Strong Convergence of Nonexpansive Nonself-Mapping in Hilbert Space

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Abstract. Let $C$ be a closed convex subset of a Hilbert space $H$, $T : C \rightarrow H$ a non-expansive nonself-mapping satisfying weakly inwardness condition such that $F(T) \neq \emptyset$. In this paper, we study the convergence of the following type sequence generated by

$$x_{n+1} = P(\alpha_n f(x_n) + (1 - \alpha_n)Tx_n)$$

where $P : H \rightarrow C$ is a projective operator, $f : C \rightarrow C$ is a fixed contractive mapping, we prove that the sequence $\{x_n\}$ converges strongly to a fixed point of $T$ under the weaker condition $Tx_{n+1} - Tx_n \rightharpoonup 0$, as $n \rightarrow \infty$. The result improves some recent results, especially Yisheng Song and Rudong Chen [5, J. Math. Anal. Appl., 321 (2006), 316-326] and Hongkun Xu [6, J. Math. Anal. Appl., 298 (2004), 279-291].

Keywords: Fixed point; nonexpansive nonself-mapping; strong convergence; Banach limit

1. Introduction and Preliminaries

Throughout the paper, let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to $x$. Similarly, $x_n \rightarrow x$ will symbolize strong convergence. Let $C$ be a nonempty closed convex subset of $H$, and $T : C \rightarrow H$ be a non-expansive nonself-mapping such that $F(T) \neq \emptyset$. In 1992, Marino and Trombetta(see reference[3]) considered the contraction $S_t : C \rightarrow C$ and $U_t : C \rightarrow C$ defined by

$$S_t(x) = tPTx + (1 - t)u \quad x \in C$$

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and

$$U_t(x) = P(tTx + (1 - t)u), \quad x \in C$$

for each $t \in [0, 1)$ and each fixed $u \in C$, where $P : H \to C$ is a projective operator, then $S_t$ and $U_t$ has a unique fixed point. By Banach contractive theorem, there is a unique $x_t \in F(S_t)$, $y_t \in F(U_t)$ such that

$$x_t = tPTx_t + (1 - t)u$$

and

$$y_t = P(tTy_t + (1 - t)u)$$

In 1995, Xu and Yin (see reference [2]) proved that if $T : C \to H$ is a non-expansive nonself-mapping, then as $t \to 0$, the sequence $\{x_t\}$ defined by (1.1) strongly converges to $P_{F(T)}u$, the sequence $\{y_t\}$ defined by (1.2) strongly converges to $P_{F(T)}u$, where $P : H \to C$ is the nearest point projection of $H$ onto $C$, i.e. for each $x \in H$, $Px$ is the unique element of $C$ that satisfies $\|x - Px\| = d(x, C) = \inf_{c \in C} \|x - c\|$. This result which has been proved by Takahashi and Kim (see reference [4]) is true in Banach space too.

In 1992, Wittmann proved the following theorem (see reference [1]):

Let $H$ be a Hilbert space, $C$ be a nonempty closed convex subset of $H$, and $T : C \to C$ be a non-expansive mapping, and $F(T) \neq \emptyset$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n \quad \text{for} \quad n = 1, 2, 3 \ldots$$

where $u \in C$ is a fixed point, $\{\alpha_n\} \in (0, 1)$ and satisfies

$$\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \text{and} \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ strongly converges to $Pu \in F(T)$, where $P : C \to F(T)$ is a projective operator.

**Definition 1.1.** $\mu$ is called a Banach limit if $\mu$ is a continuous linear functional on $l^\infty$ satisfying:

$$(i) \|\mu(e)\| = 1 = \mu(1), \quad e = (1, 1, 1, \ldots)$$

$$(ii) \mu(a_n) = \mu(a_{n+1}), \forall a_n \in (a_0, a_1, \ldots) \in l^\infty$$

$$(iii) \lim \inf_{n \to \infty} a_n \leq \mu(a_n) \leq \lim \sup_{n \to \infty} a_n, \forall a_n \in (a_0, a_1, \ldots) \in l^\infty$$

According to time and circumstances, we use $\mu(a_n)$ instead of $\mu(a_0, a_1, \ldots)$.

Further, we know the following result:

**Lemma 1.2.** (see reference [1]) For a given $a \in R$, for all $\{a_n\} \in l^\infty$ satisfies $\mu(a_n) \leq a$, if $\lim \sup_{n \to \infty} (a_{n+1} - a_n) \leq 0$, then $\lim \sup_{n \to \infty} a_n \leq a$.

The proof of the following Lemma is in [8]:
Lemma 1.3. Let \( \{\alpha_n\} \) be a sequence of nonnegative real numbers satisfying the property
\[
\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \forall n \geq 0
\]
where \( \{\gamma_n\} \in (0, 1), \sum_{n=0}^{\infty} |\delta_n| < \infty \) and satisfies \( \sum_{n=0}^{\infty} \gamma_n = \infty \), \( \limsup_{n \to \infty} \frac{\delta_n}{\gamma_n} \leq 0 \).
Then \( \lim_{n \to \infty} \alpha_n = 0 \).

We now define
\[
U_t x = P(tf(x) + (1-t)Tx), \forall x \in C,
\]
where \( P : H \to C \) is a projective operator and \( f : C \to C \), for some \( \beta \in (0, 1) \) is a fixed contractive mapping i.e. \( \forall x, y \in C \) such that \( \|f(x) - f(y)\| \leq \beta \|x - y\| \). By Banach contractive theorem, there is a unique \( x_t \in C \) such that
\[
x_t = P(tf(x_t) + (1-t)Tx_t)) \quad (1.3).
\]

We also can define the following explicit iterative sequence:
\[
x_{n+1} = P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n], n = 1, 2, \cdots \quad (1.4)
\]
where \( \alpha_n \in (0, 1) \).

2. Main Results

Recall that a nonself-mapping \( T : C \to H \) is said to satisfy the inwardness condition if \( Tx \in I_C(x) \) for all \( x \in C \), where \( I_C(x) \) is the inward set relative to \( C \) given by:
\[
I_C(x) = \{x + \lambda(y - z) : y \in C, \lambda \geq 0\},
\]
and respectively to satisfy the weak inwardness condition if \( Tx \in \overline{I_C(x)} \) for all \( x \in C \), where \( \overline{I_C(x)} \) is the closure of \( I_C(x) \) in the norm topology.

Lemma 2.1. ([5], Theorem 2.2) Let \( H \) be a Hilbert space, \( C \) be a nonempty closed convex subset of \( H \). \( T : C \to H \) is a nonexpansive mapping from \( C \) into \( H \) satisfying the weak inwardness condition and \( F(T) \neq \emptyset \), \( f : C \to C \) is a fixed contractive mapping, and the sequence \( \{x_t\} \) is defined by (1.3), where \( P \) is a nonexpansive retract of \( H \) onto \( C \). As \( t \to 0^+ \), then \( \{x_t\} \) converges strongly to some fixed point \( u \) of \( T \) such that \( u \) is the unique solution in \( F(T) \) to the following variational inequality:
\[
\langle (I - f)u, j(u - p) \rangle \leq 0, \quad \text{for all } p \in F(T).
\]

From lemma 2.1 we have the following results directly:
\[
(i) \quad Tx_t \to u, t \to 0^+
(ii) \quad tf(x_t) + (1 - t)Tx_t - x_t \to 0, t \to 0^+
\]
Theorem 2.2. Let $C$ be a closed convex subset of a Hilbert space $H$, $T : C \to H$ be a nonexpansive mapping satisfying weakly inwardness condition, and $F(T) \neq \emptyset$. Let $f : C \to C$ be a fixed contractive mapping. The sequence \{\(x_n\)\} is defined by (1.4) i.e.

\[ x_{n+1} = P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n], \quad n = 1, 2, \ldots \]

where $P : H \to C$ is projective operator and $\alpha_n \in (0, 1)$ satisfies

\[ \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty. \]

If $Tx_{n+1} - Tx_n \to 0$ as $n \to \infty$, then \{\(x_n\)\} strongly converges to $u = \lim_{t \to 0} x_t \in F(T)$, where \{\(x_t\)\} is defined by (1.3).

Proof. First we show \{\(x_n\)\} is bounded. Let $q \in F(T)$, then

\[
\|x_{n+1} - q\| = \|P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n] - Pq\|
\leq \|\alpha_n (f(x_n) - q) + (1 - \alpha_n)(x_n - q)\|
\leq \alpha_n (\|f(x_n) - f(q)\| + \|f(q) - q\|) + (1 - \alpha_n)\|x_n - q\|
\leq (1 - (1 - \beta)\alpha_n)\|x_n - q\| + \alpha_n \|f(q) - q\|
\leq \max\{\|x_n - q\|, \frac{1}{1 - \beta} \|f(q) - q\|\},
\]

By induction, $\|x_n - q\| \leq \max\{\|x_0 - q\|, \frac{1}{1 - \beta} \|f(q) - q\|\}, n \geq 0$, so we have \{\(x_n\)\} is bounded, and so are \{\(f(x_n)\)\} and \{\(Tx_n\)\}.

1. $\mu_n (f(u) - u, Tx_n - u) \leq 0$

Indeed

\[
\|x_{n+1} - PTx_n\| = \|P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n] - PTx_n\|
\leq \|\alpha_n f(x_n) + (1 - \alpha_n)Tx_n - Tx_n\|
\leq \alpha_n \|f(x_n) - Tx_n\|
\]

Since $\lim_{n \to \infty} \alpha_n = 0$, so we have $\|x_{n+1} - PTx_n\| \to 0$ as $n \to \infty$. Noting that in the Hilbert space $H$, there holds that the inequality $\|x + y\|^2 \leq \|x\|^2 + 2(y, x + y)$, for all $x, y \in H$, then we have

\[
\|x_t - x_{n+1}\|^2 = \|x_t - PTx_n + PTx_n - x_{n+1}\|^2
\leq \|x_t - PTx_n\|^2 + 2\langle PTx_n - x_{n+1}, x_t - x_{n+1} \rangle
\leq \|P(tf(x_t) + (1 - t)Tx_t) - PTx_n\|^2
+ 2 \|PTx_n - x_{n+1}\| \|x_t - x_{n+1}\|
\leq \|tf(x_t) + (1 - t)Tx_t - Tx_n\|^2
+ 2 \|PTx_n - x_{n+1}\| \|x_t - x_{n+1}\| 
\tag{2.1}
\]
In the above inequality we set \( y_t = tf(x_t) + (1-t)Tx_t \), then

\[
\|tf(x_t) + (1-t)Tx_t - Tx_n\|^2 = \|tf(x_t) - Tx_n\|^2 + (1-t)(Tx_t - Tx_n)\|^2 \\
\leq (1-t)^2 \|Tx_t - Tx_n\|^2 \\
+ 2t\langle f(x_t) - Tx_n, y_t - Tx_n \rangle \\
= (1-t)^2 \|Tx_t - Tx_n\|^2 \\
+ 2t\langle f(x_t) - Tx_n, y_t - Tx_t + Tx_t - Tx_n \rangle \\
\leq (1-t)^2 \|Tx_t - Tx_n\|^2 \\
+ 2t\langle f(x_t) - Tx_n, y_t - Tx_t \rangle \\
+ 2t\langle Tx_t - Tx_n, Tx_t - Tx_n \rangle \\
\leq (1 + t^2) \|x_t - x_n\|^2 \\
+ 2t\langle f(x_t) - Tx_n, y_t - Tx_t \rangle \\
+ 2t\langle f(x_t) -Tx_t, Tx_t - Tx_n \rangle, \\
\tag{2.2}
\]

From (2.1)(2.2) we can get

\[
2t\langle f(x_t) - Tx_t, Tx_n - Tx_t \rangle \leq (1 + t^2) \|x_t - x_n\|^2 + 2t\langle f(x_t) - Tx_n, y_t - Tx_t \rangle \\
- \mu(\|x_t - x_n\|^2) \\
= t^2 \mu(\|x_t - x_n\|^2) + 2t\mu(f(x_t) - Tx_n, y_t - Tx_t) \\
\]
i.e.

\[
\mu(f(x_t) - Tx_t, Tx_n - Tx_t) \leq \frac{t}{2} \mu(\|x_t - x_n\|^2) + \mu(f(x_t) - Tx_n, y_t - Tx_t) \\
\]

So we have \( \limsup_{t \to 0} \mu(f(x_t) - Tx_t, Tx_n - Tx_t) \leq 0 \), i.e.

\[
\mu(f(u) - u, Tx_n - u) \leq 0. \\
\]

2. We shall prove \( \limsup_{n \to \infty} \langle f(u) - u, Tx_n - u \rangle \leq 0 \).

We set \( a_n = \langle f(u) - u, Tx_n - u \rangle \), since \( \mu(a_n) \leq 0 \) for all Banach limit and \( \{Tx_n\} \) is bounded, now we take the subsequence \( \{Tx_n\} \) of \( \{Tx_n\} \), such that \( \limsup_{n \to 0}(a_{n+1} - a_n) = \lim_{j \to \infty}(a_{n_j+1} - a_{n_j}) \), from \( Tx_{n_j+1} - Tx_{n_j} \to 0 \), as \( j \to \infty \), then

\[
\limsup_{n \to \infty}(a_{n+1} - a_n) = \lim_{j \to \infty}(a_{n_j+1} - a_{n_j}) = \lim_{j \to \infty}(f(u) - u, Tx_{n_j+1} - Tx_{n_j}) = 0 \\
\]

From lemma 1.2 we have

\[
\limsup_{n \to \infty} a_n \leq 0. \\
\]
i.e.

\[
\limsup_{n \to \infty} \langle f(u) - u, Tx_n - u \rangle \leq 0. \\
\]

Let \( \theta_n = \max\{\langle f(u) - u, Tx_n - u \rangle, 0\} \), then \( \theta_n \to 0 \) as \( n \to \infty \).
3. Finally we prove $x_n \to u$ as $n \to \infty$.

\[
\|x_{n+1} - u\|^2 = \|P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n] - u\|^2 \\
\leq \|\alpha_n f(x_n) - u + (1 - \alpha_n)(Tx_n - u)\|^2 \\
= \alpha_n^2 \|f(x_n) - u\|^2 + (1 - \alpha_n)^2 \|Tx_n - u\|^2 \\
+ 2\alpha_n(1 - \alpha_n)\langle f(x_n) - u, Tx_n - u \rangle \\
= \alpha_n^2 \|f(x_n) - u\|^2 + (1 - \alpha_n)^2 \|Tx_n - u\|^2 + 2\alpha_n(1 - \alpha_n)\beta \|x_n - u\|^2 \\
+ 2\alpha_n(1 - \alpha_n)\langle f(u) - u, Tx_n - u \rangle \\
\leq \|f(x_n) - u\|^2 \cdot \alpha_n^2 + \alpha_n \cdot \|Tx_n - u\|^2 + \alpha_n \cdot \|f(x_n) - u\|^2 \\
+ 2(1 - \alpha_n)\langle f(u) - u, Tx_n - u \rangle \\
= (1 - \alpha_n^*) \|x_n - u\|^2 + \alpha_n^* \beta_n^*
\]

where

\[
\alpha_n^* = \alpha_n(2 - \alpha_n - 2\beta(1 - \alpha_n)) \\
\beta_n^* = \frac{2(1 - \alpha_n)\langle f(u) - u, Tx_n - u \rangle + \alpha_n \|f(x_n) - u\|^2}{2 - 2\beta(1 - \alpha_n)}
\]

Noting that $\lim_{n \to \infty} \alpha_n^* = 0$, $\sum_{n=0}^{\infty} \alpha_n^* = \infty$ and $\limsup_{n \to \infty} \beta_n^* \leq 0$.

From lemma 1.3 we have

\[x_n \to u, \text{ as } n \to \infty\]

The proof is complete.

\[\Box\]

\textbf{References}


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