Nonexistence of Cusp Cross-section of One-cusped Complete Complex Hyperbolic Manifolds II

Yoshinobu KAMISHIMA

Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397, Japan
kami@comp.metro-u.ac.jp

Abstract. Long and Reid have shown that some compact flat 3-manifold cannot be diffeomorphic to a cusp cross-section of any complete finite volume 1-cusped hyperbolic 4-manifold. Similar to the flat case, we give a negative answer that there exists a 3-dimensional closed Heisenberg infranilmanifold whose diffeomorphism class cannot be arisen as a cusp cross-section of any complete finite volume 1-cusped complex hyperbolic 2-manifold. This is obtained from the formula by the characteristic numbers of bounded domains related to the Burns-Epstein invariant on strictly pseudo-convex CR-manifolds [1],[3]. This paper is a sequel of our paper [8].

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INTRODUCTION

We shall consider whether every Heisenberg infranilmanifold can be arisen, up to diffeomorphism, as a cusp cross-section of a complete finite volume 1-cusped complex hyperbolic manifold. Long and Reid considered the problem that every compact Riemannian flat manifold is diffeomorphic to a cusp cross-section of a complete finite volume 1-cusped hyperbolic manifold. They have shown it is false for some compact flat 3-manifold [11]. We shall give a negative answer similarly to the flat case.
**Theorem.** Any 3-dimensional closed Heisenberg infranilmanifold with non-trivial holonomy cannot be diffeomorphic to a cusp cross-section of any complete finite volume 1-cusped complex hyperbolic 2-manifold.

D.B. McReynolds informed us that W. Neumann and A. Reid have obtained the similar result.

2. **Heisenberg infranilmanifold**

Let $\langle z, w \rangle = \bar{z}_1 \cdot w_1 + \bar{z}_2 \cdot w_2 + \cdots + \bar{z}_n \cdot w_n$ be the Hermitian inner product defined on $C^n$. The Heisenberg nilpotent Lie group $N$ is the product $R \times C^n$ with group law:

$$ (a, z) \cdot (b, w) = (a + b - \text{Im}(z, w), z + w). $$

It is easy to see that $N$ is 2-step nilpotent, i.e. $[N, N] = (R, 0) = R$, which is the central subgroup $C(N)$ of $N$. This induces a central group extension:

$$ 1 \rightarrow C(N) \rightarrow N \xrightarrow{p} C^n \rightarrow 1. $$

Let $\text{Iso}(H^{n+1}_C)/BV$ be the full group of the isometries of the complex hyperbolic space $H^{n+1}_C$. It is isomorphic to $\text{PU}(n+1, 1) \rtimes \langle \tau \rangle$ where $\tau$ is the (anti-holomorphic) involution induced by the complex conjugation. See [5], for instance. The Heisenberg rigid motions is defined as a subgroup of the stabilizer $\text{Iso}(H^{n+1}_C)_\infty$ at the point at infinity $\infty$.

**Definition 2.1.** The group of Heisenberg rigid motions $E^\tau(N)$ is defined to be $N \rtimes (U(n) \rtimes \langle \tau \rangle)$. A Heisenberg infranilmanifold (respectively orbifold) is a compact manifold (respectively orbifold) $N/\pi$ such that $\pi$ is a torsionfree (not necessarily torsionfree) discrete cocompact subgroup of $E^\tau(N)$.

3. **CR-structure on $S^{2n+1} - S^{2n-1}$**

The sphere complement $S^{2n+1} - S^{2n-1}$ is a spherical CR manifold with the transitive group $\text{Aut}_{CR}(S^{2n+1} - S^{2n-1})$ of CR transformations which is isomorphic to the unitary Lorentz group $U(n, 1)$. Note that $S^{2n+1} - S^{2n-1}$ is identified with the $(2n + 1)$-dimensional Lorentz standard space form $V^{-2n+1}_{-1}$ of constant sectional curvature $-1$, see [9]. The center $Z(U(n, 1))$ of $U(n, 1)$ is $S^1$. Then $V^{-2n+1}_{-1}$ is the total space of the principal $S^1$-bundle over the complex hyperbolic space: $S^1 \rightarrow (U(n, 1), V_{-1}^{2n+1}) \xrightarrow{\nu} (\text{PU}(n, 1), \mathbb{H}^n_C)$. If $\omega_H$ is the connection form of the above principal bundle, then it is a contact form on $V_{-1}^{2n+1}$. The kernel $\text{Null} \omega_H$ admits a CR structure such that $d\omega_H = \nu^* \Omega_H$ up to constant factor for the Kähler form $\Omega_H$ on $\mathbb{H}^n_C$. Since $U(n, 1) = S^1 \cdot SU(n, 1)$, the above
equivariant principal bundle induces the following commutative fibrations:

\[ Z \longrightarrow (\tilde{\text{SU}}(n, 1), \tilde{V}_{2n+1}^1) \longrightarrow (\hat{V}_2, \hat{V}_{2n+1}^1) \longrightarrow (\text{PU}(n, 1), \mathbb{H}_n^1) \]

(3.1)

Here \( \tilde{\text{SU}}(n, 1) \) is a lift of \( \text{SU}(n, 1) \) associated to the covering \( Z \to \tilde{V}_{2n+1}^1 \to V_{2n+1}^1 \). For a discrete subgroup \( G \subset \text{PU}(n, 1) \) such that \( \mathbb{H}_n^1/G \) is a complete finite volume complex hyperbolic orbifold, let \( \tilde{G} \subset \text{SU}(n, 1) \) be a lift where \( 1 \to Z \to \tilde{G} \to G \to 1 \) is an exact sequence. Then \( S^1 \to \tilde{V}_{2n+1}^1/\tilde{G} \to \mathbb{H}_n^1/G \) is an injective Seifert fibration (i.e. the singular fiber bundle with typical fiber \( S^1 \). The exceptional fiber is also a circle.)

4. Burns and Epstein's formula

In general, the Heisenberg infranilmanifold (its two fold cover at least) admits a spherical \( CR \)-structure, see Definition 2.1. In [2], Burns and Epstein obtained the \( CR \)-invariant \( \mu(M) \) on the 3-dimensional strictly pseudoconvex \( CR \)-manifolds \( M \) provided that the holomorphic line bundle is trivial. Let \( X \) be a compact strictly pseudoconvex complex 2-dimensional manifold with smooth boundary \( M \). Then they have shown the following equality in [3]:

\[ \int_X c_2 - \frac{1}{3} c_1^2 = \chi(X) - \frac{1}{3} \int_X \bar{c}_1^2 + \mu(M). \]

(4.1)

Here \( \bar{c}_1 \) is a lift of \( c_1 \) by the inclusion \( j^* : H^2(X, M; \mathbb{R}) \to H^2(X; \mathbb{R}) \).

5. Geometric boundary

5.1. One-cusped complex hyperbolic 2-manifold. Let \( E^r(\mathcal{N}) \) be the group of Heisenberg rigid motions on the 3-dimensional Heisenberg nilpotent Lie group \( \mathcal{N} \) and \( L : E^r(\mathcal{N}) \to U(1) \rtimes \langle \tau \rangle \) the holonomy homomorphism. Suppose that \( M = \mathcal{N}/\Gamma \) is realized as a cusp cross-section of a complete finite volume 1-cusped complex hyperbolic 2-manifold \( W = \mathbb{H}_2^2/G \). Put \( \tilde{W} = \mathbb{H}_2^2/G - M \times (0, \infty) \) so that \( \partial \tilde{W} = M \). Then \( \tilde{W} \) is homotopic to \( W \) and \( M \) is viewed as a boundary of \( \text{Int}\tilde{W} \) which supports a complete complex hyperbolic structure. The holonomy group \( L(\Gamma) \) of a 3-dimensional compact Heisenberg non-homogeneous infranilmanifold \( M = \mathcal{N}/\Gamma \) is a cyclic subgroup of order 2, 3, 4, 6 of \( U(1) \) or \( L(\Gamma) \) is isomorphic to \( \mathbb{Z}/2 \times \mathbb{Z}/2 \subset U(1) \rtimes \langle \tau \rangle \), see [4], [12] for the classification. If \( M \) has the holonomy \( \mathbb{Z}/2 \times \mathbb{Z}/2 \), then \( G \) has nontrivial summand in \( \langle \tau \rangle \) of \( \text{Iso}(\mathbb{H}_2^2) = \text{PU}(2, 1) \rtimes \langle \tau \rangle \). The two fold cover \( \mathbb{H}_2^2/G \cap \text{PU}(2, 1) \) is still a 1-cusped complex hyperbolic manifold for which the cusp cross-section is a two fold cover of \( M \) whose holonomy group becomes
$\mathbb{Z}/2 \subset U(1)$. When the holonomy group belongs to $U(1)$, the spherical CR-
structure on $M$ is canonically induced from the complex hyperbolic structure on $W$. (Note that $\tau$ does not preserve the CR-structure bundle.)

5.2. **Integral of $c_1^2$.** Let $p : \tilde{W} \rightarrow W$ be the finite covering, say of order $\ell$, whose induced covering $\tilde{M}$ of $M$ is now a (homogeneous) nilmanifold (using the separability argument if necessary). Possibly it consists of a finite number of such nilmanifolds. Since $W$ admits a complete Einstein-Kähler metric, we know that $c_2 - \frac{1}{3}c_1^2 = 0$. Moreover, since $\tilde{M}$ is a spherical CR manifold with trivial holomorphic line bundle, it follows that $\mu(\tilde{M}) = 0$ from [2]. As in §4, let $j^* : H^2(\tilde{W}, M : \mathbb{R}) \rightarrow H^2(W : \mathbb{R})$ be the map such that $j^*\tilde{c}_1(W) = c_1(W)$. Applying (4.1) to $\tilde{W}$, we have $\chi(\tilde{W}) = \frac{1}{3} \int_{\tilde{W}} c_1^2$. As $p^*(\tilde{c}_1(\tilde{W})) = \tilde{c}_1(\tilde{W})$ by naturality and $p_*[\tilde{W}] = \ell[W]$, it follows that $\int_W c_1^2 = \langle c_1^2(W), W \rangle = \langle c_1^2(\tilde{W}), \ell[\tilde{W}] \rangle$. Since $\chi(\tilde{W}) = \ell \chi(W)$, $3\chi(W) = \langle c_1^2(W), [\tilde{W}] \rangle$. It follows that

**Proposition 5.1.** If $M = N/\Gamma$ is realized as a cusp cross-section of a complete finite volume 1-cusped complex hyperbolic 2-manifold $W = \mathbb{H}_2^2/G$, then $c_1^2(W)$ is an integer in $H^4(W, M : \mathbb{Z}) = \mathbb{Z}$.

5.3. **Torsion element in $M$.** Given a CR-structure on $M$, there is the canonical splitting $TM \otimes \mathbb{C} = B^{1,0} \oplus B^{0,1}$ where $B^{1,0}$ is the holomorphic line bundle. Since $M$ is an infranilmanifold but not homogeneous, $B^{1,0}$ is nontrivial, i.e. $c_1(B^{1,0}) \neq 0$. (In fact, it is a torsion element in $H^2(M : \mathbb{Z})$, because the $\ell$-fold covering $\tilde{M}$ has the trivial holomorphic bundle.) The spherical CR-manifold $M$ has a characteristic CR-vector field $\xi$. If $e^1$ is the vector field on $M$ pointing outward to $W$, then the distribution $\langle e^1, \xi \rangle$ generates a trivial holomorphic line bundle $TC^{1,0}$ on $M$ for which $TW \otimes \mathbb{C} | M = (B^{1,0} + TC^{1,0}) \oplus (B^{0,1} + TC^{0,1})$. As $j^*(c_1(W)) = c_1(B^{1,0} + TC^{1,0}) = c_1(B^{1,0})$ and $\ell \cdot c_1(B^{1,0}) = 0$, we have $j^*\beta = \ell \cdot c_1(W)$ for some integral class $\beta \in H^2(W, M : \mathbb{Z})$.

5.4. **$H_1(M : \mathbb{Z})$.** Let $1 \rightarrow \Delta \rightarrow \Gamma \rightarrow F \rightarrow 1$ be the group extension of the fundamental group $\Gamma = \pi_1(M)$ where $\Delta$ is the maximal normal nilpotent subgroup and $F \cong \mathbb{Z}_\ell$ ($\ell = 2, 3, 4, 6$) or $F \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Recall that $\Delta$ is generated by $\{a, b, c\}$ where $[a, b] = aba^{-1}b^{-1} = e^k$ for some $k > 0$. It follows that $\Delta/[\Delta, \Delta] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_k$. Let $\gamma$ be an element of $\Gamma$ which maps to a generator
of \( \mathbb{Z}_\ell \). A calculation shows that, up to mod \([\Delta, \Delta]\),

\[
\gamma a\gamma^{-1} = a^{-1}, \quad \gamma b\gamma^{-1} = b^{-1} (\ell = 2),
\gamma a\gamma^{-1} = b, \quad \gamma b\gamma^{-1} = a^{-1}b^{-1} (\ell = 3),
\gamma a\gamma^{-1} = b, \quad \gamma b\gamma^{-1} = a^{-1} (\ell = 4),
\gamma a\gamma^{-1} = b, \quad \gamma b\gamma^{-1} = a^{-1}b (\ell = 6).
\]

(5.1)

When \( F = \mathbb{Z}_2 \times \mathbb{Z}_2 \), let \( \delta \) be an element of \( \Gamma \) which goes to another generator of \( F \). Then \( \gamma a\gamma^{-1} = a, \quad \gamma b\gamma^{-1} = b^{-1} \mod [\Delta, \Delta] \). In view of the above relation (5.1), \( \gamma \) (also \( \delta \)) becomes a torsion element of order \( m \) in \( \Gamma/[\Gamma, \Gamma] \) where \( m \) is divisible by \( \ell \). As \( \Gamma \) is generated by \( \{a, b, c, \gamma\} \) or \( \{a, b, c, \gamma, \delta\} \), it follows that

\[
H_1(M : \mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_m \oplus \begin{cases} 
\mathbb{Z}_2 \oplus \mathbb{Z}_2 & (\ell = 2) \\
\mathbb{Z}_3 & (\ell = 3) \\
\mathbb{Z}_2 & (\ell = 4) \\
1 & (\ell = 6) \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 & \end{cases}
\]

(5.2)

In any case, if \( \mathcal{N}/\Gamma \) has a nontrivial holonomy group \( F \), then \( H_1(M : \mathbb{Z}) \) is a torsion group.

5.5. Intersection number. Put \( \tilde{H}^2(\bar{W}, M : \mathbb{Z}) = H^2(\bar{W}, M : \mathbb{Z})/\text{Tor} \) where \( \text{Tor} \) is the torsion subgroup. We have a nondegenerate inner product \( \tilde{H}^2(\bar{W}, M : \mathbb{Z}) \times \tilde{H}^2(\bar{W}, M : \mathbb{Z}) \rightarrow \mathbb{Z} \) defined by the intersection form

\[
(x, y) = \langle x \cup y, [\bar{W}] \rangle.
\]

Denote by \( \bar{W} \# \pm \mathbb{CP}^2 \) the connected sum of \( \bar{W} \) with \( \mathbb{CP}^2 \pm \mathbb{CP}^2 \). Taking the connected sum with \( \pm \mathbb{CP}^2 \) if necessary, we can assume that \( \langle , \rangle \) is an indefinite form of odd type, i.e. there are nonzero elements \( x, y \in \tilde{H}^2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z}) \) such that \( \langle x, x \rangle \) is odd and \( \langle y, y \rangle = 0 \). (Compare [7].) By \( \langle \pm 1 \rangle \) we shall mean that it is generated by either \( x_+ \) or \( x_- \) of \( \tilde{H}^2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z}) \) such that \( \langle x_+, x_\pm \rangle = \pm 1 \) respectively. Moreover by the classification of nondegenerate indefinite inner product (cf. [7]), there is an isomorphism preserving the inner product from \( \tilde{H}^2(\bar{W} \# \pm \mathbb{CP}^2, M : \mathbb{Z}) \) onto

\[
m\langle 1 \rangle \oplus n\langle -1 \rangle = \langle 1 \rangle_1 \oplus \cdots \oplus \langle 1 \rangle_m \oplus \langle -1 \rangle_1 \oplus \cdots \oplus \langle -1 \rangle_n
\]

(5.3)

for \( (m, n \neq 0) \). Here \( \langle \pm 1 \rangle_i \) is the \( i \)-th copy of \( \langle \pm 1 \rangle \). Consider the commutative diagram:

\[
\begin{array}{ccc}
H^2(\bar{W}, M : \mathbb{Z}) & \xrightarrow{j^*} & H^2(\bar{W} : \mathbb{Z}) \\
D & \downarrow & D \\
H_2(\bar{W} : \mathbb{Z}) & \xrightarrow{j^*} & H_2(\bar{W}, M : \mathbb{Z}) \\
\end{array}
\]

(5.4)

\[
\begin{array}{ccc}
H^2(\bar{W}, M : \mathbb{Z}) & \xrightarrow{i^*} & H^2(M : \mathbb{Z}) \\
D & \downarrow & D \\
H_2(\bar{W} : \mathbb{Z}) & \xrightarrow{\partial} & H_1(M : \mathbb{Z}).
\end{array}
\]
It follows from (5.2) that \( j_* : \bar{H}_2(\bar{W} : \mathbb{Z}) \rightarrow \bar{H}_2(\bar{W}, M : \mathbb{Z}) \) is injective and is isomorphic if \( \mathbb{Z} \) replaces \( \mathbb{R} \). Similarly note that \( j_* : \bar{H}_2(\bar{W}# \pm \mathbb{CP}^2 : \mathbb{Z}) \rightarrow \bar{H}_2(\bar{W}# \pm \mathbb{CP}^2, M : \mathbb{Z}) \) is injective (and an isomorphism for the coefficient \( \mathbb{R} \)). Identified the generators of \( H_2(\bar{W}# \pm \mathbb{CP}^2 : \mathbb{Z}) \) with the basis (5.3), we may choose the generators \([V_i] \in \bar{H}_2(\bar{W}# \pm \mathbb{CP}^2, M : \mathbb{Z})\) such that

\[
(5.5) \quad j_*(\langle \pm 1 \rangle_i) = \ell_i[V_i]
\]

for some \( \ell_i \in \mathbb{Z} \).

5.6. Canonical bundle. The circle bundle \( L : S^1 \rightarrow \bar{V}_1/G \rightarrow \mathbb{H}_2^2/G = W \) is represented by the Kähler form \( \Omega \) of the Kähler-hyperbolic metric, i.e. \( [\Omega] = c_1(L) \in H^2(W : \mathbb{Z}) \). Hence \( W = \mathbb{H}_2^2/G \) is projective-algebraic, i.e. \( W \subset \mathbb{CP}^N \).

Let \( \bar{W}_j : \bar{W}# \pm \mathbb{CP}^2 \) be represented by \( c_1([V]) \) for some divisor \( V \) in \( W \), i.e. \( D(c_1(W)) = [V] \in \bar{H}_2(\bar{W}, M : \mathbb{Z}) \), compare [6]. Embed \( V \) into \( \bar{W}# \pm \mathbb{CP}^2 \) and suppose that

\[
[V] = \sum_i a_i[V_i] \in \bar{H}_2(\bar{W}# \pm \mathbb{CP}^2, M : \mathbb{Z}).
\]

As \( D \circ i^*c_1(W) = \partial[V] \), it follows \( \partial\partial([V]) = 0 \) by the argument of § 5.3. We observe that \( \partial[V] \) maps into \( \mathbb{Z}_m \) in \( H_1(M : \mathbb{Z}) \) (cf. (5.2)) and so does each \( \partial[V_i] \). It may occur that \( \partial a_i[V_i] = \partial a_j[V_j] \) for some \( i, j \). So we can write

\[
[V] = k[V_1] + j, x \text{ where } x \in H_2(\bar{W}# \pm \mathbb{CP}^2 : \mathbb{Z}) \text{ and } V_1 \text{ satisfies that}
\]

1. \( \partial V_1 = S^1 \) and \( \ell[S^1] = 0 \) in \( \mathbb{Z}_m \subset H_1(M : \mathbb{Z}) \).

2. \( \ell \) is minimal with respect to Property (1).

3. \( (k, l) \) is relatively prime.

5.7. Realization of \( c_1 \). As \( \ell \partial[V_i] = 0 \) in \( H_1(M : \mathbb{Z}) \), there is a surface \( U \) in \( W \) whose cycle \( [U] \in H_2(\bar{W}# \pm \mathbb{CP}^2 : \mathbb{Z}) \) represents \( j_*[U] = \ell[V_i] \).

Let \( [U] = a_1(\pm 1)_1 + a_2(\pm 1)_2 + \ldots + a_s(\pm 1)_s \). Then, \( \ell[V_i] = a_1\ell_1[V_1] + a_2\ell_2[V_2] + \ldots + a_s\ell_s[V_s] \). Since each \( [V_i] \) is a generator of \( H_2(\bar{W}# \pm \mathbb{CP}^2, M : \mathbb{Z}) \), it follows that \( \ell = a_1\ell_1 \) and \( a_j = 0 \) (\( j \neq 1 \)). Hence \( [U] = a_1(\pm 1)_1 \). On the other hand, note that \( \langle \pm 1 \rangle_1 \) is a cycle of \( \bar{H}_2(\bar{W}# \pm \mathbb{CP}^2, M : \mathbb{Z}) \) for which \( j_*(\langle \pm 1 \rangle_1) = \ell_1[V_1] \) by (5.5). Noting that \( \ell \) is minimal by Property (2) of § 5.6, \( \ell_1 \) is divisible by \( \ell \). Therefore \( \ell_1 = \pm \ell \) and \( a_1 = \pm 1 \) so that \( [U] = \pm \langle \pm 1 \rangle_1 \). In particular, the intersection number

\[
(U) \cdot [U] = \pm 1.
\]

Put \( y = \frac{k}{\ell}[U] + x \in H_2(\bar{W}# \pm \mathbb{CP}^2 : \mathbb{R}) \). Calculate

\[
y \cdot y = \frac{k^2}{\ell^2}[U] \cdot [U] + 2k \frac{[U]}{\ell} \cdot x = \pm \frac{k^2}{\ell^2} + 2k \frac{[U]}{\ell} \cdot x = x \cdot x,
\]
(5.7) \[ \ell(y \cdot y) = \pm \frac{k^2}{\ell} \mod \mathbb{Z} \]

Noting that \((k, \ell) = 1\) by Property (3) of § 5.6, if \(\ell \neq 1\), \(y \cdot y\) cannot be an integer.

As \(j_*(\frac{k}{\ell} [U]) = k[V_1]\), note that \(j_*y = k[V_1] + j_*x = [V]\). Consider the following diagram:

\[
\begin{array}{c}
H^2(W^\# \pm \mathbb{CP}^2 : \mathbb{R}) \xrightarrow{j^*} H^2(W^\# \pm \mathbb{CP}^2, M : \mathbb{R}) \\
\| \xrightarrow{j_*+id} \| \\
\overline{H}_2(W : \mathbb{R}) + \langle 1 \rangle + \langle -1 \rangle \xrightarrow{j_*+id} \overline{H}_2(W, M : \mathbb{R}) + \langle 1 \rangle + \langle -1 \rangle.
\end{array}
\]

Let \(y = y_0 + t\langle 1 \rangle + s\langle -1 \rangle\) for some \(y_0 \in \overline{H}_2(W : \mathbb{R}), s, t \in \mathbb{R}\). As \(j_*y = [V]\), it follows that \([V] = j_*y_0 + t\langle 1 \rangle + s\langle -1 \rangle\). Noting \([V] \in \overline{H}_2(W, M : \mathbb{Z})\), we have that \([V] = j_*y_0\) and \(t = s = 0\). In particular, this implies that \(y = y_0 \in \overline{H}_2(W : \mathbb{R})\). By the commutative diagram (5.4) and using the fact \(D(c_1(W)) = [V]\) (cf. §5.6), the element \(D^{-1}(y) \in H^2(W, M : \mathbb{R})\) satisfies that \(j^*(D^{-1}(y)) = c_1(W)\).

On the other hand, recall from the argument of [3] that the integral \(\langle \overline{c}_1(W), [\overline{W}] \rangle\) does not depend on the choice of lift \(\overline{c}_1(W)\) to \(c_1(W)\), so we can choose \(\overline{c}_1(W) = D^{-1}(y) \in H^2(W, M : \mathbb{R})\) (cf. §5.2). By definition, \(y \cdot y = \langle \overline{c}_1(W)^2, [\overline{W}] \rangle\) which is an integer by Proposition 5.1. This contradiction proves Theorem.

**Remark 5.2.** Neumann and Reid have shown that if an infranil 3-manifold arises as a cusp cross-section of a 1-cusped complex hyperbolic 2-manifold, then the rational Euler number must be 1/3-integral. There are infranil manifolds which do not satisfy this condition.

**References**


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