A New Class of Positively Quadrant Dependent Bivariate Distributions with Pareto

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Abstract

This paper presents a new class of bivariate distributions with Pareto of Marshall-Olkin type. The approach in this paper uses a shock model to derive a class of positively quadrant dependent of bivariate distributions with Pareto. Further this class generalizes the bivariate Pareto distributions which obtained by Muliere and Scarsini (1987). The dependence structure and main other properties of the bivariate distributions with Pareto are investigated. The difference among the dependent and independent cases of bivariate Pareto distributions is discussed.

Keywords: Bivariate pareto distributions, Minimum distribution, Positively Quadrant dependent, Regression dependent.

1 Introduction

The study of bivariate distributions was mostly confined to the normal, exponential and Weibull cases see for example Barlow and Proschan (1981). Partly for this reason general properties of bivariate distributions are not often discussed. It is well known that sometimes failure rate can occur for more than one reason and the multivariate distributions is a nice tool for modeling such situation. For example, assume that $T_1$ and $T_2$ are the times at that two specific components of an electronic system fail. If these components will fail at the same time with probability $p$, then their common failure time may be distributed according to some univariate distributions. On the other hand these components will fail at different times with probability $1 - p$, and in this case their failure times should be distributed according to some bivariate distributions.

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Pareto distributions have wide applications in different fields such as the field of income and wealth modeling as well as failure times, birth rate, mortality rates and reliability models.

The multivariate Pareto distributions have important applications in modeling problems involving distributions of incomes when incomes exceed a certain limit. For example, income from several sources, it is not at all clear that we will be able to confidently visualize marginal features of the distribution. Rather we might be able to speculate that for given levels of income sources $2, 3, \ldots, k$ the income from source 1 will have a Pareto-like distribution with parameters perhaps depending on the level of income from the other sources, see (Arnold, 1993, El-Gohary, 2005).

The reliability analysis and electronics widely use the univariate distributions such as exponential, Weibull, linear failure rate, Pareto and others, see for example Barlow and Proschan (1981); Sarhan and El-Gohary (2003) and El-Gohary (2004).

Several basic multivariate parametric families of distributions such as exponential, Weibull, linear failure rate distributions, and shock models that give rise to them are considered by Barlow and Proschan (1981); Lee (1979); Patra, and Dey, (1999) and El-Gohary (2005). Earlier, Marshall and Olkin (1967a,b,1981) considered a shock model to derive a bivariate exponential distribution. Generalization of bivariate exponential distribution is proposed by Marshall and Olkin (1967b). In the other hand the early study of bivariate distributions with Pareto considered by Mardia (1962), Johnson and Kotz (1972), and its recently studies are presented by Arnold (1993); Gupta (2001); El-Gohary (2005) and others. El-Gohary (2006) used the multivariate distributions with pareto to study the competing risk reliability models.

The objects of this paper are to suggest a new family of bivariate Pareto distributions with general form which can allowed to both marginal are univariate Pareto form as a special case, generalizes Muliere and Scarsini (1987)’s class of bivariate Pareto and to discuss the main properties of this family such as quadrant dependent, regression dependent and others.

The paper is organized as follows. Section two deals with the introduction of a new family of bivariate distributions with Pareto using a shock models. Section three introduces the probability density function of this family of bivariate Pareto distributions as well as its main properties. Section four presents the expectation, covariance and correlation of the bivariate distributions with Pareto. Section five concerned with the dissuasion of the difference among the dependent and independent cases.
2 The Bivariate Pareto model

In many reliability situations it is more realistic to assume some positively dependence form among the life time system components. The positive dependence between components life time arises from common environmental stresses and shocks from components depending common sources of power and the like. This section presents a general class of bivariate distributions with Pareto as a shock model.

2.1 Bivariate Pareto distributions as a Shock Model

We consider a bivariate Pareto distributions of type II for life lengths of two nonindependent components. Suppose three independent sources of shocks are present in the system environment. A shock from the first source destroys the system first component at a random time \( T_1 \) where survival function of \( T_1 \) is given by

\[
\bar{F}_{T_1}(t_1) = P(T_1 > t_1) = \left( \frac{a_1}{a_1 + t_1} \right)^{\theta_1}, \quad t_1 > 0, \quad a_1, \theta_1 > 0 \quad (2.1)
\]

A shock from the second source destroys the system second component at a random time \( T_2 \), where survival function of \( T_2 \) is given by

\[
\bar{F}_{T_2}(t_2) = P(T_2 > t_2) = \left( \frac{a_2}{a_2 + t_2} \right)^{\theta_2}, \quad t_2 > 0, \quad a_2, \theta_2 > 0 \quad (2.2)
\]

Finally, a shock from the third source destroys both of the first and second system components at a random time \( T \) where survival function of \( T \) is given by

\[
\bar{F}_T(t) = P(T > t) = \left( \frac{a}{a + t} \right)^{\theta}, \quad t > 0, \quad a, \theta > 0 \quad (2.3)
\]

Therefore, the random life lengths \( X_1, X_2 \) of the system first and second component should satisfy

\[
X_1 = \min(T_1, T), \quad X_2 = \min(T_2, T) \quad (2.4)
\]

The joint survival functions \( \bar{F}_{T_i, T}(t_i, t), (i = 1, 2) \) of the independent random variables \((T, T_i), (i = 1, 2)\) are presented in the following lemma.

**Lemma 2.1** The joint distribution function of the random variables \( (T_i, T) \) are given by

\[
\bar{F}_{T_i, T}(t_i, t) = \left( \frac{a_i}{a_i + t_i} \right)^{\theta_i} \left( \frac{a}{a + t} \right)^{\theta}, \quad (i = 1, 2) \quad (2.5)
\]

**Proof.** Based on the survival functions (2.1), (2.2) and (2.3) of \( T_i, (i = 1, 2) \) and \( T \) respectively, we can prove this lemma as follows.

\[
\bar{F}_{T_i, T}(t_i, t) = P(T_i > t_i, T > t), \quad (i = 1, 2)
\]
Using the independence of the random variables \(T_1, T_2\) and \(T\) we get

\[
\bar{F}_{T_1,T}(t_i, t) = P(T_i > t_i) P(T > t) = \left(\frac{a_i}{a_i + t_i}\right)^{\theta_i} \left(\frac{a}{a + t}\right)^{\theta_i}, \quad t_i, t \geq 0, \theta_i > 0, (i = 1, 2) \tag{2.6}
\]

In addition to the joint survival functions of \((T_1, T)\) and \((T_2, T)\), we can easily obtained their joint distributions in the form

\[
F_{T_i,T}(t_i, t) = 1 - \left(\frac{a_i}{a_i + t_i}\right)^{\theta_i} - \left(\frac{a}{a + t}\right)^{\theta_i} + \left(\frac{a_i}{a_i + t_i}\right)^{\theta_i} \left(\frac{a}{a + t}\right)^{\theta_i}, \quad t, t_i \geq 0, \theta, \theta_i > 0, (i = 1, 2) \tag{2.7}
\]

Next, the following lemma gives the joint survival function of the bivariate random variable \((X_1, X_2)\) with Pareto distribution of type II.

**Lemma 2.2** The joint survival function of the bivariate random variable \((X_1, X_2)\) is given by

\[
\bar{F}_{X_1,X_2}(x_1, x_2) = \left(\frac{a}{a + z_0}\right)^{\theta} \prod_{i=1}^{2} \left(\frac{a_i}{a_i + x_i}\right)^{\theta_i}, \quad z_0 = \max(x_1, x_2) \tag{2.8}
\]

**Proof.** The proof of this lemma is based on the use of the bivariate random variable \((X_1, X_2)\) definition and survival functions of \(T, T_i, i = 1, 2\). That is

\[
\bar{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) = P\left[\min(T_1, T) > x_1, \min(T_2, T) > x_2\right] \tag{2.9}
\]

Therefore

\[
\bar{F}(x_1, x_2) = P\left[T_1 > x_1, T_2 > x_2, T > \max(x_1, x_2)\right] \tag{2.10}
\]

Since \(T_1, T_2\) and \(T\) are independent random variables and using (2.1), (2.2) and (2.3), then one gets the formula (2.8) of the joint survival function of bivariate random variable \((X_1, X_2)\).

Further the joint distribution function of the bivariate random variable \((X_1, X_2)\) can easily obtained in the form

\[
F_{X_1,X_2}(x_1, x_2) = \left[1 - \left(\frac{a}{a + z_0}\right)^{\theta}\right] \prod_{i=1}^{2} \left[1 - \left(\frac{a_i}{a_i + x_i}\right)^{\theta_i}\right], \quad z_0 = \min(x_1, x_2) \tag{2.11}
\]

The following lemma provides the marginal survival functions of the random variables \(X_i, i = 1, 2\) of the bivariate random variable \((X_1, X_2)\).

**Lemma 2.3** The marginal survival function of the random variable \(X_i, i = 1, 2\) is given by
\[ F_{X_i}(x_i) = \left( \frac{a_i}{a_i + x_i} \right)^{\theta_i} \left( \frac{a}{a + x_i} \right)^{\theta}, \quad (i = 1, 2) \]  \hspace{1cm} (2.12)

**Proof.** Based on the definition of the marginal survival function of the random variable \( X_i \) we have

\[ \bar{F}_{X_i}(x_i) = P(X_i > x_i), \quad (i = 1, 2) \]  \hspace{1cm} (2.13)

Using the definition of \( X_i, (i = 1, 2), \) we get

\[ \bar{F}_{X_i}(x_i) = P[\min(T_i, T) > x_i] = P[T_i > x_i, T > x_i], \quad x_i \geq 0, \ a, a_i > 0 \ (i = 1, 2) \]  \hspace{1cm} (2.14)

Therefore using (2.1), (2.2) and (2.3) one gets the formula (2.12), which completes the proof.

One can noted that the marginal survival and marginal distributions of the random variables \( X_1 \) and \( X_2 \) have a Pareto form only if \( a_1 = a_2 = a. \)

### 3 Bivariate Pareto Distributions

The aim of this section is to present the joint probability density function (pdf) of the bivariate random variable with type II Pareto distributions and its marginal densities. Some main properties of this random variable such as quadrant dependent, regression dependent and others are given.

#### 3.1 Joint pdf of Pareto Distributions

The following theorem provides an approach of obtaining the joint bivariate probability density function of bivariate Pareto when the component of the random variable can be equal with positive probability.

**Theorem 3.1** If the bivariate survival function \( \bar{F}_{X_1, X_2}(x_1, x_2) \) of the random variable \( (X_1, X_2) \) takes the following form:

\[ \bar{F}_{X_1, X_2}(x_1, x_2) = \left( \frac{a}{a + z} \right)^{\theta} \prod_{i=1}^{2} \left( \frac{a_i}{a_i + x_i} \right)^{\theta_i}, \text{ where } z = \max(x_1, x_2) \]  \hspace{1cm} (3.1)

then the joint density function (pdf) of the bivariate random variable \( (X_1, X_2) \) is given by

\[ f_{X_1, X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & x_1 > x_2 \\ f_2(x_1, x_2), & x_1 < x_2 \\ f_0(x_1, x_2), & x_1 = x_2 = x \end{cases} \]  \hspace{1cm} (3.2)
where

\[ f_1(x_1, x_2) = \]

\[
\frac{\theta_2}{a_2 (a_2 + x_2)} \left[ \frac{\theta_1}{a_1 (a_1 + x_1)} \right]^{\theta_1 + 1} \left[ \frac{a_1}{a_1 + x_1} \right]^{\theta_1} \left( \frac{a}{a + x_1} \right)^{\theta + 1} \left( \frac{a_1}{a_1 + x_1} \right)^{\theta_1} + \frac{\theta}{a} \left( \frac{a}{a + x_1} \right)^{\theta + 1} \left( \frac{a_1}{a_1 + x_1} \right)^{\theta_1} \]

\[ f_2(x_1, x_2) = \]

\[
\frac{\theta_1}{a_1 (a_1 + x_1)} \left[ \frac{\theta_2}{a_2 (a_2 + x_2)} \right]^{\theta_2 + 1} \left[ \frac{a_2}{a_2 + x_2} \right]^{\theta_2} + \frac{\theta}{a} \left( \frac{a}{a + x_2} \right)^{\theta + 1} \left( \frac{a_2}{a_2 + x_2} \right)^{\theta_2} \]

\[ f_0(x_1, x_2) = \frac{\theta}{a} \left( \frac{a_1}{a_1 + x} \right)^{\theta_1} \left( \frac{a_2}{a_2 + x} \right)^{\theta_2} \left( \frac{a}{a + x} \right)^{\theta + 1} \] (3.3)

**Proof.** The proof of this theorem is based on obtaining the forms of \( f_1(x_1, x_2) \) and \( f_2(x_1, x_2) \) by differentiating the joint survival function \( F_{X_1,X_2}(x_1, x_2) \) that given by (3.1) with respect to both \( x_1 \) and \( x_2 \) twice times. In the other hand, the function \( f_0(x_1, x_2) \) will be obtained from the identity

\[
\int_0^\infty \int_0^{x_1} f_1(x_1, x_2) dx_2 dx_1 + \int_0^\infty \int_{x_1}^\infty f_2(x_1, x_2) dx_2 dx_1 + \int_0^\infty f_0(x, x) dx = 1, \]

(3.4)

Since

\[
\int_0^{x_1} f_1(x_1, x_2) dx_2 dx_1 = 1 + \int_0^\infty \left( \frac{a_2}{a_2 + x_1} \right)^{\theta_2} \left( \frac{a}{a + x_1} \right)^{\theta} \frac{d}{dx_1} \left( \frac{a_1}{a_1 + x_1} \right)^{\theta_1} \left( \frac{a}{a + x_1} \right)^{\theta} dx_1, \]

(3.5)

and

\[
\int_{x_1}^\infty f_2(x_1, x_2) dx_2 dx_1 = - \int_0^\infty \left( \frac{a_2}{a_2 + x_1} \right)^{\theta_2} \left( \frac{a}{a + x_1} \right)^{\theta} \frac{d}{dx_1} \left( \frac{a_1}{a_1 + x_1} \right)^{\theta_1} \left( \frac{a}{a + x_1} \right)^{\theta} dx_1, \]

(3.6)

Substituting from (3.5) and (3.6) into (3.4) one can get

\[
\int_0^\infty \left\{ f_0(x, x) - \frac{\theta}{a} \left( \frac{a_1}{a_1 + x} \right)^{\theta_1} \left( \frac{a_2}{a_2 + x} \right)^{\theta_2} \left( \frac{a}{a + x} \right)^{\theta + 1} \right\} dx = 0, \]

(3.7)

Therefore, since this above integral satisfies for all positive value of \( x \) then the function \( f_0(x, x) \) is given by

\[
f_0(x, x) = \frac{\theta}{a} \left( \frac{a_1}{a_1 + x} \right)^{\theta_1} \left( \frac{a_2}{a_2 + x} \right)^{\theta_2} \left( \frac{a}{a + x} \right)^{\theta + 1}, \quad x > 0 \]

(3.8)

which completes the proof.

The following lemma provides the marginal pdfs of the bivariate random variable with Pareto of type II. Further these pdfs have a Pareto forms only if \( a_1 = a_2 = a \).
Lemma 3.1 The marginal pdf’s of the bivariate random variable \((X_1, X_2)\) which has pdf given by (3.2) are given by

\[
f_{X_i}(x_i) = \frac{\theta}{a} \frac{a_i}{a + x_i} \left( \frac{a_i}{a_i + x_i} \right)^{\theta_i} + \frac{\theta_i}{a_i} \frac{a_i}{a_i + x_i} \left( \frac{a_i}{a_i + x_i} \right)^{\theta_i + 1} \left( \frac{a}{a + x_i} \right)^{\theta}, \quad (i = 1, 2)
\]

(3.9)

Proof. The proof of this lemma can be reached by integrating the pdf with respect to \(x_i\) as follows

\[
f_{X_1}(x_1) = \int_0^{x_1} f_1(x_1, x_2) dx_2 + \int_{x_1}^{\infty} f_2(x_1, x_2) dx_2 + f_0(x_1, x_2)
\]
\[
f_{X_2}(x_2) = \int_0^{x_2} f_2(x_1, x_2) dx_1 + \int_{x_2}^{\infty} f_1(x_1, x_2) dx_1 + f_0(x_1, x_2)
\]

(3.10)

Substituting from (3.3) into (3.10) and after some algebraic manipulation one gets the formula (3.9), which completes the proof.

Note the marginal distributions of the random variables \(X_i, \quad (i = 1, 2)\) have Pareto distributions with parameters \((a, \theta_i + \theta)\), \((i = 1, 2)\) only if \(a_i = a, \quad (i = 1, 2)\).

Next we study the bivariate dependence of the random variable \((X_1, X_2)\), the distribution of minimum random variable, the regression dependence and many main other properties of bivariate distributions with Pareto.

Definition 3.2 Random variables \(X_1\) and \(X_2\) are positively quadrant dependent (PQD) if the inequality (Lai and Xie, 2000)

\[
P\left[ X_1 \leq x_1, X_2 \leq x_2 \right] \geq P\left[ X_1 \leq x_1 \right] P\left[ X_2 \leq x_2 \right], \quad \text{for all } x_1, x_2
\]

(3.11)

holds.

The following lemma presents the bivariate dependence of the distributions with Pareto.

Lemma 3.3 The two random variables \(X_1\) and \(X_2\) with bivariate distribution (2.11) are positively quadrant dependent (PQD).

Proof. The proof of this lemma can be achieved by using the bivariate distributions of \((X_1, X_2)\) and its marginal distributions taking into account the positively quadrant dependent definition. Since

\[
P\left[ X_1 \leq x_1, X_2 \leq x_2 \right] = \left[ 1 - \left( \frac{a}{a + z_0} \right)^{\theta} \right] \prod_{i=1}^{2} \left[ 1 - \left( \frac{a_i}{a_i + x_i} \right)^{\theta_i} \right], \quad z_0 = \min (x_1, x_2)
\]

(3.12)

and

\[
P\left[ X_1 \leq x_1 \right] P\left[ X_2 \leq x_2 \right] = \prod_{i=1}^{2} \left[ 1 - \left( \frac{a_i}{a_i + x_i} \right)^{\theta_i} \right] \left[ 1 - \left( \frac{a}{a + x_i} \right)^{\theta} \right]
\]

(3.13)
Using (3.12) and (3.13) we can easily verify that

\[
P[X_1 \leq x_1, X_2 \leq x_2] \prod_{i=1}^{2} \left[1 - \left(\frac{a}{a + x_i}\right)^{\theta_i}\right] = \left[1 - \left(\frac{a}{a + z_0}\right)^{\theta}\right] \prod_{i=1}^{2} P[X_i \leq x_i]
\]

(3.14)

from which we can easily verify that the inequality (3.11) holds for all \(x_1, x_2 > 0\). Therefore the bivariate random variable \((X_1, X_2)\) is positively quadrant dependent, which completes the proof.

An important property of independent Pareto random variables \(X_1\) and \(X_2\) is that \(\min(X_1, X_2)\) is Pareto if \(a_1 = a_2 = a\). The following lemma gives the distribution of minimum random variable of bivariate distribution with Pareto.

**Lemma 3.4** The bivariate Pareto random variable \((X_1, X_2)\) has a minimum Pareto distributed with parameters \((a, \theta + \theta_1 + \theta_2)\) if the parameters \(a, a_i, (i = 1, 2)\) are equal, i.e \(a_1 = a_2 = a\).

**Proof.** The proof of this lemma can be reached as follows:

\[
P\left(\min(X_1, X_2) > x\right) = P\left(T_1 > x, T_2 > x, T > x\right)
\]

Using the definitions of \(T_1, T_2\) and \(T\) and their independence we have

\[
P\left(\min(X_1, X_2) > x\right) = \left(\frac{a}{a + x}\right)^{\theta} \prod_{i=1}^{2} \left(\frac{a_i}{a_i + x}\right)^{\theta_i}
\]

(3.15)

Apply the condition \(a_1 = a_2 = a\), one gets the survival function of \(\min(X_1, X_2)\) has a Pareto distribution with parameters \((a, \theta + \theta_1 + \theta_2)\), which completes the proof.

In what follows we discuss the bivariate regression dependent of bivariate distributions with Pareto.

**Theorem 3.2** The random variable \(X_1\) of the bivariate distributions with Pareto defined by the survival distribution (3.1) is positively regression dependent \(PRD(X_1|X_2)\) on \(X_2\).

**Proof.** The proof of this theorem can be reach by applying the definition of positively regression dependent. The random variable \(X_1\) is positively regression dependent of \(X_2\) if \(P(X_1 \leq x_1|X_2 = x_2)\) is non-increasing function of \(x_2\) for all \(x_1\) (Tong,1980).

\[
P(X_1 \leq x_1|X_2 = x_2) = \left\{\int_{0}^{x_2} f(\xi, x_2)d\xi\right\} / f_{X_2}(x_2)
\]

(3.16)
Substituting from (3.2) into (3.16), after some lengthy manipulation one gets

\[ P(X_1 \leq x_1 | X_2 = x_2) = \left\{ \left( \frac{a_1}{a_1 + x_1} \right)^{\theta_1} f_{X_2}(x_2) + \frac{\theta_2}{a_2} \left( \frac{a_2}{a_2 + x_2} \right)^{\theta_2+1} \right\} f_{X_2}^{-1}(x_2) \]

which can be written in the form

\[ P(X_1 \leq x_1 | X_2 = x_2) = \left( \frac{a_1}{a_1 + x_1} \right)^{\theta_1} + \Psi(x_2) \quad (3.17) \]

where

\[ \Psi(x_2) = \frac{\theta_2(a + x_2)^{\theta+1}}{a^{\theta}(a_2 + x_2) + \theta_2(a + x_2)} \]

This function is non-increasing function of \( x_2 \), since we can verify that \( \Psi(x_2) < \Psi(x_2^*) \) for all \( x_2 > x_2^* \) and for all fixed \( x_1 \), which completes the proof.

The next theorem provides some important properties of conditional probabilities of bivariate distributions with Pareto.

**Theorem 3.3** For the bivariate random variable \((X_1, X_2)\) which is Pareto distributed with the survival distribution (3.1) its conditional probabilities satisfy the following properties:

1. The random variable \( X_1 \) is a right-tail increasing function of \( X_2 \), if \( X_1 > X_2 \)
2. The random variable \( X_2 \) is a right-tail increasing function of \( X_1 \), if \( X_2 > X_1 \)
3. The random variable \( X_1 \) is a left-tail decreasing function of \( X_2 \), if \( X_1 < X_2 \)
4. The random variable \( X_2 \) is a left-tail decreasing function of \( X_1 \), if \( X_2 > X_1 \)

**Proof.** For a bivariate random variable \((X_1, X_2)\) the variable \( X_1 \) is said to be a right-tail increasing function in \( X_2 \) if \( P(X_1 > x_1 | X_2 > x_2) \) is non-decreasing function of \( X_2 \) for all \( x_1 \). Now for \( X_1 > X_2 \) one gets

\[ P(X_1 > x_1 | X_2 > x_2) = \bar{F}_{X_1}(x_1) \left( \frac{a}{a + x_2} \right)^{-\theta} \quad (3.19) \]

which we can easily verify that its non-decreasing function of \( x_2 \) for all \( x_1 \).

In a similar manner, one can uses the condition \( X_2 > X_1 \) to get

\[ P(X_2 > x_2 | X_1 > x_1) = \bar{F}_{X_2}(x_2) \left( \frac{a}{a + x_1} \right)^{-\theta} \quad (3.20) \]
which we can easily verify that its non-decreasing function of \( x_1 \) for all \( x_2 \).

In the other hand, for a bivariate random variable \((X_1, X_2)\) the variable \( X_1 \) is said to be left-tail increasing function in \( X_2 \) if \( P(X_1 \leq x_1 | X_2 \leq x_2) \) is non-increasing function of \( X_2 \) for all \( x_1 \). Applying this definition when \( X_1 < X_2 \) one gets

\[
P(X_1 \leq x_1 | X_2 \leq x_2) = \bar{F}_{X_1}(x_1)\left[1 - \left(\frac{a}{a + x_2}\right)^\theta\right]^{-1}
\]  

which we can verify that \( X_2 \) is a left-tail increasing function in \( X_1 \), which completes the proof.

**Corollary 3.5** Based on Theorem 3.1, we can now state a similar characterization result for Marshall and Olkin type of bivariate exponential distribution. In this context consider the nonlinear transformation \( X_1 = a(e^{U_1} - 1), X_2 = a(e^{U_2} - 1) \) for the bivariate random variable \((X_1, X_2)\) in Theorem 3.1. Then the bivariate random variable \((U_1, U_2)\) has the following joint density function

\[
f_{U_1, U_2}(u_1, u_2) = \begin{cases} 
\theta_2(\theta_1 + \theta) \exp\left[-(\theta_1 + \theta)u_1 - \theta_2 u_2\right], & u_1 > u_2 \\
\theta_1(\theta_2 + \theta) \exp\left[-\theta_1 u_1 - (\theta_2 + \theta)u_2\right], & u_1 < u_2 \\
\theta \exp\left[-(\theta_1 + \theta_2 + \theta)u\right], & u_1 = u_2 = u
\end{cases}
\]  

**Proof.** The proof of this corollary follows by calculate the Jacobian matrix

\[
J = \frac{\partial(x_1, x_2)}{\partial(u_1, u_2)},
\]

of the nonlinear transformation

\[
U_1 = \log\left(1 + \frac{X_1}{a}\right), \quad U_2 = \log\left(1 + \frac{X_2}{a}\right),
\]  

and substitute into the density equation which relates between the density functions of \((X_1, X_2)\) and \((U_1, U_2)\) as

\[
f_{X_1, X_2}(x_1, x_2) = |J| f_{U_1, U_2}(u_1, u_2)
\]

using (3.2) and (3.3) into the above density function, the resulting probability density function \( f_{U_1, U_2}(u_1, u_2) \) is given by the form (3.22), which completes the proof. The next section provides the expectation, covariance and correlation of bivariate random variable with Pareto distribution of type II.

**Lemma 3.6** Based on Theorem 3.1 Muliere and Scarsini (1987)’s bivariate Pareto distribution can be obtained as a special case from (3.2) by setting \( a_1 = a_2 = a \) and using the linear transformation \( X = X_1 + a, Y = X_2 + a \) into (3.2).
Proof. The proof of this lemma can be reached directly by setting \( a_1 = a_2 = a \) and using the linear transformation \( X = X_1 + a, Y = X_2 + a \) into (3.2) to get the joint density function obtained by Muliere and Scarsini (1987)'s in the form

\[
f_{X,Y}(x, y) = \begin{cases} 
\frac{\theta_2(\theta_1+\theta)}{a^2} \left( \frac{x}{a} \right)^{-(\theta+\theta_1+1)} \left( \frac{y}{a} \right)^{-(\theta_2+1)}, & \text{if } x > y \\
\frac{\theta}{a} \left( \frac{x}{a} \right)^{-(\theta+\theta_1+\theta_2+1)}, & \text{if } x = y \\
\frac{\theta_1(\theta_2+\theta)}{a^2} \left( \frac{y}{a} \right)^{-(\theta+\theta_2+1)} \left( \frac{x}{a} \right)^{-(\theta_1+1)}, & \text{if } y > x 
\end{cases}
\] (3.24)

which completes the proof. Therefore the new class of bivariate distributions with Pareto generalizes known result in this area.

4 Bivariate Pareto Expectations and Covariance

This section presents expressions for the expectations of \( X_i, X_i^2 \ (i = 1, 2) \) and \( X_1X_2 \) of bivariate random variable \((X_1, X_2)\) that has survival distribution with Pareto (3.1). Also the covariance and correlation of this bivariate random variable is proved to be positive.

Lemma 4.1 For the bivariate random variable \((X_1, X_2)\) with pareto distribution (3.1), the expectations of \( X_i, X_i^2 \ (i = 1, 2) \) and \( X_1X_2 \) and variance of \( X_i, (i = 1, 2) \) are given by

\[
E(X_i) = \frac{a}{\theta + \theta_i - 1}, \quad E(X_i^2) = \frac{a^2}{(\theta + \theta_i - 2)(\theta + \theta_i - 1)} \quad i = 1, 2, \quad (4.1)
\]

\[
E(X_1X_2) = \frac{(2\theta + \theta_1 + \theta_2 - 2)a^2}{(\theta + \theta_1 - 1)(\theta + \theta_2 - 1)(\theta + \theta_1 + \theta_2 - 2)} \quad (4.2)
\]

and

\[
\sigma^2_{X_i} = \frac{a^2}{(\theta + \theta_i - 1)(\theta + \theta_i - 2)}, \quad \theta + \theta_i > 2, \quad i = 1, 2 \quad (4.3)
\]

Proof. Now the expectation of \( X_i, (i = 1, 2) \) can be calculate from the expectation definition in the form:

\[
E(X_i) = \int_{0}^{\infty} \int_{0}^{x_1} x_1 f_1(x_1, x_2) dx_2 dx_1 + \int_{0}^{\infty} \int_{x_1}^{\infty} x_1 f_2(x_1, x_2) dx_2 dx_1 + \int_{0}^{\infty} x f_0(x, x) dx \quad (4.4)
\]
Substituting (3.3) into (4.4) and after some algebraic manipulation we get

\[ E(X_i) = \frac{a}{\theta + \theta_i - 1}, \quad i = 1, 2 \quad (4.5) \]

Also the expression \( X_1X_1 \) can be obtained using the definition

\[ E(X_1X_2) = \int_0^\infty \int_0^{x_1} x_1x_2f_1(x_1, x_2)dx_2dx_1 + \int_0^\infty \int_{x_1}^\infty x_1x_2f_2(x_1, x_2)dx_2dx_1 + \int_0^\infty x^2f_0(x, x)dx \quad (4.6) \]

Substituting (3.3) into (4.6) and after lengthy algebraic manipulation we get

\[ E(X_1X_2) = \frac{2\theta + \theta_1 + \theta_2 - 2}{(\theta + \theta_1 - 1)(\theta + \theta_2 - 1)(\theta + \theta_1 + \theta_2 - 2)} \quad (4.7) \]

Similarly we can prove that the expectation of \( X_i^2 \) and the variance of \( X_i \) as given by (4.1) and (4.3) respectively.

**Lemma 4.2** The covariance and correlation of the bivariate distribution \((X_1, X_2)\) with Pareto with the bivariate survival (3.1) is given by:

\[ \text{Cov}(X_1, X_2) = \frac{a^2 \theta}{(\theta + \theta_1 - 1)(\theta + \theta_2 - 1)(\theta + \theta_1 + \theta_2 - 2)} > 0, \quad (4.8) \]

and

\[ \rho(X_1, X_2) = \frac{\theta}{\theta + \theta_1 + \theta_2 - 2} \sqrt{\frac{(\theta + \theta_1 - 1)(\theta + \theta_2 - 2)}{(\theta + \theta_1 - 1)(\theta + \theta_2 - 1)}} \quad (4.9) \]

where \( \theta + \theta_i > 2, \quad i = 1, 2 \).

**Proof.** The proof of this lemma can be reach using the covariance definition which is

\[ \text{Cov}(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) \quad (4.10) \]

Substituting (4.1) and (4.2) into (4.10), one gets the formula (4.8) of the covariance of \((X_1, X_2)\). Hence the covariance of \((X_1, X_2)\) is positive.

Substituting (4.3) and (4.8) into

\[ \rho(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} \]

one gets the formula (4.9) of the correlation of \((X_1, X_2)\), which completes the proof.

Finally the following section discusses the difference among the dependent and independent cases of bivariate distribution with Pareto.
5 Comparison with independent Case

It is common practice in reliability theory to assume that the components of a system have independent life times. It is important to investigate the effect of this assumption when the system components have lives a bivariate distribution with Pareto.

Consider the marginal survival of bivariate with Pareto that given by (2.12) and suppose we operate under the assumption that the joint distribution \( F_{X_1,X_2}(x_1, x_2) \) is \( F_{X_1}(x_1)F_{X_2}(x_2) \) when in fact, \( F_{X_1,X_2}(x_1, x_2) \) is given by (3.1). Clearly, the difference

\[
F_{X_1,X_2}(x_1, x_2) - F_{X_1}(x_1)F_{X_2}(x_2) = \left\{ \prod_{i=1}^{2} \left( \frac{a_i}{a_i + x_i} \right)^{\theta_i} \right\} \left\{ \left( \frac{a}{a + z} \right)^{\theta} - \prod_{i=1}^{2} \left( \frac{a}{a + x_i} \right)^{\theta} \right\}
\]

(5.1)

where \( z = \max(x_1, x_2) \).

For every \( x_1, x_2 > 0 \) this difference is positive which means that the probability that both items survive is actually greater than the assumption of independence.

We Note that the difference is decreasing function of \( x_1 \) if \( x_1 < x_2 \) and decreasing function of \( x_2 \) if \( x_1 > x_2 \), so that for the special case \( a_1 = a_2 = a \) we find

\[
F_{X_1,X_2}(x_1, x_2) - F_{X_1}(x_1)F_{X_2}(x_2) \leq \left( \frac{a}{a + z} \right)^{\theta + \theta_1 + \theta_2} \left[ 1 - \left( \frac{a}{a + z} \right)^{\theta} \right]
\]

Further the maximum difference between the dependent and independent survival times of the system occurs at

\[
z = a \left[ \left( 1 + \frac{\theta}{\theta + \theta_1 + \theta_2} \right)^{\frac{1}{\theta_1 + \theta_2}} - 1 \right]
\]

(5.2)

Thus we can easily conclude that the maximum difference among the dependent and independent case of bivariate distribution with Pareto depends upon the parameter \( \theta \) as well as both of \( \theta_i \), \( (i = 1, 2) \). Further this difference tends to zero as any of \( \theta, \theta_i \) tends to infinity.

6 Conclusion

The new family of bivariate with Pareto distributions has many different applications in the reliability theory and the field of income. This family is a positively quadrant dependent. It is proved that this family tends to bivariate exponential Marslall and Olkin type with nonlinear transformation. The
distribution of minimum of the bivariate with Pareto has a Pareto form if the distribution parameters satisfy the condition $a_1 = a_2 = a$. Further many different properties of the new family are discussed. Finally the maximum difference among the dependent and independent cases is presented.

References


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