Fixed Point Theorems for a k -set Contraction Map on a Nearly-convex Subset of a Locally Convex Space¹

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Abstract

In this paper, the first part, we establish the fixed point theorems for a k -set contraction map on the class $Q(X, Y)$, (see [1]) . The second part, we generalize the KKM property on a convex set [2] to the KKM^* property on a nearly-convex set, and then we establish fixed point theorems, matching theorems and variational inequalities for this class.

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1 Introduction and Preliminaries

In 1930, Schauder had shown that a continuous compact map $f : X \rightarrow X$ defined on a closed convex subset X of a Banach space has a fixed point. This theorem enormous influence on fixed point theory, variational inequalities and equilibrium problems. Recently there are appeared some results on fixed point so-called Kakutani factorizable multi-functions defined on convex sets. In this paper, we will invoke nearly-convexity of constraint regions in place of convexity. The first part, we establish the fixed point theorem and coincidence theorem for a k -set contraction map on the class $Q(X, Y)$. The second part, we generalize the KKM property on a convex set to the KKM^* property on a nearly-convex set, and then we establish the fixed point theorem for a k -set contraction map on the family $KKM^*(X, X)$, which not need a compact map.

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Let X and Y be two sets, and let $T : X \rightarrow 2^Y$ be a set-valued mapping. We shall use the following notations in the sequel.

- (i) $T(x) = \{y \in Y : y \in T(x)\}$,
- (ii) $T(A) = \cup_{x \in A} T(x)$,
- (iii) $T^{-1}(y) = \{x \in X : y \in T(x)\}$,
- (iv) $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$,
- (v) T is said to lower semicontinuous if for each open subset B of Y , $T^{-1}(B)$ is open in X , and
- (vi) if D is a nonempty subset of X , then $\langle D \rangle$ denote the class of all nonempty finite subset of D .

For the case that X and Y are two topological spaces. Then T is said to be closed if its graph $\mathcal{G}_T = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed. T is said to be compact if the image $T(X)$ of X under T is contained in a compact subset of Y . A subset D of X is said to be compactly closed (resp. compactly open) in X if for any compact subset K of X , the set $D \cap K$ is closed (resp. closed) in K . Obviously, D is compactly open in X if and only if its complement D^c is compactly closed in X .

We now introduce a new class of nearly-convex sets. A nonempty subset X of a Hausdorff topological vector space E is said to be nearly-convex [4] if for every compact subset A of X and every neighborhood V of the origin 0 of E , there is a continuous mapping $h_{A,V} : A \rightarrow X$ such that $x \in h_{A,V}(x) + V$ for each $x \in A$ and $co(h_{A,V}(A)) \subset X$. We call $h_{A,V}$ a continuous convex-inducing mapping.

Remark 1 (i) *In general, the continuous convex-inducing mapping $h_{A,V}$ is not unique. If $U \subset V$, then it is clear that any $h_{A,U}$ can be regarded as an $h_{A,V}$.*

(ii) *It is clear that the convex set is nearly-convex, but the inverse is not true. For a counterexample,*

Let (M, d) be a metric space, $M = \mathbb{R}^2$, we define the metric $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $x = (x_1, x_2)$, $y = (y_1, y_2) \in M$. Then the set $B(0) = \{x \in M : d(x, 0) < 1\} \cup \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$ is a nearly-convex set, but not convex.

Let E will denote a Hausdorff topological vector space, and $B(E)$ the family of nonempty bounded subsets.

Let $\mathcal{P} = \{P \mid P \text{ is a family of seminorms which determines the topology on } E\}$. Let $\mathcal{R}^+$ be the set of all nonnegative real numbers. A mapping $\Phi : B(E) \rightarrow \mathcal{R}^+$ is called a measure of noncompactness [3] provided the following conditions hold:

- (i) $\Phi(\overline{\text{co}}(\Omega)) = \Phi(\Omega)$ for each $\Omega \in B(E)$, where $\overline{\text{co}}(\Omega)$ denotes the closure of the convex hull of Ω ,
- (ii) $\Phi(\Omega) = 0$ if and only if Ω is precompact,
- (iii) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$, for each $A, B \in B(E)$, and
- (iv) $\Phi(\lambda\Omega) = \lambda\Phi(\Omega)$, for each $\lambda \geq 0$, $\Omega \in B(E)$.

The above notion is a generalization of the set measure of noncompactness; if $\{p : P\}$, $P \in \mathcal{P}$ is a family of seminorms which determines the topology on E , then for each $p \in P$ and $\Omega \in E$, we define the set-measure of noncompactness $\alpha_p : 2^E \rightarrow \mathcal{R}^+$ by

$$\alpha_p(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ can be covered by a finite number of sets and each}$$

$$p\text{-diameter of the sets is less than } \varepsilon\}$$

$$\text{where the } p\text{-diameter of a set } D = \sup\{p(x - y) : x, y \in D\}.$$

Definition 1 A mapping $T : X \rightarrow 2^E$ is said to be a k -set contraction map, if there exists $P \in \mathcal{P}$ such that for each $p \in P$, $\alpha_p(T(\Omega)) \leq k\alpha_p(\Omega)$ with $k \in (0, 1)$ for each bounded subset Ω of X and $T(X)$ is bounded.

Let X be a subset of a Hausdorff topological vector space E and Y a topological space, we now define a new class $Q(X, Y)$ of the set-valued mappings from X into Y as follows:

$$T \in Q(X, Y)$$

$\iff$ for any compact convex subset K of X and any continuous mapping

$$f : T(K) \rightarrow K, \text{ the composition } f(T|_K) : K \rightarrow K \text{ has a fixed point.}$$

We next generalize the KKM property (see, [2]) on a convex set of a topological vector space to the following form for a nearly-convex set X .

Definition 2 Let X be a nearly-convex subset of a topological vector space E , and Y a topological space. If $T, F : X \rightarrow 2^Y$ are two set-valued mappings such that for each finite subset A of X and every neighborhood V of the origin 0 of E , there exists a continuous convex-inducing mapping $h_{A,V} : A \rightarrow X$ such that $T(\text{co}(h_{A,V}(A))) \subset F(A)$, then we call F a generalized KKM^* mapping with respect to T .

Let $T : X \rightarrow 2^Y$ be a set-valued mapping such that if $F : X \rightarrow 2^Y$ is a generalized KKM^* mapping with respect to T then the family $\{\overline{Fx} : x \in X\}$ has the finite intersection property, then we say that T has the KKM^* property. Denote

$$KKM^*(X, Y) = \{T : X \rightarrow 2^Y \mid T \text{ has the } KKM^* \text{ property}\}.$$

Remark 2 In particular, for the case $\text{co}(A) \subset X$, we may let the convex-inducing mapping $h_{A,V}$ be the identity mapping.

Remark 3 In general, $Q(X, Y)$ and $KKM^*(X, Y)$ may not be comparable. (see, [1])

2 Fixed point theorem for the class $Q(X, Y)$

In this section, we establish a fixed point theorem for a k -set contraction map on the class $Q(X, Y)$, which not need to be a compact map.

Theorem 1 Let X be a nonempty bounded nearly-convex subset of a Hausdorff topological vector space E . Assume that T is a k -set contraction map, $0 < k < 1$. Then X contains a precompact nearly-convex subset.

Proof. Since T is a k -set contraction map, $0 < k < 1$, there exists $P \in \mathcal{P}$ such that for each $p \in P$, we have $\alpha_p(T(A)) \leq k\alpha_p(A)$ for each subset A of X . Take $y \in X$. Let

$$X_0 = X, \quad X_1 = \text{co}(T(X_0) \cup \{y\}) \cap X, \quad \text{and}$$

$$X_{n+1} = \text{co}(T(X_n) \cup \{y\}) \cap X, \quad \text{for each } n \in N.$$

Then,

- (1) X_n is nearly-convex, for each $n \in N$,
- (2) $X_{n+1} \subset X_n$, for each $n \in N$
- (3) $T(X_n) \subset X_{n+1}$, for each $n \in N$ and
- (4) $\alpha_p(X_{n+1}) \leq \alpha_p(T(X_n)) \leq k\alpha_p(X_n) \leq \dots \leq k^{n+1}\alpha_p(X_0)$, for each $n \in N$.

Thus $\alpha_p(X_n) \rightarrow 0$, as $n \rightarrow \infty$, and hence $X_\infty = \bigcap_{n \geq 1} X_n$ is a nonempty precompact nearly-convex set. $\square$

Remark 4 *In the process of the proof of Theorem 1, we call the set X_∞ a precompact-inducing nearly-convex subset of X .*

Corollary 1 *Let X be a nonempty bounded convex subset of a Hausdorff topological vector space E . Assume that T is a k -set contraction map, $0 < k < 1$. Then X contains a precompact-inducing convex subset.*

Theorem 2 *Let X be a nonempty bounded nearly-convex subset of a Hausdorff topological vector space E , and let $T \in Q(X, X)$. Assume that:*

- (i) *T is a k -set contraction map, $0 < k < 1$ and closed, and*
- (ii) *the precompact-inducing nearly-convex subset X_∞ of X , $\overline{T(X_\infty)} \subset X_\infty$*

Then T has a fixed point in X .

Proof. Let $\mathcal{N} = \{U_i : i \in I\}$ be a local base of E such that U_i is symmetric and open for each $i \in I$, and let $V \in \mathcal{N}$.

By ii, since $\alpha_p(\overline{T(X_n)}) \rightarrow 0$, as $n \rightarrow \infty$, hence $\overline{T(X_\infty)} = \bigcap_{n \geq 1} \overline{T(X_n)}$ is a nonempty compact subset of X_∞ . And, since X_∞ is nearly-convex, there is a continuous mapping $h_{\overline{T(X_\infty)}, V} : \overline{T(X_\infty)} \rightarrow X_\infty$ such that $x \in h_{\overline{T(X_\infty)}, V}(x) + V$ for each $x \in \overline{T(X_\infty)}$ and $co(h_{\overline{T(X_\infty)}, V}(\overline{T(X_\infty)})) \subset X_\infty$.

Let $Z = co(h_{\overline{T(X_\infty)}, V}(\overline{T(X_\infty)}))$. Then Z is a compact and convex subset of X , and $T(Z) \subset X_\infty$. Since $T \in Q(X, X)$ and $h_{\overline{T(X_\infty)}, V}$ is continuous, the composition $h_{\overline{T(X_\infty)}, V} \circ T|_Z : Z \rightarrow 2^Z$ has a fixed point, say $x_V \in h_{\overline{T(X_\infty)}, V}(T(x_V))$. Let $x_V = h_{\overline{T(X_\infty)}, V}(y_V)$ for some $y_V \in T(x_V) \subset T(Z) \subset \overline{T(X_\infty)}$. Then $y_V \in x_V + V = h_{\overline{T(X_\infty)}, V}(y_V) + V$. Since $\overline{T(X_\infty)}$ is compact, we may assume that $\{y_V\}$ converges to $\bar{x}$, and then $\{x_V\}$ also converges to $\bar{x}$. The closedness of T implies $\bar{x} \in T(\bar{x})$. $\square$

Follow above Theorem 2, we immediate have the following corollary.

Corollary 2 [1] *Let X be a nonempty nearly-convex subset of a Hausdorff topological vector space E , and let $T \in Q(X, X)$ be compact and closed with $\overline{T(X)} \subset X$. Then T has a fixed point in X .*

We next establish the following coincidence theorem.

Theorem 3 *Let X be a nonempty bounded convex subset of a Hausdorff topological vector space E , and let $T, G : X \rightarrow 2^X$ be two set-valued mappings. Assume that:*

- (i) $T \in Q(X, X)$ is a k -set contraction map, $0 < k < 1$ and closed with $\overline{T(X)} \subset X$,
- (ii) for each $y \in G(X)$, $G^{-1}(y)$ is convex, and
- (iii) for the precompact-inducing convex subset X_∞ of X , $\overline{T(X_\infty)} \subset \cup\{intG(x) : x \in X_\infty\}$.

Then there exists an $x_0 \in X$ such that $T(x_0) \cap G(x_0) \neq \emptyset$.

Proof. By the same process of the proof of Theorem 2, we get a compact subset $\overline{T(X_\infty)}$ of X .

By (iii), there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of X_∞ such that $\overline{T(X_\infty)} \subset \cup_{i=1}^n intG(x_i)$. Let $\{\lambda_i\}_{i=1}^n$ be a partition of the unity subordinated to $\{intG(x_i) : i = \{1, 2, \dots, n\}\}$, and let $P = co\{x_1, x_2, \dots, x_n\}$. Define $f : \overline{T(X_\infty)} \rightarrow P$ by

$$f(y) = \sum_{i=1}^n \lambda_i(y)x_i = \sum_{i \in N_y} \lambda_i(y)x_i, \quad \text{for each } y \in \overline{T(X_\infty)},$$

where $i \in N_y$ iff $\lambda_i(y) \neq 0$ iff $y \in intG(x_i) \subset G(x_i)$,

Then $x_i \in G^{-1}(y)$ for each $i \in N_y$. Clearly, f is continuous, and by (ii), we have

$$f(y) \in co\{x_i : i \in N_y\} \subset G^{-1}(y), \quad \text{for each } y \in \overline{T(X_\infty)}.$$

Since P is compact convex subset of X and $T \in Q(X, X)$, the composition $f|T(P) \circ T|_P : P \rightarrow P$ has a fixed point $x_0 \in P \subset X$. So $x_0 \in fT(x_0)$ and $f^{-1}(x_0) \subset G(x_0)$. So, we complete the proof. $\square$

3 Fixed point theorem for the class $KKM^*(X, Y)$

In this section, we establish the fixed point theorem for a k -set contraction map on the class $KKM^*(X, Y)$, which not need to be a compact map.

The following Lemma 1 will play important role for this paper.

Lemma 1 *Let X be a nonempty nearly-convex subset of a Hausdorff topological vector space E , Y a topological space. Then $T|_D \in KKM^*(D, Y)$ whenever $T \in KKM^*(X, Y)$ and D is a nonempty nearly-convex subset of X .*

Proof. Let $F : D \rightarrow 2^Y$ be a generalized KKM^* mapping with respect to $T|_D$. Then for any finite subset A of D and any neighborhood V of the origin 0 of E , there exists a continuous convex-inducing mapping $h_{A,V} : A \rightarrow D$ such that $T|_D(\text{co}(h_{A,V}(A))) \subset F(A)$.

Define $F' : X \rightarrow 2^Y$ by

$$F'(x) = \begin{cases} F(x) & x \in D, \\ Y & x \in X \setminus D. \end{cases}$$

It is clear that for any finite subset B of X and any neighborhood U of the origin 0 of E , there exists a continuous convex-inducing mapping $h_{B,U} : B \rightarrow X$ such that $T(\text{co}(h_{B,U}(B))) \subset F'(B)$. Indeed,

- (1) if $B \not\subset D$, then there exists some $x \in B \setminus D$, and hence $F'(x) = Y$. So the result is obvious.
- (2) if $B \subset D$, since F be a generalized KKM^* mapping with respect to $T|_D$, the inclusion is true.

Thus F' is a generalized KKM^* mapping with respect to T . Since $T \in KKM^*(X, Y)$, hence the family $\{\overline{F'x} : x \in X\}$ has finite intersection property, and so does the family $\{\overline{Fx} : x \in D\}$. So $T|_D \in KKM^*(D, Y)$. $\square$

Lemma 2 *Let X be a nonempty nearly-convex subset of a Hausdorff topological vector space E , Y and Z are two topological spaces. Then $fT \in KKM^*(X, Z)$ whenever $T \in KKM^*(X, Y)$ and $f \in C(Y, Z)$.*

Proof. Let F be a generalized KKM^* mapping with respect to fT . Then for any $A = \{x_1, x_2, \dots, x_n\} \in \langle X \rangle$ and any neighborhood of the origin 0 of E , there exists a continuous convex-inducing mapping $h_{A,V} : A \rightarrow X$ such that $fT(\text{co}(h_{A,V}(A))) \subset \cup_{i=1}^n F(x_i)$. So, $T(\text{co}(h_{A,V}(A))) \subset \cup_{i=1}^n f^{-1}F(x_i)$, which says that $f^{-1}F$ is a generalized KKM^* mapping with respect to T . Since $T \in KKM^*(X, Y)$, the family $\{f^{-1}F(x) : x \in X\}$ has the finite intersection property, and so does the family $\{F(x) : x \in X\}$. This shows that $fT \in KKM^*(X, Z)$. $\square$

Theorem 4 *Let X be a nonempty bounded nearly-convex subset of a locally convex space E , and let $T \in KKM^*(X, X)$ is a k -set contraction, $0 < k < 1$ and closed with $\overline{T(X)} \subset X$. Then T has a fixed point in X .*

Proof. Let $\mathcal{N} = \{U_i : i \in I\}$ be a local base of E such that U_i is symmetric, open and convex for each $i \in I$, and let $V \in \mathcal{N}$.

And, by the same process of the proof of Theorem 2, we get a compact subset $\overline{T(X_\infty)}$ of X . Since $\overline{T(X_\infty)} \subset \overline{T(X)} \subset X$ and X is nearly-convex, there

is a continuous mapping $h_{\overline{T(X_\infty)}, V} : \overline{T(X_\infty)} \rightarrow X$ such that $x \in h_{\overline{T(X_\infty)}, V}(x) + V$ for each $x \in \overline{T(X_\infty)}$ and $co(h_{\overline{T(X_\infty)}, V}(\overline{T(X_\infty)})) \subset X$.

Let $Z = co(h_{\overline{T(X_\infty)}, V}(\overline{T(X_\infty)}))$. Then Z is a compact and convex subset of X , and $T(Z) \subset X$. Since $T \in KKM^*(X, X)$ and X_∞ is a nearly-convex subset of X , by Lemma 1, we have $T|_{X_\infty} \in KKM^*(X_\infty, X)$. Next, since $h_{\overline{T(X_\infty)}, V}$ is continuous, by Lemma 2, we have $h_{\overline{T(X_\infty)}, V} \circ T|_{X_\infty} \in KKM^*(X_\infty, Z)$. Put $F = h_{\overline{T(X_\infty)}, V} \circ T|_{X_\infty}$. Then F is closed, since T is closed, $h_{\overline{T(X_\infty)}, V}$ is continuous and $\overline{T(X_\infty)}$ is compact.

We now claim that for each $U_i \in \mathcal{N}$, there exists an $x_i \in X_\infty$ such that

$$(x_i + U_i + V) \cap F(x_i) \neq \phi.$$

If the above statement is not true, then there exists $U \in \mathcal{N}$ such that $(x + U + V) \cap F(x) = \phi$, for all $x \in X_\infty$.

Let $K = h_{\overline{T(X_\infty)}, V}(\overline{T(X_\infty)})$. Then $K \subset h_{\overline{T(X_\infty)}, V}(\overline{T(X_\infty)}) \subset Z$. Define $G : X_\infty \rightarrow 2^Z$ by

$$G(x) = K \setminus (x + \frac{1}{2}U + V), \text{ for each } x \in X_\infty.$$

Then

- (1) $G(x)$ is compact, for each $x \in X_\infty$, and
- (2) G is a generalized KKM^* mapping with respect to F .

We prove (2) by contradiction. Let $A = \{x_1, x_2, \dots, x_n\} \subset X_\infty$. Then there exists $V' \in \mathcal{N}$ such that for any continuous convex-inducing $h_{A, V'} : A \rightarrow X_\infty$ one has $F(co(h_{A, V'}(A))) \not\subseteq G(A)$. Let $U' \in \mathcal{N}$ such that $U' \subset \frac{1}{2}U \cap V'$. Then $F(co(h_{A, U'}(A))) \not\subseteq G(A)$. So, there is $u \in co(h_{A, U'}(A))$ and $v \in F(u)$ such that $v \notin \cup_{i=1}^n G(x_i)$. From the definition of G , it follows that $v \in (x_i + \frac{1}{2}U + V)$, for each $i \in \{1, 2, \dots, n\}$. Hence $v \in (x_i + \frac{1}{2}U + V) \in (h_{A, U'}(x_i) + \frac{1}{2}U + V + U') \subset (h_{A, U'}(x_i) + U + V)$, for each $i \in \{1, 2, \dots, n\}$, since X_∞ is nearly-convex. Thus, $h_{A, U'}(x_i) \in (v + U + V)$, for each $i \in \{1, 2, \dots, n\}$, and hence $u \in co(h_{A, U'}(A)) \in v + U + V$. So, $v \in u + U + V$ and $v \in F(u)$, we conclude that $F(u) \cap (u + U + V) \neq \phi$, a contradiction. Therefore, G is a generalized KKM^* mapping with respect to F .

Since $F \in KKM^*(X_\infty, Z)$ and G is a generalized KKM^* mapping with respect to F , the family $\{G(x) : x \in X_\infty\}$ has the finite intersection property, and so we conclude that $\cap_{x \in X_\infty} G(x) \neq \phi$. Choose $\eta \in \cap_{x \in X_\infty} G(x)$, then $\eta \in K \setminus (x + \frac{1}{2}U + V)$, for each $x \in X_\infty$. Since $\eta \in \cap_{x \in X_\infty} G(x) \subset K \subset h_{\overline{T(X_\infty)}, V}(\overline{T(X_\infty)}) \subset \overline{T(X_\infty)} + V \subset \overline{X_\infty} + V$, hence there is an $x_0 \in X_\infty$ such that $\eta \in x_0 + \frac{1}{2}U + V$. But $\eta \in K \setminus (x_0 + \frac{1}{2}U + V)$, a contradiction. Therefore, we have proved for each $U_i \in \mathcal{N}$, there exists an $x_i \in X_\infty$ such that

$(x_i + U_i + V) \cap F(x_i) \neq \phi$. Let $y_i \in (x_i + U_i + V) \cap F(x_i)$. Since $\{y_i\} \subset K$ and K is compact, we may assume that $\{y_i\}$ converges to some $y_0 \in K$, and since $\{x_i\} \subset \overline{X_\infty}$, we assume that $\{x_i\}$ converges to x_0 . The closedness of F implies that $(x_0, y_0) \in \mathcal{G}_F$, so we have $y_0 \in x_0 + \overline{V}$ and $y_0 \in F(x_0) = h_{\overline{T(X_\infty)}, V}(T(x_0))$. Choose $z_0 \in T(x_0)$ such that $y_0 = h_{\overline{T(X_\infty)}, V}(z_0)$. Noting that $z_0 \in h_{\overline{T(X_\infty)}, V}(z_0) + V = y_0 + V \subset x_0 + V + \overline{V} \subset (x_0 + V + V + V)$, and hence $T(x_0) \cap x_0 + V + V + V \neq \phi$ for any $V \in \mathcal{N}$, which just as before, implies T has a fixed point. $\square$

Corollary 3 *Let X be a nonempty bounded convex subset of a locally convex space E , and let $T \in KKM(X, X)$ is a k -set contraction, $0 < k < 1$ and closed with $\overline{T(X)} \subset X$. Then T has a fixed point in X .*

Corollary 4 *Let X be a nonempty nearly-convex subset of a locally convex space E , and let $T \in KKM^*(X, X)$ is compact and closed. Then T has a fixed point in X .*

Corollary 5 [2] *Let X be a nonempty convex subset of a locally convex space E , and let $T \in KKM(X, X)$ is compact and closed. Then T has a fixed point in X .*

We now establish the following KKM^* -type theorem, which is equivalent to the matching theorem after it.

Theorem 5 *Let X be a nonempty bounded nearly-convex subset of a Hausdorff topological vector space E . If $T, F : X \rightarrow 2^X$ are two set-valued mappings satisfying the following:*

- (i) $T \in KKM^*(X, X)$ is a k -set contraction map, $0 < k < 1$ with $\overline{T(X)} \subset X$,
- (ii) for any $x \in X$, $F(x)$ is compactly closed in X , and
- (iii) F is a generalized KKM^* mapping with respect to T ,

then

$$\overline{T(X_\infty)} \cap (\cap \{F(x) : x \in X_\infty\}) \neq \phi,$$

where X_∞ is the precompact-inducing nearly-convex subset of X .

Proof. Let $\mathcal{N} = \{U_i : i \in I\}$ be a local base of E such that U_i is symmetric and open for each $i \in I$. By the same process of the proof of Theorem 2, we get a compact subset $\overline{T(X_\infty)}$ of X , and $T|_{X_\infty} \in KKM^*(X_\infty, X)$, since $T \in KKM^*(X, X)$.

Define $H : X_\infty \rightarrow 2^X$ by

$$H(x) = \overline{T(X_\infty)} \cap F(x), \text{ for each } x \in X_\infty.$$

It follows from (iii) that F is a generalized KKM^* mapping with respect to $T|_{X_\infty}$, and hence for any $x \in X_\infty$ and any $V \in \mathcal{N}$, there exists a continuous convex-inducing $h_{\{x\},V} : \{x\} \rightarrow X_\infty$ such that $T(\text{co}(h_{\{x\},V}(\{x\}))) \subset F(x)$ and $\text{co}(h_{\{x\},V}(x)) \subset X_\infty$. So $H(x) \neq \emptyset$.

By (ii), $H(x)$ is compact in X , for each $x \in X_\infty$. We now claim that H is a generalized KKM^* mapping with respect to $T|_{X_\infty}$. Let $A \in \langle X_\infty \rangle$. By (iii), for any $V \in \mathcal{N}$, there exists a continuous convex-inducing mapping $h_{A,V} : A \rightarrow X_\infty$ such that $T(\text{co}(h_{A,V}(A))) \subset F(A)$ and $\text{co}(h_{A,V}(A)) \subset X_\infty$. So, $T(\text{co}(h_{A,V}(A))) \subset F(A) \cap \overline{T(X_\infty)} = H(A)$. Thus, we have shown that H is a generalized KKM^* mapping with respect to $T|_{X_\infty}$. Since $T|_{X_\infty} \in KKM^*(X_\infty, X)$, the family $\{H(x) : x \in X_\infty\}$ has the finite intersection property. And, since $H(x)$ is compact, hence $\bigcap_{x \in X_\infty} H(x) \neq \emptyset$, that is, $\overline{T(X_\infty)} \cap (\bigcap \{F(x) : x \in X_\infty\}) \neq \emptyset$. $\square$

Corollary 6 *Let X be a nonempty bounded convex subset of a Hausdorff topological vector space E . If $T, F : X \rightarrow 2^X$ are two set-valued mappings satisfying the following:*

- (i) $T \in KKM(X, X)$ is a k -set contraction map, $0 < k < 1$ with $\overline{T(X)} \subset X$,
- (ii) for any $x \in X$, $F(x)$ is compactly closed in X , and
- (iii) F is a generalized KKM mapping with respect to T ,

then there exists a nonempty convex subset Y of X such that

$$\overline{T(X_\infty)} \cap (\bigcap \{F(x) : x \in X_\infty\}) \neq \emptyset,$$

where X_∞ is the precompact-inducing nearly-convex subset of X .

As a consequence of the above theorems, we get the following generalization of the Ky Fan matching theorem.

Theorem 6 *Let X be a nonempty bounded nearly-convex subset of a Hausdorff topological vector space E . If $T, H : X \rightarrow 2^X$ are two set-valued mappings satisfying the following:*

- (i) $T \in KKM^*(X, X)$ is a k -set contraction map, $0 < k < 1$ with $\overline{T(X)} \subset X$,
- (ii) for any $x \in X$, $H(x)$ is compactly open in X , and
- (iii) for the precompact-inducing nearly-convex subset X_∞ of X , $\overline{T(X_\infty)} \subset H(X_\infty)$,

then the precompact-inducing nearly-convex subset X_∞ of X satisfies the following condition:

$$T(X_\infty) \cap (\cap \{H(x) : x \in M\}) \neq \phi, \text{ for some } M \in \langle X_\infty \rangle.$$

Proof. Let $\mathcal{N} = \{U_i : i \in I\}$ be a local base of E such that U_i is symmetric and open for each $i \in I$. And, by the same process of the proof of Theorem 2, we get a compact subset $\overline{T(X_\infty)}$ of X , and $T|_{X_\infty} \in KKM^*(X_\infty, X)$, since $T \in KKM^*(X, X)$.

We claim that there exists $M \in \langle X_\infty \rangle$ such that $T(X_\infty) \cap (\cap \{H(x) : x \in M\}) \neq \phi$. On the contrary, assume that $T(X_\infty) \cap (\cap \{H(x) : x \in M\}) = \phi$ for any $M \in \langle X_\infty \rangle$, then $T(X_\infty) \subset \cap_{x \in M} H^c(x)$. Since X_∞ is nearly-convex, hence for any $V \in \mathcal{N}$, there exists a continuous convex-inducing mapping $h_{M,V} : M \rightarrow X_\infty$ such that $co(h_{M,V}(M)) \subset X_\infty$. So $T(co(h_{M,V}(M))) \subset T(X_\infty) \subset \cap_{x \in M} H^c(x)$. This implies H^c is a generalized KKM^* mapping with respect to T . By (ii), for any $x \in X$, $H^c(x)$ is compactly closed in X . Follows Theorem 5, we have $\overline{T(X_\infty)} \cap (\cap \{H^c(x) : x \in X_\infty\}) \neq \phi$, which implies $\overline{T(X_\infty)} \not\subset \cup_{x \in X_\infty} H(x)$, a contradiction to (iii). We complete the proof. $\square$

Corollary 7 Let X be a nonempty bounded convex subset of a Hausdorff topological vector space E . If $T, H : X \rightarrow 2^X$ are two set-valued mappings satisfying the following:

- (i) $T \in KKM(X, X)$ is a k -set contraction map, $0 < k < 1$ with $\overline{T(X)} \subset X$,
- (ii) for any $x \in X$, $H(x)$ is compactly open in X , and
- (iii) for the precompact-inducing convex subset X_∞ of X , $\overline{T(X_\infty)} \subset H(X_\infty)$,

then the precompact-inducing convex subset X_∞ of X satisfies the following condition:

$$T(X_\infty) \cap (\cap \{H(x) : x \in M\}) \neq \phi, \text{ for some } M \in \langle X_\infty \rangle.$$

As a consequence of the above Corollary 6, we have the following generalized variational inequality.

Theorem 7 Let X be a nonempty bounded convex subset of a Hausdorff topological vector space E , and let $T \in KKM^*(X, X)$ be a k -set contraction map, $0 < k < 1$ with $\overline{T(X)} \subset X$. If $\varphi, \psi : X \times X \rightarrow (-\infty, \infty)$ are two real-valued mappings satisfying the following:

- (i) $\psi(x, y) \leq 0$, for each $(x, y) \in \mathcal{G}_T$,
- (ii) for fixed $x \in X$, the mapping $y \mapsto \varphi(x, y)$ is lower semicontinuous on K for each compact subset K of X , and
- (iii) for fixed $y \in X$, the set $\{x \in X : \psi(x, y) > 0\}$ contains the convex hull of the set $\{x \in X : \varphi(x, y) > 0\}$

then for the precompact-inducing convex subset X_∞ of X , there exists an $\bar{y} \in X_\infty$ such that

$$\varphi(x, \bar{y}) \leq 0 \text{ for each } x \in X_\infty.$$

Proof. Define $F, S : X \rightarrow 2^X$ by

$$S(x) = \{y \in X : \psi(x, y) \leq 0\} \text{ for each } x \in X, \text{ and}$$

$$F(x) = \{y \in X : \varphi(x, y) \leq 0\} \text{ for each } x \in X.$$

By assumption (i), we have $\mathcal{G}_T \subset \mathcal{G}_S$, and by assumption (ii), $F(x)$ is compactly closed for each $x \in X$. The condition (iii) implies that for each finite subset A of X , $S(\text{co}(A)) \subset F(A)$, and then $T(\text{co}(A)) \subset F(A)$, that is; F is a generalized KKM mapping with respect to T .

So, all the conditions in Corollary 6 are satisfied, and so for the precompact-inducing convex subset X_∞ of X , we have that $\overline{T(X_\infty)} \cap (\cap \{F(x) : x \in X_\infty\}) \neq \emptyset$. Let $\bar{y} \in \overline{T(Y)} \cap (\cap \{F(x) : x \in Y\})$, and hence we have $\varphi(x, \bar{y}) \leq 0$ for each $x \in Y$ $\square$

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