International Mathematical Forum, 2, 2007, no. 27, 1341 - 1344

A Note on the Classification of Nine Dimensional Lie Algebras with Nontrivial Levi Decomposition

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Abstract

We point out an omission in the classification of nine dimensional real indecomposable Lie algebras having a nontrivial Levi decomposition.

Mathematics Subject Classification: 17B05, 17B10

Keywords: Lie algebra, Levi decomposition, classification.

1 Introduction

It is well known that one of the main obstructions for the classification of real Lie algebras with a nontrivial Levi decomposition is the absence of classifications of solvable Lie algebras in dimensions $n \geq 7$. Since radicals of a Lie algebra constitute a module over the Levi part, a systematic classification depends on the analysis of derivations of solvable algebras. For real Lie algebras in dimension $n \leq 5$, classifications have been proposed by different authors (see e.g. [6] for a review on this work). The classification of six dimensional real solvable Lie algebras was systematically initiated in 1962 by G. B. Mubarakzyanov, proving various results on the decomposition of these structures [2]. In particular, it followed that an indecomposable solvable Lie algebra must have a nilradical of dimension four or five. The isomorphism classes of solvable Lie algebras with a five dimensional nilradical were given in [3]. The remaining case, algebras with a four dimensional nilradical, was finished by P. Turkowski some years later [4]. It was noted that one isomorphism class, called $\mathfrak{g}_{6,92}^{*}$, was missing from the list in [3]. Over a basis $\{X_1, .., X_6\}$, the nonvanishing brackets are given by

$$\begin{array}{l}[X_2, X_4] = [X_3, X_5] = X_1,\ [X_1, X_6] = 2pX_1,\\ [X_2, X_6] = pX_2 + qX_4, \quad [X_3, X_6] = pX_3 + qX_5,\\ [X_4, X_6] = pX_4 - qX_2, \quad [X_5, X_6] = pX_5 - qX_3,\end{array}$$

where p, q are real parameters. We observe that if $q = 0$, then $\mathfrak{g}_{6,92}^{*}$ is isomorphic to the Lie algebra $\mathfrak{g}_{6,92}^{\alpha,\mu_0,\nu_0}$ given by:

$$\begin{array}{l}[X_2, X_4] = [X_3, X_5] = X_1,\ [X_1, X_6] = \alpha X_1,\\ [X_2, X_6] = \frac{\alpha}{2}X_2 + \nu_0 X_3, \quad [X_3, X_6] = \frac{\alpha}{2}X_3 - \mu_0 X_2,\\ [X_4, X_6] = \frac{\alpha}{2}X_4 + \mu_0 X_5, \quad [X_5, X_6] = \frac{\alpha}{2}X_5 - \nu_0 X_4,\end{array}$$

for the values $\alpha = 2p, \mu_0 = \nu_0 = 0$ [3]. Actually, since $\alpha \neq 0$ by indecomposability, we can normalize α to 2.[1] Thus the algebras for which $q \neq 0$ are those not covered in [3]. If $q \neq 0$, the transformation $X_6' = \frac{1}{q}X_6$ shows that we can always suppose $q = 1$, that is, $\mathfrak{g}_{6,92}^{*}(p, q) \simeq \mathfrak{g}_{6,92}^{*}\left(\frac{p}{q}, 1\right)$. Therefore the Lie algebra $\mathfrak{g}_{6,92}^{*}$ to be added to the Mubarakzyanov list is:

$$\begin{array}{l}[X_2, X_4] = [X_3, X_5] = X_1,\ [X_1, X_6] = 2pX_1,\\ [X_2, X_6] = pX_2 + X_4, \quad [X_3, X_6] = pX_3 + X_5,\\ [X_4, X_6] = pX_4 - X_2, \quad [X_5, X_6] = pX_5 - X_3,\end{array}$$

for $p \in \mathbb{R}$, as proven in [4].

In [5] nine dimensional indecomposable Lie algebras with nontrivial Levi decomposition were classified, obtaining 63 isomorphism classes. In particular, it was shown that there is only one Lie algebra with radical isomorphic to $\mathfrak{g}_{6,92}^{*}$, called $L_{9,7}^{p}$ and given by the brackets

$$\begin{array}{llll}[X_1, X_2] = X_3, & [X_1, X_3] = -X_2, & [X_2, X_3] = X_1, & [X_1, X_4] = \frac{1}{2}X_7,\\ [X_1, X_5] = \frac{1}{2}X_6, & [X_1, X_6] = -\frac{1}{2}X_5, & [X_1, X_7] = -\frac{1}{2}X_4, & [X_2, X_4] = \frac{1}{2}X_5,\\ [X_2, X_5] = -\frac{1}{2}X_4, & [X_2, X_6] = \frac{1}{2}X_7, & [X_2, X_7] = -\frac{1}{2}X_6, & [X_3, X_4] = \frac{1}{2}X_6,\\ [X_3, X_5] = -\frac{1}{2}X_7, & [X_3, X_6] = -\frac{1}{2}X_4, & [X_3, X_7] = \frac{1}{2}X_5, & [X_4, X_6] = X_8,\\ [X_5, X_7] = X_8, & [X_4, X_9] = pX_4 + X_6, & [X_5, X_9] = pX_5 + X_7, & \\ [X_6, X_9] = pX_6 - X_4, & [X_7, X_9] = pX_7 - X_5, & [X_8, X_9] = 2X_8. & \end{array}$$

over a basis $\{X_1, .., X_9\}$. As further proved in [5], the complexification $L_{9,7}^{p} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the complex Lie algebra

$$L_{9,53}^{p} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}(2, \mathbb{C}) \overrightarrow{\oplus}_{2D_{\frac{1}{2}} \oplus D_0} \mathfrak{g}_{6,92}^{p,-1,-1},$$

[1] In fact this also coincides with the Lie algebra $\mathfrak{g}_{6,82}^{\alpha,\lambda,\lambda_1}$ of the list [3] for the values $\alpha = 2, \lambda = \lambda_1 = 0$. This is due to some overlappings within the original list for some special values of the parameters. See also [1].

However, as seen before, the case $q = 0$ leads to the algebra $\mathfrak{g}_{6,92}^{2,0,0} = \mathfrak{g}_{6,82}^{2,0,0}$, which allows another real form with compact Levi part that has accidentally been omitted in the classification. Consider the real Lie algebra with Levi decomposition $\mathfrak{so}(3) \overrightarrow{\oplus}_{R_4 \oplus D_0} \mathfrak{g}_{6,92}^{2,0,0}$ and brackets

$$\begin{array}{llll}
[X_1, X_2] = X_3, & [X_1, X_3] = -X_2, & [X_2, X_3] = X_1, & [X_1, X_4] = \frac{1}{2}X_7, \\
[X_1, X_5] = \frac{1}{2}X_6, & [X_1, X_6] = -\frac{1}{2}X_5, & [X_1, X_7] = -\frac{1}{2}X_4, & [X_2, X_4] = \frac{1}{2}X_5, \\
[X_2, X_5] = -\frac{1}{2}X_4, & [X_2, X_6] = \frac{1}{2}X_7, & [X_2, X_7] = -\frac{1}{2}X_6, & [X_3, X_4] = \frac{1}{2}X_6, \\
[X_3, X_5] = -\frac{1}{2}X_7, & [X_3, X_6] = -\frac{1}{2}X_4, & [X_3, X_7] = \frac{1}{2}X_5, & [X_4, X_6] = X_8, \\
[X_5, X_7] = X_8, & [X_4, X_9] = X_4, & [X_5, X_9] = X_5, & [X_6, X_9] = X_6, \\
[X_7, X_9] = X_7, & [X_8, X_9] = 2X_8. & &
\end{array}$$

over the basis $\{X_1, .., X_9\}$. This algebra is clearly non-isomorphic to $L_{9,7}$, since the radicals $\mathfrak{g}_{6,92}^{2,0,0}$ and $\mathfrak{g}_{6,92}^{*}$ are non-isomorphic[2]. We denote this missing algebra by $L_{9,7}^{*}$.

Lemma 1. *The complexification of $L_{9,7}^{*}$ is isomorphic to the complexification of the Lie algebra $L_{9,52} = \mathfrak{sl}(2, \mathbb{R}) \overrightarrow{\oplus}_{2D_{\frac{1}{2}} \oplus D_0} \mathfrak{g}_{6,82}^{2,0,0}$.*

Proof. Consider the change of basis given by

$$\begin{pmatrix} X_1' \\ X_2' \\ X_3' \\ X_4' \\ X_5' \\ X_6' \\ X_7' \\ X_8' \\ X_9' \end{pmatrix} = \begin{pmatrix} -2i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta i & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta & \gamma i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta i & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 & \gamma i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \end{pmatrix}$$

where $2\beta^2 = 1$, $2\gamma^2 + 1 = 0$ and $i = \sqrt{-1}$. Over this basis the nontrivial brackets of $L_{9,7}^{*} \otimes_{\mathbb{R}} \mathbb{C}$ are

$$\begin{array}{llll}
[X_1', X_2'] = 2X_2', & [X_1', X_3'] = -2X_3', & [X_2', X_3'] = X_1', & [X_1', X_4'] = X_4', \\
[X_1', X_5'] = -X_5', & [X_1', X_6'] = X_6', & [X_1', X_7'] = -X_7', & [X_2', X_5'] = X_4', \\
[X_2', X_7'] = X_6', & [X_3', X_4'] = X_5', & [X_3', X_6'] = \frac{1}{2}X_7', & [X_4', X_5'] = X_8', \\
[X_6', X_7'] = X_8', & [X_4', X_9'] = X_4', & [X_5', X_9'] = X_5', & [X_6', X_9'] = X_6', \\
[X_7', X_9'] = X_7', & [X_8', X_9'] = 2X_8'. & &
\end{array}$$

These commutation relations are exactly those given for the Lie algebra $L_{9,52}$ in [5]. □

[2]This can be seen directly analyzing the derivations. The algebra of derivations of $\mathfrak{g}_{6,92}^{2,0,0}$ has dimension 16, while that of $\mathfrak{g}_{6,92}^{*}$ has dimension 10.

It should be remarked that this additional real form satisfies

$$\dim H^2(L^*_{9,7}, L^*_{9,7}) = 1,$$

which implies that its deformations belong to the parameterized family $L^p_{9,7}$.

ACKNOWLEDGEMENTS. The author benefited from useful discussions with V. I. Lagno.

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Received: September 27, 2006