Strong Convergence of Nonexpansive Nonself-Mapping in Hilbert Space¹

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Abstract. Let C be a closed convex subset of a Hilbert space H , $T : C \rightarrow H$ a non-expansive nonself-mapping satisfying weakly inwardness condition such that $F(T) \neq \emptyset$. In this paper, we study the convergence of the following type sequence generated by

$$x_{n+1} = P(\alpha_n f(x_n) + (1 - \alpha_n)Tx_n)$$

where $P : H \rightarrow C$ is a projective operator, $f : C \rightarrow C$ is a fixed contractive mapping. we prove that the sequence $\{x_n\}$ converges strongly to a fixed point of T under the weaker condition $Tx_{n+1} - Tx_n \rightarrow 0$, as $n \rightarrow \infty$. The result improves some recent results, especially Yisheng Song and Rudong Chen [5, J. Math. Anal. Appl., **321** (2006), 316-326] and Hongkun Xu [6, J. Math. Anal. Appl., **298** (2004), 279-291].

Keywords: Fixed point; nonexpansive nonself-mapping; strong convergence; Banach limit

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper, let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x . Similarly, $x_n \rightarrow x$ will symbolize strong convergence. Let C be a nonempty closed convex subset of H , and $T : C \rightarrow H$ be a non-expansive nonself-mapping such that $F(T) \neq \emptyset$. In 1992, Marino and Trombetta (see reference [3]) considered the contraction $S_t : C \rightarrow C$ and $U_t : C \rightarrow C$ defined by

$$S_t(x) = tPTx + (1 - t)u \quad x \in C$$

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and

$$U_t(x) = P(tTx + (1-t)u) \quad x \in C$$

for each $t \in [0, 1)$ and each fixed $u \in C$, where $P : H \rightarrow C$ is a projective operator, then S_t and U_t has a unique fixed point. By Banach contractive theorem, there is a unique $x_t \in F(S_t)$, $y_t \in F(U_t)$ such that

$$x_t = tPTx_t + (1-t)u \quad (1.1)$$

and

$$y_t = P(tTy_t + (1-t)u) \quad (1.2)$$

In 1995, Xu and Yin (see reference [2]) proved that if $T : C \rightarrow H$ is a non-expansive nonself-mapping, then as $t \rightarrow 0$, the sequence $\{x_t\}$ defined by (1.1) strongly converges to $P_{F(T)}u$, the sequence $\{y_t\}$ defined by (1.2) strongly converges to $P_{F(T)}u$, where $P : H \rightarrow C$ is the nearest point projection of H onto C , i.e. for each $x \in H$, Px is the unique element of C that satisfies $\|x - Px\| = d(x, C) = \inf_{c \in C} \|x - c\|$. This result which has been proved by Takahashi and Kim (see reference [4]) is true in Banach space too.

In 1992, Wittmann proved the following theorem (see reference [1]): Let H be a Hilbert space, C be a nonempty closed convex subset of H , and $T : C \rightarrow C$ be a non-expansive mapping, and $F(T) \neq \emptyset$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n \quad \text{for } n = 1, 2, 3 \dots$$

where $u \in C$ is a fixed point, $\{\alpha_n\} \in (0, 1)$ and satisfies

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ strongly converges to $Pu \in F(T)$, where $P : C \rightarrow F(T)$ is a projective operator.

Definition 1.1. μ is called a Banach limit if μ is a continuous linear functional on l^∞ satisfying:

- (i) $\|\mu(e)\| = 1 = \mu(1)$, $e = (1, 1, 1 \dots)$
- (ii) $\mu(a_n) = \mu(a_{n+1})$, $\forall a_n \in (a_0, a_1, \dots) \in l^\infty$
- (iii) $\liminf_{n \rightarrow \infty} a_n \leq \mu(a_n) \leq \limsup_{n \rightarrow \infty} a_n$, $\forall a_n \in (a_0, a_1, \dots) \in l^\infty$

According to time and circumstances, we use $\mu(a_n)$ instead of $\mu(a_0, a_1, \dots)$.

Further, we know the following result:

Lemma 1.2. (see reference [1]) For a given $a \in R$, for all $\{a_n\} \in l^\infty$ satisfies $\mu(a_n) \leq a$, if $\limsup_{n \rightarrow \infty} (a_{n+1} - a_n) \leq 0$, then $\limsup_{n \rightarrow \infty} a_n \leq a$.

The proof of the following Lemma is in [8]:

Lemma 1.3. *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \forall n \geq 0$$

where $\{\gamma_n\} \in (0, 1)$, $\sum_{n=0}^{\infty} |\delta_n| < \infty$ and satisfies $\sum_{n=0}^{\infty} \gamma_n = \infty$, $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$.

Then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

We now define

$$U_t x = P(tf(x) + (1 - t)Tx), \forall x \in C,$$

where $P : H \rightarrow C$ is a projective operator and $f : C \rightarrow C$, for some $\beta \in (0, 1)$ is a fixed contractive mapping i.e. $\forall x, y \in C$ such that $\|f(x) - f(y)\| \leq \beta \|x - y\|$. By Banach contractive theorem, there is a unique $x_t \in C$ such that

$$x_t = P(tf(x_t) + (1 - t)Tx_t) \quad (1.3).$$

We also can define the following explicit iterative sequence:

$$x_{n+1} = P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n], n = 1, 2, \dots \quad (1.4)$$

where $\alpha_n \in (0, 1)$.

2. MAIN RESULTS

Recall that a nonself-mapping $T : C \rightarrow H$ is said to satisfy the inwardness condition if $Tx \in I_C(x)$ for all $x \in C$, where $I_C(x)$ is the inward set relative to C given by:

$$I_C(x) = \{x + \lambda(y - z) : y \in C, \lambda \geq 0\},$$

and respectively to satisfy the weak inwardness condition if $Tx \in \overline{I_C(x)}$ for all $x \in C$, where $\overline{I_C(x)}$ is the closure of $I_C(x)$ in the norm topology.

Lemma 2.1. *([5], Theorem 2.2) Let H be a Hilbert space, C be a nonempty closed convex subset of H . $T : C \rightarrow H$ is a nonexpansive mapping from C into H satisfying the weak inwardness condition and $F(T) \neq \emptyset$, $f : C \rightarrow C$ is a fixed contractive mapping, and the sequence $\{x_t\}$ is defined by (1.3), where P is a nonexpansive retract of H onto C . As $t \rightarrow 0^+$, then $\{x_t\}$ converges strongly to some fixed point u of T such that u is the unique solution in $F(T)$ to the following variational inequality:*

$$\langle (I - f)u, j(u - p) \rangle \leq 0, \quad \text{for all } p \in F(T).$$

From lemma 2.1 we have the following results directly:

- (i) $Tx_t \rightarrow u, t \rightarrow 0^+$
- (ii) $tf(x_t) + (1 - t)Tx_t - x_t \rightarrow 0, t \rightarrow 0^+$

Theorem 2.2. *Let C be a closed convex subset of a Hilbert space H , $T : C \rightarrow H$ be a nonexpansive mapping satisfying weakly inwardness condition, and $F(T) \neq \emptyset$. Let $f : C \rightarrow C$ is a fixed contractive mapping. The sequence $\{x_n\}$ is defined by (1.4) i.e.*

$$x_{n+1} = P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n], n = 1, 2, \dots$$

where $P : H \rightarrow C$ is projective operator and $\alpha_n \in (0, 1)$ satisfies

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \text{ and, } \sum_{n=1}^{\infty} \alpha_n = \infty.$$

If $\|Tx_{n+1} - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$, then $\{x_n\}$ strongly converges to $u = \lim_{t \rightarrow 0} x_t \in F(T)$, where $\{x_t\}$ is defined by (1.3).

Proof. First we show $\{x_n\}$ is bounded. Let $q \in F(T)$, then

$$\begin{aligned} \|x_{n+1} - q\| &= \|P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n] - Pq\| \\ &\leq \|\alpha_n(f(x_n) - q) + (1 - \alpha_n)(x_n - q)\| \\ &\leq \alpha_n(\|f(x_n) - f(q)\| + \|f(q) - q\|) + (1 - \alpha_n)\|x_n - q\| \\ &\leq (1 - (1 - \beta)\alpha_n)\|x_n - q\| + \alpha_n\|f(q) - q\| \\ &\leq \max\{\|x_n - q\|, \frac{1}{1-\beta}\|f(q) - q\|\}, \end{aligned}$$

By induction, $\|x_n - q\| \leq \max\{\|x_0 - q\|, \frac{1}{1-\beta}\|f(q) - q\|\}$, $n \geq 0$, so we have $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$ and $\{Tx_n\}$.

$$1. \mu_n \langle f(u) - u, Tx_n - u \rangle \leq 0$$

Indeed

$$\begin{aligned} \|x_{n+1} - PTx_n\| &= \|P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n] - PTx_n\| \\ &\leq \|\alpha_n f(x_n) + (1 - \alpha_n)Tx_n - Tx_n\| \\ &\leq \alpha_n \|f(x_n) - Tx_n\| \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, so we have $\|x_{n+1} - PTx_n\| \rightarrow 0$ as $n \rightarrow \infty$. Noting that in the Hilbert space H , there holds that the inequality $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$, for all $x, y \in H$, then we have

$$\begin{aligned} \|x_t - x_{n+1}\|^2 &= \|x_t - PTx_n + PTx_n - x_{n+1}\|^2 \\ &\leq \|x_t - PTx_n\|^2 + 2\langle PTx_n - x_{n+1}, x_t - x_{n+1} \rangle \\ &\leq \|P[t f(x_t) + (1 - t)Tx_t] - PTx_n\|^2 \\ &\quad + 2\|PTx_n - x_{n+1}\| \|x_t - x_{n+1}\| \\ &\leq \|t f(x_t) + (1 - t)Tx_t - Tx_n\|^2 \\ &\quad + 2\|PTx_n - x_{n+1}\| \|x_t - x_{n+1}\| \end{aligned} \tag{2.1}$$

In the above inequality we set $y_t = tf(x_t) + (1-t)Tx_t$, then

$$\begin{aligned}
 \|tf(x_t) + (1-t)Tx_t - Tx_n\|^2 &= \|t(f(x_t) - Tx_n) + (1-t)(Tx_t - Tx_n)\|^2 \\
 &\leq (1-t)^2 \|Tx_t - Tx_n\|^2 \\
 &\quad + 2t\langle f(x_t) - Tx_n, y_t - Tx_n \rangle \\
 &= (1-t)^2 \|Tx_t - Tx_n\|^2 \\
 &\quad + 2t\langle f(x_t) - Tx_n, y_t - Tx_t + Tx_t - Tx_n \rangle \\
 &= (1-t)^2 \|Tx_t - Tx_n\|^2 \\
 &\quad + 2t\langle f(x_t) - Tx_n, y_t - Tx_t \rangle \\
 &\quad + 2t\langle f(x_t) - Tx_t, Tx_t - Tx_n \rangle \\
 &\quad + 2t\langle Tx_t - Tx_n, Tx_t - Tx_n \rangle \\
 &\leq (1+t^2) \|x_t - x_n\|^2 \\
 &\quad + 2t\langle f(x_t) - Tx_n, y_t - Tx_t \rangle \\
 &\quad + 2t\langle f(x_t) - Tx_t, Tx_t - Tx_n \rangle,
 \end{aligned} \tag{2.2}$$

From (2.1)(2.2) we can get

$$\begin{aligned}
 2t\langle f(x_t) - Tx_t, Tx_n - Tx_t \rangle &\leq (1+t^2) \|x_t - x_n\|^2 + 2t\langle f(x_t) - Tx_n, y_t - Tx_t \rangle \\
 &\quad + 2 \|PTx_n - x_{n+1}\| \|x_t - x_{n+1}\| - \|x_t - x_{n+1}\|^2
 \end{aligned}$$

Now we apply Banach limit to the above inequality,

$$\begin{aligned}
 2t\mu\langle f(x_t) - Tx_t, Tx_n - Tx_t \rangle &\leq (1+t^2)\mu(\|x_t - x_n\|^2) + 2t\mu\langle f(x_t) - Tx_n, y_t - Tx_t \rangle \\
 &\quad - \mu(\|x_t - x_{n+1}\|^2) \\
 &= t^2\mu(\|x_t - x_n\|^2) + 2t\mu\langle f(x_t) - Tx_n, y_t - Tx_t \rangle
 \end{aligned}$$

i.e.

$$\mu\langle f(x_t) - Tx_t, Tx_n - Tx_t \rangle \leq \frac{t}{2}\mu(\|x_t - x_n\|^2) + \mu\langle f(x_t) - Tx_n, y_t - Tx_t \rangle$$

So we have $\limsup_{t \rightarrow 0} \mu\langle f(x_t) - Tx_t, Tx_n - Tx_t \rangle \leq 0$, i.e.

$$\mu\langle f(u) - u, Tx_n - u \rangle \leq 0.$$

2. we shall prove $\limsup_{n \rightarrow \infty} \langle f(u) - u, Tx_n - u \rangle \leq 0$.

We set $a_n = \langle f(u) - u, Tx_n - u \rangle$, since $\mu(a_n) \leq 0$ for all Banach limit and $\{Tx_n\}$ is bounded, now we take the subsequence $\{Tx_{n_j}\}$ of $\{Tx_n\}$, such that $\limsup_{n \rightarrow 0} (a_{n+1} - a_n) = \lim_{j \rightarrow \infty} (a_{n_j+1} - a_{n_j})$, from $Tx_{n_j+1} - Tx_{n_j} \rightarrow 0$, as $j \rightarrow \infty$, then

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} (a_{n+1} - a_n) &= \lim_{j \rightarrow \infty} (a_{n_j+1} - a_{n_j}) \\
 &= \lim_{j \rightarrow \infty} \langle f(u) - u, Tx_{n_j+1} - Tx_{n_j} \rangle = 0
 \end{aligned}$$

From lemma 1.2 we have

$$\limsup_{n \rightarrow \infty} a_n \leq 0.$$

i.e.

$$\limsup_{n \rightarrow \infty} \langle f(u) - u, Tx_n - u \rangle \leq 0.$$

Let $\theta_n = \max\{\langle f(u) - u, Tx_n - u \rangle, 0\}$, then $\theta_n \rightarrow 0$ as $n \rightarrow \infty$.

3. Finally we prove $x_n \rightarrow u$ as $n \rightarrow \infty$.

$$\begin{aligned}
 \|x_{n+1} - u\|^2 &= \|P[\alpha_n f(x_n) + (1 - \alpha_n)Tx_n] - u\|^2 \\
 &\leq \|\alpha_n(f(x_n) - u) + (1 - \alpha_n)(Tx_n - u)\|^2 \\
 &= \alpha_n^2 \|f(x_n) - u\|^2 + (1 - \alpha_n)^2 \|Tx_n - u\|^2 \\
 &\quad + 2\alpha_n(1 - \alpha_n)\langle f(x_n) - u, Tx_n - u \rangle \\
 &\leq (1 - \alpha_n)^2 \|x_n - u\|^2 + \alpha_n^2 \|f(x_n) - u\|^2 \\
 &\quad + 2\alpha_n(1 - \alpha_n)(\langle f(x_n) - f(u), Tx_n - u \rangle + \langle f(u) - u, Tx_n - u \rangle) \\
 &\leq (1 - \alpha_n)^2 \|x_n - u\|^2 + \alpha_n^2 \|f(x_n) - u\|^2 + 2\alpha_n(1 - \alpha_n)\beta \|x_n - u\|^2 \\
 &\quad + 2\alpha_n(1 - \alpha_n)\langle f(u) - u, Tx_n - u \rangle \\
 &\leq [(1 - 2\alpha_n + \alpha_n^2 + 2\alpha_n(1 - \alpha_n)\beta) \|x_n - u\|^2 + \alpha_n(\|f(x_n) - u\|^2 \\
 &\quad + 2(1 - \alpha_n)\langle f(u) - u, Tx_n - u \rangle)] \\
 &= (1 - \alpha_n^*) \|x_n - u\|^2 + \alpha_n^* \beta_n^*,
 \end{aligned}$$

where

$$\begin{aligned}
 \alpha_n^* &= \alpha_n(2 - \alpha_n - 2\beta(1 - \alpha_n)) \\
 \beta_n^* &= \frac{2(1 - \alpha_n)\langle f(u) - u, Tx_n - u \rangle + \alpha_n \|f(x_n) - u\|^2}{2 - 2\beta(1 - \alpha_n)}
 \end{aligned}$$

Noting that $\lim_{n \rightarrow \infty} \alpha_n^* = 0$, $\sum_{n=0}^{\infty} \alpha_n^* = \infty$ and $\limsup_{n \rightarrow \infty} \beta_n^* \leq 0$.

From lemma 1.3 we have

$$x_n \rightarrow u, \text{ as } n \rightarrow \infty$$

The proof is complete. □

REFERENCES

1. R. Witmann, *approximation of fixed points of nonexpansive mappings*, Archs. Math., **58** (1992), 486-491.
2. H.K. Xu and X.M. Yin, *Strong convergence theorems for nonexpansive nonself-mappings*, Nonlinear Anal., **24** (1995), 223-228.
3. G. Marino and G. Trombetta, *On approximating fixed points*, Indian J. Anal., **34** (1992), 91-98.
4. W. Takahashi and G.E. Kim, *Strong convergence of approximants to fixed points of nonexpansive nonself-mappings in Banach space*, Nonlinear Anal. **32**(1998)447-454.
5. Yisheng Song and Rudong Chen, *Viscosity approximation method for nonexpansive nonself-mappings*, J. Math. Anal. Appl. **321** (2006), 316-326.
6. H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298** (2004), 279-291.

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