ψ is normalized then $(a, c)^{-1} = (-a, -c)$ for any $(a, c) \in G$

Lemma 1.7 For a 2-divisible abelian topological group A and an abelian group C , every element of $H^2(C, A)$ can be represented by a normalized cocycle.

proof : if A is a 2-divisible abelian group and ψ is a central cocycle, then $q(c) = \psi(c, -c)/2$ gives a normalized cocycle.

Remark 1.8 . Note that by either of the following conditions :

- (1) $q(0) = 0$ for central cocycle ,
- (2) $q(-c) = -q(c)$ for all c and normalized cocycles,

if the coboundary of a 1-chain q is added to a central (normalized) cocycle, then the result is still central (normalized).

2 Lie Ring

In this section we define the topological Lie ring of nilpotency class two and will find similar results as in the previous part (for more on Lie ring see [5]).

Definition 2.1 A topological Lie ring means an abelian topological group with the following conditions:

- (1) the commutator $[,]$ is bilinear
- (2) $[a, a] = 0 \Leftrightarrow [a, b] = -[b, a]$, for all a, b
- (3) $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$

As usual we denote the commutator $ab - ba$ of elements a, b in the lie ring by $[a, b]$. A commutative Lie ring means zero commutator.

Definition 2.2 A topological Lie ring n is of nilpotency class two if there exists a central extension of Lie rings

$$0 \rightarrow a \rightarrow n \rightarrow c \rightarrow 0$$

with a continuous section $c \rightarrow n$, where a and c are commutative Lie rings. Note that the extension corresponds to a 2-cocycle $\phi \in C^2(C, A)$ and every element of n can be written as $n = (a, c)$, $a \in a, c \in c$. Now we define a new operation on n :

$$(2.2) \quad (a_1, c_1) \cdot (a_2, c_2) = (a_1 + a_2 + \phi(c_1, c_2), c_1 + c_2), \quad a_1, a_2 \in A, \quad c_1, c_2 \in C$$

and

$$[(a_1, c_1), (a_2, c_2)] = [\eta(c_1, c_2) - \phi(0, 0), 0]$$

where $\eta(c_1, c_2) = \phi(c_1, c_2) - \phi(c_2, c_1)$.

The commutator properties are equivalent to the fact that $\eta : C \times C \rightarrow A$ is a skew-symmetric bihomomorphism, that is, $\eta(c_1, c_2) = -\eta(c_2, c_1)$. This implies that $\eta(0, 0) = 0$. In other word η is a 2-cocycle, $\eta \in C^2(c, A)$ with the trivial action of c on A .

In this construction

$$a \rightarrow a \times \{c\}, a \mapsto (a - \phi(0, 0), 0)$$

and

$$c \rightarrow n/a, c \mapsto (0, c) + a$$

are Lie ring isomorphisms.

If we choose another representation of coset than $(0, c)$ for any function $q : C \rightarrow A$

$$c \rightarrow n/a, c \mapsto (0, c) + (q(c) - \phi(0, 0)) + a = (q(c), c) + a$$

From (2.2) we get the same Lie ring structure in n . The same Lie ring structure in n can be defined by 2-cocycle :

$$\phi(c_1, c_2) + q(c_2) - q(c_1 + c_2)$$

obtaining from ϕ by adding a 1-chain. Note that the coboundary of a 1-chain $r \in C^1(c, A)$ with the trivial action of c on A , is $r(c_1, c_2) = r[c_1, c_2] = 0$. Thus, the Lie ring structure in n is uniquely determined by elements of $H_{sym}^2(C, A) \oplus H^2(c, A)$ with the trivial actions of C and c on A , where $H_{sym}^2(C, A)$ denotes the subgroup of second cohomology classes defined by symmetric cocycles. Again note that we need the central extensions in order the cocycles be non-degenerate.

Definition 2.3 A skew-symmetric bihomomorphism $\eta : C \times C \rightarrow A$ is non-degenerate iff for any $c_1 \neq 0$, there exists $c_2 \in C$ such that $\eta(c_1, c_2) \neq 0$

For a non-degenerate cocycle η , element $(a_1, a_2) \in n$ with $c_1 \neq 0$ can not be central since its commutator with $(0, c_2)$ is not 0, (c_2 as in definition 2.3).

In the next result C_{sym}^2 will be the symmetric cocycles and $H_{sym}^2(C, A)$ the subgroup of 2-cocycles defined by symmetric cocycles.

Theorem 2.4 Let A be a 2-divisible abelian topological group and C an abelian topological group and c a commutative Lie ring with underlying abelian group of which is C . Let c and C act trivially on A . Then

$$L_c : C^2(C, A) \rightarrow C_{sym}^2(C, A) \oplus C^2(c, A), \quad \psi \mapsto (\phi, \eta)$$

is an isomorphism, factor of which by coboundaries of 1-chain is an isomorphism

$$L_h : H^2(C, A) \rightarrow H_{sym}^2(C, A) \oplus H^2(c, A)$$

Proof. Consider the following map:

$$\begin{aligned} L_c : C^2(C, A) &\rightarrow C_{sym}^2(C, A) \oplus C^2(c, A) \\ \psi &\mapsto (\phi, \eta) \end{aligned}$$

where

$$(2.4.1) \quad \phi(c_1, c_2) = \frac{\psi(c_1, c_2) + \psi(c_2, c_1)}{2}$$

$$(2.4.2) \quad \eta(c_1, c_2) = \psi(c_1, c_2) + \psi(c_2, c_1)$$

Note that ϕ is a cocycle and symmetric,

$$\phi(c_1, c_2) = \phi(c_2, c_1)$$

Also η is Skew-symmetric.

Now we show that η is a bihomomorphism. By the cocycle identities :

$$\psi(c_1 + c_2, c_3) + \psi(c_1, c_2) = \psi(c_1, c_2 + c_3) + \psi(c_2, c_3)$$

$$\psi(c_3 + c_2, c_1) + \psi(c_3, c_1) = \psi(c_3, c_1 + c_2) + \psi(c_1, c_2)$$

$$-\psi(c_1 + c_3, c_2) - \psi(c_1, c_3) = -\psi(c_1, c_3 + c_2) - \psi(c_3, c_2)$$

Adding these ,we get

$$\psi(c_1 + c_2, c_3) + \psi(c_3, c_1) - \psi(c_1, c_3) = \psi(c_3, c_1 + c_2) + \psi(c_2, c_3) - \psi(c_3, c_2)$$

$$\text{Hence, } \eta(c_1 + c_2, c_3) = \eta(c_1, c_3) + \eta(c_2, c_3)$$

The map L_c , by linearity, is a continuous homomorphism . Now L_c induces a continuous homomorphism $L_h : H^2(C, A) \rightarrow H_{sym}^2(C, A) \oplus H^2(c, A)$; since adding a coboundary to ψ adds the same coboundary to ϕ and $H^2(c, A) = C^2(c, A)$. On the other hand ,we define

$$E_c : C_{sym}^2(C, A) \oplus C^2(c, A) \rightarrow C^2(C, A)$$

$$(2.4.3) \quad \psi(c_1, c_2) = \phi(c_1, c_2) + \frac{\eta(c_1, c_2)}{2}$$

Since every bihomomorphism from $C \times C$ to A is an element of $C^2(C, A)$ so η is a cocycle and by linearity ϕ is a cocycle. Similarly, E_c is a continuous homomorphism and induces

$$E_h : H_{sym}^2(C, A) \oplus H^2(c, A) \rightarrow H^2(C, A)$$

It is easy to show that $L_h \circ E_h = Id$ and vice versa. Hence L_h is an isomorphism.

Remark 2.5.

- (1) Let G be an abelian topological group of nilpotency class two with a 2-divisible center and $L(G)$ the Lie ring of nilpotency class two with 2-divisible center defined on the underlying set of G by cocycles (2.4.1), (2.4.2)..By Theorem 2.3 this construction does not depend on the choice of the cocycle defining a group structure in G .
- (2) For a Lie ring n of nilpotency class two with 2-divisible center, denote $E(n)$ the group of nilpotency class two the underlying set n by a cocycle (2.4.3)

By theorem 2.4 , this construction does not depend on the choice of the cocycles ϕ , η defining a Lie ring structure in n . Denote the categories of abelian topological groups of nilpotency class two with 2-divisible center and Lie rings of nilpotency class two with 2-divisible center by $\mathcal{G}_*$, $\mathcal{L}_*$ respectively.

Theorem 2.6 *There is an isomorphism between $\mathcal{G}_*$, $\mathcal{L}_*$.*

Proof. Let $G_1, G_2 \in \mathcal{G}_*$, $f : G_1 \rightarrow G_2$ be a continuous homomorphism and consider f as a map from $L(G_1)$ to $L(G_2)$ and denote it by $L(f)$. Similarly, for a continuous homomorphism $f : g_1 \rightarrow g_2$, $E(f)$ is f considered as a map from $E(n_1)$ to $E(n_2)$.

Note that by lemma 1.7 for a 2-divisible abelian topological group A , and an abelian group C every element of $H^2(C, A)$ can be represented by a normalized cocycle. Choose cocycles ψ_1, ψ_2 so that they define the group structure on G_1 and G_2 , respectively. So

$$\phi_1(c_1, c_2) = \frac{\psi_1(c_1, c_2) + \psi_1(c_2, c_1)}{2}$$

$$\phi_2(c_1, c_2) = \frac{\psi_2(c_1, c_2) + \psi_2(c_2, c_1)}{2}$$

are normalized. Now we show that if f is a group homomorphism, then $L(f)$ is a Lie ring homomorphism. Consider the following identities:

$$\begin{aligned} f(a_1 + a_2 + \psi_1(c_1, c_2), c_1 + c_2) &= f(a_1, c_1) \cdot f(a_2, c_2) \\ &= (a_1, \gamma_1) \cdot (a_2, \gamma_2) \\ &= (a_1 + a_2 + \psi_2(\gamma_1, \gamma_2)) \end{aligned}$$

$$\begin{aligned} f(-a_1 - a_2 - \psi_1(c_1, c_2), -c_1 - c_2) &= (f(a_2, c_2) \cdot f(a_1, c_1))^{-1} \\ &= ((a_2, \gamma_2) \cdot (a_1, \gamma_1))^{-1} \\ &= (-a_1 - a_2 - \psi_2(\gamma_1, \gamma_2), -\gamma_1 - \gamma_2) \end{aligned}$$

Multiplying the left and the right hand sides, we get

$$(2.5.1) \quad f(\eta_1(c_1, c_2), 0) = (\eta_2(\gamma_1, \gamma_2), 0)$$

Now dividing the central element by 2 and inversion we get,

$$f(-\frac{\eta_1(c_1, c_2)}{2}, 0) = (-\frac{\eta_2(\gamma_1, \gamma_2)}{2}, 0)$$

So

$$f(a_1 + a_2 + \phi_1(c_1 + c_2), c_1 + c_2) = (a_1 + a_2 + \phi_2(\gamma_1, \gamma_2), \gamma_1 + \gamma_2)$$

Hence

$$L(f)(n_1 + n_2) = L(f)n_1 + L(f)n_2 \quad n_1 = (a_1, c_1), \quad n_2 = (a_2, c_2)$$

This formula together with (2.5.1) gives

$$L(f)([n_1, n_2]) = [L(f)n_1, L(f)n_2],$$

that is $L(f)$ is a Lie ring homomorphism. Similarly for f a Lie ring homomorphism, $E(f)$ is a group homomorphism.

References

- [1] R. C. Alperin, H. Sahleh, *Hopf's formula and the Schur multiplier for topological groups*, Kyungpook Math. Journal. vol 31, no 1 (1991), 35-71.
- [2] E. Hewitt, K. A. Ross, *Abstract Harmonic Analysis I*, Academic Press, New York, 1963.
- [3] S.T. Hu, *Cohomology theory in topological groups*, Michigan Math. J. 1(1) (1952), 11-59.
- [4] A. Mihailovs, *The orbit method for finite groups of nilpotency class two of odd order*, preprint
- [5] J.R. Robinson, *A course in the theory of groups*, Springer-verlag, New York, 1982.
- [6] A. Shtern, *Remarks on pseudocohomology and the real continuous bounded cohomology of connected locally compact groups*, Annals of Global Analysis and Geometry 20 (2001), 199-221

Received: October 31, 2006