A New Proof of the Beez-Cartan Theorem

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Abstract

The present paper contains a completely new, alternative proof of the Beez-Cartan theorem. It is not obtained, as in the traditional textbooks, in the context of Differential Geometry, but rather with simple methods of Multilinear Algebra. For that purpose, we first give the definitions needed to characterize the Levi-Civita connection, and then we prove its existence and uniqueness by describing an effective algorithm to calculate the \( n^3 \) coordinates of the linear forms which determine it. The final section consists of an analysis of the Heisenberg algebra, which lets us give a concrete example of the results obtained.

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1 Introduction

There is no doubt that E. Cartan’s method of moving frames \([4]\) constitutes one of the most important synthetic achievements in modern Mathematics. By introducing it, we are able to obtain a description of geometry or physics systems independent from the reference chosen or the local inertia system, which permits a global analysis of their properties. Used with profusion between 1940 and 1950 \([10]\), that method was almost abandoned until the beginning of the seventies, when Harrison and Eastbrook \([8]\) started to develop the theory again in the context of integration problems in partial differential equations \([13, 14]\). For instance, using differential forms, it is possible to derive non relativist Maxwell’s Electrodynamics equations in an effective and brief way \([11, 13]\). In addition, at the moment some authors \([1, 5]\) are solving coadjoint representation problems in Lie algebras and groups by means of moving
frames. There are also plenty of applications related to practical questions arisen in Quantum Theory, Elementary Particles Physics (gauge symmetries), or even in Robotic [3], where it is very convenient to describe robots dynamics respect to a reference which moves with the articulations at a time.

One of the classical applications of Cartan’s method is the study of the rigidity properties of the regular submanifolds of a differential riemannian manifold $M$ of zero or positive curvature, that is, its characterization by means of the isometries group of $M$. In this sense, the rigidity Beez-Killing-Cartan theorem claims that two isometric hypersurfaces in $\mathbb{R}^n$ such that the rank of the second fundamental form is at least three differ just in a movement. That theorem has been adapted to more general submanifolds, and also when immersions are not regular [2, 9]. Several authors are introducing new techniques which may be eventually useful to prove the result again [6, 9]. In that context, the following pages show that an algebraic approach is possible too. We first define some concepts axiomatically in order to preserve their meaning in Differential Geometry.

Let $V$ be a finite dimensional vector space over $\mathbb{R}$. Let $B = \{\bar{e}_1, \ldots, \bar{e}_n\}$ be a basis for $V$, and $B^* = \{\sigma^1, \ldots, \sigma^n\}$ its dual.

**Definition 1.** A linear connection over $V$ is a mapping $d : V \rightarrow V^* \otimes V$ such that, when applied to vectors of the basis $B$, is:

$$d(\bar{e}_i) = \sum_{j=1}^{n} \omega^i_j \otimes \bar{e}_j,$$

where $\omega^i_j$ are linear forms.

**Notation:** From now on, we will follow the agreement usually introduced in Multilinear Algebra and Differential Geometry for eliminating summatory symbols, understanding that crossed indices are the ones which add. We will also write $d\bar{e}_i$ instead of $d(\bar{e}_i)$, because of historical reasons. In order to obtain a matrical expression of the linear connection, we will abbreviate $\omega^i_j \otimes \bar{e}_j$ by $\omega^i_j \bar{e}_j$.

It is clear that the linear connection is determined if and only if the $n^2$ linear forms $\omega^i_j$ are known. Then, if $\bar{x} = \lambda^i \bar{e}_i$, since $d$ is a linear operator, $d\bar{x} = d\lambda^i \bar{e}_i = \lambda^i d\bar{e}_i$. Thus, written in matrix form, the linear connection is given by:

$$d\bar{x} = (\lambda^1, \lambda^2, \cdots, \lambda^n) \begin{pmatrix} \omega^1_1 & \omega^2_1 & \cdots & \omega^n_1 \\ \omega^1_2 & \omega^2_2 & \cdots & \omega^n_2 \\ \vdots & \vdots & \ddots & \vdots \\ \omega^1_n & \omega^2_n & \cdots & \omega^n_n \end{pmatrix} \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \vdots \\ \bar{e}_n \end{pmatrix}$$

(1)
**Definition 2.** We will call exterior differential to a linear operator \( d : \Lambda^p \rightarrow \Lambda^{p+1} \) which verifies the properties:

1. \( d \circ d = 0 \)
2. If \( \alpha \in \Lambda^p \), \( d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \)
3. \( d\omega = 0 \), if \( \omega \in \Lambda^0 = \mathbb{R} \)

**Definition 3.** The exterior differential we have just defined lets us generalize the concept of linear connection to another operator, which we will keep on representing by \( d \), such that:

\[
d(\omega_h \otimes \bar{x}) = d\omega_h \otimes \bar{x} + (-1)^h \omega_h \wedge d\bar{x},
\]

(2)

where \( d \) applied to \( h \)-linear forms is the exterior differential, and \( d \) applied to vectors is the linear connection. In particular, if \( h = 0 \), then the new operator coincides with the initial linear connection.

**Definition 4.** We will call fundamental tensor \( dP \) to the mapping

\[
dP = \sigma^i \otimes \bar{e}_i.
\]

Let us note that \( dP \) is well defined, because it does not depend on the basis \( B \) chosen. In addition, the fundamental tensor has the property that, when applied to vectors in \( V \), gives back the identity. In fact, if \( \bar{x} = \lambda^j \bar{e}_j \), then:

\[
dP(\bar{x}) = (\sigma^i \otimes \bar{e}_i)(\bar{x}) = \sigma^i(\bar{x})\bar{e}_i = \sigma^i(\lambda^j \bar{e}_j)\bar{e}_i = \lambda^j \sigma^i(\bar{e}_j)\bar{e}_i = \lambda^j \delta^i_j \bar{e}_i,
\]

where \( \delta^i_j \) is Kronecker’s delta. Therefore, \( dP(\bar{x}) = \lambda^j \bar{e}_j \).

In matrix form, \( dP \) is written:

\[
dP = \begin{pmatrix} \sigma^1 & \sigma^2 & \ldots & \sigma^n \end{pmatrix} \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \vdots \\ \bar{e}_n \end{pmatrix}
\]

**Definition 5.** We will call torsion tensor \( T \) to \( d \) applied to fundamental tensor, in other words,

\[
T = ddP = d(\sigma^i \otimes \bar{e}_i).
\]

From (2) we obtain:

\[
T = d\sigma^i \otimes \bar{e}_i - \sigma^i \wedge d\bar{e}_i = d\sigma^i \otimes \bar{e}_i - \sigma^i \wedge d\omega^i_j \bar{e}_j = (d\sigma^i - \sigma^j \wedge d\omega^i_j) \bar{e}_j.
\]
If $\langle \cdot, \cdot \rangle$ is an inner product defined over $V$, and $B = \{\bar{e}_1, \ldots, \bar{e}_n\}$ is an orthonormal basis for $V$, then there exists an unique linear connection whose torsion tensor is zero and whose matrix respect to $B$ is antisymmetric.

**Definition 6.** We will call Levi-Civita connection to the unique linear connection $\bar{d}e_i = \omega^j_i \bar{e}_j$ such that:

1. $T = d\sigma = 0$.
2. $\omega^j_i = -\omega^i_j$.

## 2 The Levi-Civita connection

We now prove the existence and uniqueness of the Levi-Civita connection.

As in the previous definitions, let $B = \{\bar{e}_1, \ldots, \bar{e}_n\}$ be an orthonormal basis for $V$, $B^* = \{\sigma^1, \ldots, \sigma^n\}$ its dual, and let $d$ be the Levi-Civita connection. The problem consists of determining its linear forms $\omega^j_i$. For that purpose, we consider the coordinates of each one respect from $B^*$: let $\Gamma^j_{ik}$ denote the $k$-th coordinate of $\omega^j_i$.

Then, the above definition is equivalent to:

1. $T = (d\sigma^i - \sigma^j \wedge d\omega^j_i)\bar{e}_i = 0$
2. $\Gamma^j_{ik} = -\Gamma^i_{jk}$ (antisymmetry),

from which we can derive:

$$d\sigma^i - \sigma^j \wedge d\omega^j_i = 0$$
$$d\sigma^i = \sigma^j \wedge d\omega^j_i = \sigma^j \wedge d(\Gamma^i_{jk} \sigma^k) = \Gamma^i_{jk} \sigma^j \wedge \sigma^k$$

Writing it explicitly, a more useful expression is obtained:

$$d\sigma^i = \Gamma^i_{1k} \sigma^1 \wedge \sigma^k + \Gamma^i_{2k} \sigma^2 \wedge \sigma^k + \cdots + \Gamma^i_{nk} \sigma^n \wedge \sigma^k =$$
$$= (\Gamma^i_{11} \sigma^1 \wedge \sigma^1 + \Gamma^i_{12} \sigma^1 \wedge \sigma^2 + \cdots + \Gamma^i_{1n} \sigma^1 \wedge \sigma^n) +$$
$$+ (\Gamma^i_{21} \sigma^2 \wedge \sigma^1 + \Gamma^i_{22} \sigma^2 \wedge \sigma^2 + \cdots + \Gamma^i_{2n} \sigma^2 \wedge \sigma^n) +$$
$$+ \cdots + (\Gamma^i_{n1} \sigma^n \wedge \sigma^1 + \Gamma^i_{n2} \sigma^n \wedge \sigma^2 + \cdots + \Gamma^i_{nn} \sigma^n \wedge \sigma^n)$$

Well now, wedge product is anticommutative, so:
\[ d\sigma^i = \sigma^1 \wedge \left[ \Gamma_{11}^i \sigma^1 + (\Gamma_{12}^i - \Gamma_{21}^i) \sigma^2 + \cdots + (\Gamma_{1n}^i - \Gamma_{n1}^i) \sigma^n \right] + \sigma^2 \wedge \left[ \Gamma_{22}^i \sigma^2 + (\Gamma_{23}^i - \Gamma_{32}^i) \sigma^3 + \cdots + (\Gamma_{2n}^i - \Gamma_{n2}^i) \sigma^n \right] + \cdots + \sigma^{n-1} \wedge \left[ \Gamma_{(n-1)(n-1)}^i \sigma^{n-1} + (\Gamma_{(n-1)n}^i - \Gamma_{n(n-1)}^i) \sigma^n \right] + \Gamma_{nn}^i \sigma^n \wedge \sigma^n \]

and finally, because of the connection’s antisymmetry:

\[ d\sigma^i = (\Gamma_{jk}^i - \Gamma_{kj}^i) \sigma^j \wedge \sigma^k, \quad j < k. \quad (4) \]

Here \( d \) is the exterior differential, so we know \( d\sigma^i \) and \( \sigma^j \wedge \sigma^k \). Therefore, \( A_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i \) may be calculated.

Since \( A_{jk}^i \) are defined for all \( i, j, k \in \{1, \ldots, n\} \), there is no problem on changing indices’s role by permuting them cyclically:

\[
\begin{align*}
\Gamma_{jk}^i - \Gamma_{kj}^i &= A_{jk}^i \\
\Gamma_{ki}^i - \Gamma_{ik}^i &= A_{ki}^i \\
\Gamma_{ij}^i - \Gamma_{ji}^i &= A_{ij}^i
\end{align*}
\]

Adding the first two equalities and subtracting the third one, we obtain:

\[
\Gamma_{jk}^i - \Gamma_{kj}^i + \Gamma_{ki}^i - \Gamma_{ik}^i + \Gamma_{ij}^i = A_{jk}^i + A_{ki}^i - A_{ij}^i
\]

Then, if the antisymmetric condition 2 is applied to the last three coordinates, we have

\[
\Gamma_{jk}^i - \Gamma_{kj}^i + \Gamma_{ki}^i + \Gamma_{jk}^i - \Gamma_{kj}^i = 2\Gamma_{jk}^i
\]

Thus,

\[
\Gamma_{jk}^i = \frac{A_{jk}^i + A_{ki}^i - A_{ij}^i}{2}, \quad (5)
\]

and it is enough to give to \( i, j, k \) the proper values to obtain the \( n^3 \) coordinates which determine all connection’s linear forms.

### 3 Gauss equation. First and second fundamental forms

Now let \((E^{n+1}, <\cdot, \cdot>)\) be the \( n+1 \)-dimensional Euclidean vector space with the usual inner product. We consider an hyperplane \( H \) such that the exterior derivative and the canonical inclusion’s dual \( i : H \to E^{n+1} \) commute, that is:

\[
i^* \circ d = d \circ i^*.
\]
Let \( \{\bar{e}_1, \ldots, \bar{e}_n, \bar{e}_{n+1}\} \) be a basis for \( V \), and \( \{\bar{\omega}^1, \ldots, \bar{\omega}^n, \bar{\omega}^{n+1}\} \) its dual. Then, 
\( \bar{\omega}^{n+1}(\bar{x}) = 0 \) \( \forall \bar{x} \in H \).

In \( E^{n+1} \), linear connections are defined exactly as we did in definition 6:
\[
d\bar{e}_i = \omega^j_i \bar{e}_j \quad j = 1, \ldots, n + 1
\]

We will characterize the Levi-Civita connection as the unique linear connections which annihilates the fundamental tensor and whose associated matrix (1) is antisymmetric:
\[
d\bar{\omega}^i - \bar{\omega}^j \wedge \bar{\omega}^j_i = 0 \quad j = 1, \ldots, n + 1
\]

If we denote \( \omega^j_i = i^*(\bar{\omega}^j_i) \) and \( \omega^i = i^*(\bar{\omega}^i) \), then
\[
i^*(d\bar{\omega}^{n+1}) = d(i^*(\bar{\omega}^{n+1})) = 0,
\]
and separating \( \bar{\omega}^{n+1} \) from the other forms, it remains:
\[
d\omega^i = \omega^j_i \wedge \omega^j \quad i, j = 1, \ldots, n,
\]
\[
d\omega^{n+1} = \omega^j \wedge \omega^{n+1}_j \quad j = 1, \ldots, n + 1.
\] (6)

Now let us suppose that the curvature of \( d \) is null in \( E^{n+1} \), that is,
\[
dd\bar{e}_i = 0 \quad i = 1, \ldots, n + 1,
\]
Then,
\[
0 = ddd\bar{e}_i = d(\omega^j_i \otimes \bar{e}_j) = d\omega^j_i \otimes \bar{e}_j - \omega^j_i \wedge d\bar{e}_j = d\omega^j_i \otimes \bar{e}_j - \omega^j_i \wedge \omega^j_i \bar{e}_j =
\]
\[
= (d\omega^j_i - \omega^h_i \wedge \omega^j_i) \bar{e}_j
\]

Since coordinates respect from a given basis are unique, we obtain
\[
d\omega^j_i = \omega^h_i \wedge \omega^j_i = \omega^h_i \wedge \omega^j_i + \omega^{n+1}_i \wedge \omega^{n+1}_i,
\] (7)
formula usually known as Gauss equation.

We study now how the linear connection transforms basis vectors:
\[
d\bar{e}_1 = \bar{\omega}^1_1 \bar{e}_1 + \bar{\omega}^1_2 \bar{e}_2 + \cdots + \bar{\omega}^n_1 \bar{e}_n + \bar{\omega}^{n+1}_1 \bar{e}_{n+1}
\]
\[
d\bar{e}_2 = \bar{\omega}^1_2 \bar{e}_1 + \bar{\omega}^2_2 \bar{e}_2 + \cdots + \bar{\omega}^n_2 \bar{e}_n + \bar{\omega}^{n+1}_2 \bar{e}_{n+1}
\]
\[
\vdots
\]
\[
d\bar{e}_n = \bar{\omega}^1_n \bar{e}_1 + \bar{\omega}^2_n \bar{e}_2 + \cdots + \bar{\omega}^n_n \bar{e}_n + \bar{\omega}^{n+1}_n \bar{e}_{n+1}
\]
\[
d\bar{e}_{n+1} = \bar{\omega}^1_{n+1} \bar{e}_1 + \bar{\omega}^2_{n+1} \bar{e}_2 + \cdots + \bar{\omega}^n_{n+1} \bar{e}_n + \bar{\omega}^{n+1}_{n+1} \bar{e}_{n+1}
\]
Considering connection linear forms’ antisymmetry, the above system may be rewritten as:

\[
\begin{align*}
d\bar{e}_1 &= 0 + \bar{\omega}_1^2 \bar{e}_2 + \cdots + \bar{\omega}_1^n \bar{e}_n + \bar{\omega}_1^{n+1} \bar{e}_{n+1} \\
d\bar{e}_2 &= -\bar{\omega}_2^2 \bar{e}_1 + 0 + \cdots + \bar{\omega}_2^n \bar{e}_n + \bar{\omega}_2^{n+1} \bar{e}_{n+1} \\
&\vdots \\
d\bar{e}_n &= -\bar{\omega}_n^n \bar{e}_1 - \bar{\omega}_n^n \bar{e}_2 - \cdots - 0 + \bar{\omega}_n^{n+1} \bar{e}_{n+1} \\
d\bar{e}_{n+1} &= -\bar{\omega}_n^{n+1} \bar{e}_1 - \bar{\omega}_n^{n+1} \bar{e}_2 - \cdots - \bar{\omega}_n^n \bar{e}_n + 0
\end{align*}
\]

Finally, if we restrict \(d\) to \(H\), it results:

\[
\begin{align*}
d\bar{e}_1 &= 0 + \omega_1^2 \bar{e}_2 + \cdots + \omega_1^n \bar{e}_n \\
d\bar{e}_2 &= -\omega_2^2 \bar{e}_1 + 0 + \cdots + \omega_2^n \bar{e}_n \\
&\vdots \\
d\bar{e}_n &= -\omega_n^n \bar{e}_1 - \omega_n^n \bar{e}_2 - \cdots - \omega_n^{n-1} \bar{e}_{n-1} + 0,
\end{align*}
\]

which may be expressed in matrix form as:

\[
\begin{pmatrix}
  d\bar{e}_1 \\
  d\bar{e}_2 \\
  \vdots \\
  d\bar{e}_n
\end{pmatrix} =
\begin{pmatrix}
  0 & \omega_1^2 & \cdots & \omega_1^n \\
  -\omega_2^2 & 0 & \cdots & \omega_2^n \\
  \vdots & \vdots & \ddots & \vdots \\
  -\omega_n^n & -\omega_n^n & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
  \bar{e}_1 \\
  \bar{e}_2 \\
  \vdots \\
  \bar{e}_n
\end{pmatrix}
\]

(8)

**Definition 7.** We will call first fundamental form to the quadratic form

\[
I = \sum_{i=1}^{n} \omega^i \otimes \omega^i
\]

Reducing \(I\) to its diagonal form, we are able to calculate the basis \(\{\omega^1, \ldots, \omega^n\}\), and, by means of an analogous method to the one described in section two, the Levi-Civita connection linear forms may be known.

**Definition 8.** We will call second fundamental form to the quadratic form

\[
II = - \langle d\bar{e}_{n+1}, dP \rangle
\]

where \(\langle \ldots, \ldots \rangle\) is the usual inner product.

**Proposition 1.** The second fundamental form is symmetric and its expression in the basis \(\{\omega^1, \ldots, \omega^n\}\) is given by

\[
II = \Gamma^{n+1}_{jk} \omega^k \otimes \omega^j
\]
Proof. Just by substituting $de_{n+1}$ and fundamental tensor formulae in the previous definition, we have:

$$II = - < d\bar{e}_{n+1}, dP > = - < \omega^j_{n+1} \bar{e}_j, \omega^i \bar{e}_i > = - < -\omega^j_{n+1} \bar{e}_j, \omega^i \bar{e}_i >$$

Now we consider the coordinates of $\omega^j_{n+1}$ respect from the dual basis. If $\omega^j_{n+1} = \Gamma^j_{n+1} \omega^k$, then:

$$II = < \Gamma^j_{n+1} \omega^k \bar{e}_j, \omega^i \bar{e}_i > = \Gamma^j_{n+1} \omega^k \otimes \omega^i < \bar{e}_i, \bar{e}_j >$$

Since $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is an orthonormal basis,

$$II = \Gamma^j_{n+1} \omega^k \otimes \omega^i$$

the expression we wanted to obtain.

It remains to prove the symmetry of $II$, that is:

$$\Gamma^j_{n+1} = \Gamma^j_{n+1} \quad j, k = 1 \ldots n.$$ 

Let be $\omega^j_{n+1} = \Gamma^j_{n+1} \omega^k$. Since $d$ and $i^*$ commute, by (6)

$$0 = \omega^j \wedge \omega^j_{n+1} = \omega^j \wedge (\Gamma^j_{n+1} \omega^k) = \Gamma^j_{n+1} \omega^j \wedge \omega^k$$

Making the same calculus which allowed us to obtain (3),

$$(\Gamma^j_{n+1} - \Gamma^j_{n+1}) \omega^j \wedge \omega^k = 0,$$

and it must be $\Gamma^j_{n+1} = \Gamma^j_{n+1}$.

\[\Box\]

4 The Beez-Cartan rigidity theorem

Theorem 1. If the rank of the second fundamental form is equal to or greater than three, then the first fundamental forms determines the second one.

Proof. Deriving the second form respect from each element of the basis, we have

$$\frac{\partial II}{\partial \omega^i} = 2 \Gamma^j_{n+1} \omega^j = 2 \omega^i_{n+1}. \quad (9)$$

Because of the symmetry proved in the above proposition, the spectral theorem claims the existence of a basis of linear forms respect from which $II$ is diagonal and $I$ remains invariant. Let $\{\sigma^1, \sigma^2, \ldots, \sigma^n\}$ be that basis:

$$I = \sum_{i=1}^{n} \sigma^i \otimes \sigma^i \quad (10)$$

$$II = \sum_{i=1}^{n} a_i \sigma^i \otimes \sigma^i \quad (11)$$
Then, the number of nonzero \( a_i \) is the rank of \( II \) and coincides with the number of linearly independent \( \omega^{n+1}_i \), since: \( \omega^{n+1}_i = a_i \sigma^i \).

By assumption, there are at least three linearly independent \( \omega^{n+1}_i \): let us denote them by \( \omega^{n+1}_i, \omega^{n+1}_j, \omega^{n+1}_k \). The Gauss equation obtained in (7) will be written as:

\[
d\omega^j_i - \omega^h_i \wedge \omega^j_h = -a_i a_j \sigma^i \wedge \sigma^j = -\lambda_{ij} \sigma^i \wedge \sigma^j
\]

Just as we remarked when defining \( I \), the first fundamental form determines all the connection linear forms. Therefore, from Gauss equation the following products are known:

\[
a_i a_j = \lambda_{ij}, \quad a_i a_k = \lambda_{ik}, \quad a_j a_k = \lambda_{jk}
\]  

(12)

By multiplying the first two equations and dividing by the third one, we have:

\[
a_i^2 = \frac{\lambda_{ij} \lambda_{ik}}{\lambda_{jk}},
\]

and finally

\[
a_i = \pm \sqrt{\frac{\lambda_{ij} \lambda_{ik}}{\lambda_{jk}}},
\]  

(13)

If we choose every positive solution (otherwise we would be changing the space orientation), the coefficients of \( II \) in the basis \( \{ \sigma^1, \sigma^2, \ldots, \sigma^n \} \) are calculated uniquely.

**Remark 1.** The condition that rank of the second fundamental form is equal to or greater than three is sufficient but not necessary. If there exist only two linearly independent \( \omega^i_j \), then the unique equation we are able to derive from (12) is

\[
a_i a_j = \lambda_{ij},
\]

which determines \( II \) except for a proportional factor. Let us note that this factor could not be arbitrary, since \( \omega^i_j \) must satisfy Gauss equation.
5 A practical case: The Heisenberg algebra

We will complete our study by illustrating the sense of the concepts and techniques used with a practical example introduced by Werner Heisenberg in order to quantize the harmonic oscillator.

Let $\mathbb{R}^3$ be the three-dimensional Euclidean vector space with the usual inner product. Let $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ be an orthonormal basis, and $B^* = \{\omega^1, \omega^2, \omega^3\}$ its dual. We define there an exterior differential product. Let $\mathcal{R}$ be the three-dimensional Euclidean vector space with the usual inner product.

Moreover, we have just defined $\omega$. So it is enough to calculate $d\omega$ and its dual. We define there an exterior differential $d$ such that:

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = d\omega^3 = 0$$ (14)

Then, if $d$ is the Levi-Civita connection in $\mathbb{R}^3$:

$$d\bar{e}_1 = \omega^1_1 \bar{e}_1 + \omega^1_2 \bar{e}_2 + \omega^1_3 \bar{e}_3$$
$$d\bar{e}_2 = \omega^2_1 \bar{e}_1 + \omega^2_2 \bar{e}_2 + \omega^2_3 \bar{e}_3$$
$$d\bar{e}_3 = \omega^3_1 \bar{e}_1 + \omega^3_2 \bar{e}_2 + \omega^3_3 \bar{e}_3,$$

or, equivalently,

$$d\bar{e}_1 = \omega^1_2 \bar{e}_2 + \omega^1_3 \bar{e}_3$$
$$d\bar{e}_2 = -\omega^2_1 \bar{e}_1 + \omega^2_3 \bar{e}_3$$
$$d\bar{e}_3 = -\omega^3_1 \bar{e}_1 - \omega^3_2 \bar{e}_2$$ (15)

So it is enough to calculate $\omega_1^1, \omega_1^2, \omega_1^3$. As in the previous two sections, let $\Gamma_{ij}^{k}$ be the coordinates of these linear forms. From (4) we can derive:

$$d\omega^1 = (\Gamma_{12}^1 - \Gamma_{21}^1)\omega^1 \wedge \omega^2 + (\Gamma_{13}^1 - \Gamma_{31}^1)\omega^1 \wedge \omega^3 + (\Gamma_{23}^1 - \Gamma_{32}^1)\omega^2 \wedge \omega^3$$
$$d\omega^2 = (\Gamma_{12}^2 - \Gamma_{21}^2)\omega^1 \wedge \omega^2 + (\Gamma_{13}^2 - \Gamma_{31}^2)\omega^1 \wedge \omega^3 + (\Gamma_{23}^2 - \Gamma_{32}^2)\omega^2 \wedge \omega^3$$
$$d\omega^3 = (\Gamma_{12}^3 - \Gamma_{21}^3)\omega^1 \wedge \omega^2 + (\Gamma_{13}^3 - \Gamma_{31}^3)\omega^1 \wedge \omega^3 + (\Gamma_{23}^3 - \Gamma_{32}^3)\omega^2 \wedge \omega^3$$ (16)

Moreover, we have just defined $d$ applied to $\omega^1, \omega^2, \omega^3$. Evaluating (14) for the special case (16) we obtain:

$$\omega^2 \wedge \omega^3 = (\Gamma_{12}^1 - \Gamma_{21}^1)\omega^1 \wedge \omega^2 + (\Gamma_{13}^1 - \Gamma_{31}^1)\omega^1 \wedge \omega^3 + (\Gamma_{23}^1 - \Gamma_{32}^1)\omega^2 \wedge \omega^3$$
$$0 = (\Gamma_{12}^2 - \Gamma_{21}^2)\omega^1 \wedge \omega^2 + (\Gamma_{13}^2 - \Gamma_{31}^2)\omega^1 \wedge \omega^3 + (\Gamma_{23}^2 - \Gamma_{32}^2)\omega^2 \wedge \omega^3$$
$$0 = (\Gamma_{12}^3 - \Gamma_{21}^3)\omega^1 \wedge \omega^2 + (\Gamma_{13}^3 - \Gamma_{31}^3)\omega^1 \wedge \omega^3 + (\Gamma_{23}^3 - \Gamma_{32}^3)\omega^2 \wedge \omega^3$$

The latter two equations imply:

$$\Gamma_{11}^2 = -\Gamma_{21}^1 = -\Gamma_{12}^1 = 0,$$
$$\Gamma_{31}^3 = -\Gamma_{31}^1 = -\Gamma_{13}^1 = 0,$$
$$\Gamma_{32}^3 = -\Gamma_{32}^1 = \Gamma_{23}^2 = 0,$$
$$\Gamma_{12}^2 = \Gamma_{21}^2 = 0,$$
$$\Gamma_{13}^3 = \Gamma_{31}^3 = 0,$$
$$\Gamma_{23}^3 = \Gamma_{32}^3 = 0.$$
But, since $d\omega^1 = \omega^2 \wedge \omega^3$, we also have $\Gamma^1_{23} - \Gamma^1_{32} = 1$, or

$$\Gamma^3_{12} = \Gamma^2_{13} + 1 \quad (17)$$

Thus, $\omega^2_1, \omega^3_1, \omega^3_2$ are given by:

$$\omega^2_1 = \Gamma^2_{13} \omega^3$$

$$\omega^3_1 = \Gamma^3_{12} \omega^2 = (\Gamma^2_{13} + 1) \omega^2$$

$$\omega^3_2 = \Gamma^3_{21} \omega^1 = (\Gamma^2_{13} + 1) \omega^1,$$

and everything is reduced to calculate the coefficient $\Gamma^2_{13}$. By means of the result obtained in (5):

$$\Gamma^2_{13} = \frac{A^2_{13} + A^1_{32} - A^3_{21}}{2} = -\frac{1}{2}$$

Finally,

$$\omega^2 = -\frac{1}{2} \omega^3$$

$$\omega^3_1 = \Gamma^3_{12} \omega^2 = \frac{1}{2} \omega^2$$

$$\omega^3_2 = \Gamma^3_{21} \omega^1 = \frac{1}{2} \omega^1$$

Now let us consider $H = L(\{\bar{e}_1, \bar{e}_2\})$ the subspace spanned by the first two vectors of the basis, which is an ideal of the Heisenberg algebra. The first fundamental form is written

$$I = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3,$$

and the second one is given by

$$II = -\Gamma^3_{12} \omega^1 \otimes \omega^2 - \Gamma^3_{21} \omega^2 \otimes \omega^1 = -\frac{1}{2} \omega^1 \otimes \omega^2 - \frac{1}{2} \omega^2 \otimes \omega^1,$$

so $\text{rank}(II) = 2$. Nevertheless, it is clear that $I$ determines $II$, precisely because of the formula obtained in (17).

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