Pointed Nodal Plane Curves with the Expected Number of Moduli

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Abstract. Here we describe families of plane curves passing through s fixed points and with the expected number of moduli as s -pointed curves.

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For all integers g, s such that either $g \geq 2$ and $s \geq 0$ or $g = 1$ and $s \geq 1$ or $g = 0$ and $s \geq 3$ let $\mathcal{M}_{g,s}$ denote the moduli space of all pairs $(C, (P_1, \dots, P_s))$, where C is a smooth genus g curve and $(P_1, \dots, P_s)$ is an ordered set of s distinct points of C . From now on we assume $g \geq 2$. Let A be a smooth and connected projective surface and Γ a locally closed subset of $\text{Hilb}(A)$ such the normalization of every $T \in \Gamma$ is a smooth genus g curve. Fix s general points $Q_1, \dots, Q_s \in A$ and set $\Gamma(Q_1, \dots, Q_s) := \{T \in \Gamma : Q_i \in T_{\text{reg}} \text{ for all } i\}$. For any $T \in \Gamma(Q_1, \dots, Q_s)$ the normalization map $\phi : C \rightarrow T$ induces a pointed curve $(C, \phi^{-1}(Q_1), \dots, \phi^{-1}(Q_s))$. Hence for a fixed choice of the points $Q_1, \dots, Q_s$ the set Γ induces a morphism $\eta_s : \Gamma(Q_1, \dots, Q_s) \rightarrow \mathcal{M}_{g,s}$. We are interested in the integer $\dim(\text{Im}(\eta_s))$, i.e. in the number of moduli of the family $\Gamma(Q_1, \dots, Q_s)$, mainly when $A = \mathbf{P}^2$. In the next statement for any non-empty algebraic set W the integer $\dim(W)$ will denote the maximal dimension of one of its irreducible components. In the statements and proofs of Theorems 1 and 2 we will only apply this convention when W is equidimensional. We will first prove the following result.

Proposition 1. *Let A be a smooth and connected projective surface and Γ a non-empty locally closed subset of $\text{Hilb}(A)$ such the normalization of every $T \in$*

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Γ is a smooth genus $g \geq 2$ curve. Fix a positive integer s and s general points $Q_1, \dots, Q_s \in A$. Set $\epsilon_0 := \dim(\text{Aut}(A))$, $I_s := \{h \in \text{Aut}(A) : h(Q_i) = Q_i \text{ for all } i\}$ and $\epsilon_s := \dim(I_s)$. Assume that Γ has the expected number of moduli, i.e. that the normalization map induces a morphism $\eta_0 : \Gamma \in \mathcal{M}_g$ such that $\dim(\text{Im}(\eta_0)) = \min\{3g-3, \dim(\Gamma) - \epsilon_0\}$. Then $\Gamma(Q_1, \dots, Q_s)$ has the expected number of moduli, i.e.

- (i) $\Gamma(Q_1, \dots, Q_s) = \emptyset$ if $\dim(\Gamma) < s$;
- (ii) $\dim(\Gamma(Q_1, \dots, Q_s)) = \dim(\Gamma) - s$ and $\dim(\text{Im}(\eta_s)) = \min\{3g-3+s, \dim(\Gamma) - s - \epsilon_s\}$ if $\dim(\Gamma) \geq s$.

The following result is an easy consequence of Proposition 1 and part (3) of [3], Th. 4.2.

Theorem 1. Fix integers d, g, s such that $g \geq 2$, $s \geq 0$ and $(d-1)(d-2)/2 \geq g$. Fix s general points $Q_1, \dots, Q_s \in \mathbf{P}^2$ and let M be the family of all integral degree d curves $T \subset \mathbf{P}^2$ such that T has exactly $(d-1)(d-2)/2 - g$ ordinary nodes as its only singularities and $Q_i \in T_{\text{reg}}$ for all i . Set $\epsilon_s = 8 - 2s$ if $s \leq 3$ and $\epsilon_s = 0$ if $s \geq 4$. Then M has the expected number of moduli, i.e. $M = \emptyset$ if $s \geq g + 3d$, $\dim(M) = g - s + 3d - 1$ if $s \leq g + 3d - 1$ and $\dim(\text{Im}(\eta_s)) = \min\{3g-3+s, g-s+3d-1-\epsilon_s\}$ if $s \leq g + 3d - 1$.

We need the integer ϵ_s because $\dim(\text{Aut}(\mathbf{P}^2)) = 8$ and any 4 points of $\mathbf{P}^2$, no 3 of them collinear, are projectively equivalent.

Then we will consider pointed k -gonal curve. For all integers g, k such that $g \geq 2$ and $k \geq 2$ set $\mathcal{M}_g^{1,k} := \{C \in \mathcal{M}_g : C \text{ has a } g_k^1\}$. Hence $\mathcal{M}_g^{1,k} = \mathcal{M}_g$ if $g \leq 2k-2$, while $\mathcal{M}_g^{1,k}$ is an integral $(2g+2k-5)$ -dimensional variety if $g \geq 2k-3$. The following result is an easy consequence of Proposition 1 and [1], Th. 0.2.

Theorem 2. Fix integers g, d, k, s such that $g \geq 2k-3 \geq 1$, $s \geq 0$ and $d \geq (g+k+2)/2$. Fix $s+1$ general points $Q_0, \dots, Q_s$ in $\mathbf{P}^2$ and let M_s denote the set of all integral degree d curve $T \subset \mathbf{P}^2$ such $Q_i \in T_{\text{reg}}$ for $1 \leq i \leq s$, Q_0 is an ordinary point with multiplicity $d-k$ of T and T has further $(d-2)(d-1)/2 - g - (d-k)(d-k-1)/2$ ordinary nodes as its only singularities. Set $\epsilon_s = 6 - 2s$ if $s \leq 2$ and $\epsilon_s = 0$ if $s \geq 3$.

- (i) $M_s = \emptyset$ if $s \geq 4d - k + g$;
- (ii) Assume $s \leq 4d - k + g - 1$. Then $\dim(M_s) = 4d - k + g - 1 - s$ and $\dim(\text{Im}(\eta_s)) = \min\{2g+2k-5, 4d-k+g-1-s-\epsilon_s\}$.

To use [1] and [3] we work over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) = 0$.

Proof of Proposition 1. We will use induction on s , the beginning case of the induction being the case $s = 0$ which is true by assumption. Assume $s > 0$ and that the result is true for the integer $s' := s-1$. Fix any irreducible component M of $\Gamma(Q_1, \dots, Q_{s-1})$ and set $M(Q_s) := \{T \in M : Q_s \in T_{\text{reg}}\}$. We fix $Q_1, \dots, Q_{s-1}$ and one of the finitely many M 's and then move Q_s . We get

$M(Q_s) = \emptyset$ if $\dim(M) = 0$ and $\dim(M(Q_s)) = \dim(M) - 1$ if $\dim(M) > 0$. Hence $\Gamma(Q_1, \dots, Q_s) = \emptyset$ if $\dim(\Gamma) < s$, while $\dim(\Gamma(Q_1, \dots, Q_s)) = \dim(\Gamma) - s$ if $\dim(\Gamma) \geq s$. We only need to prove the second assertion of part (ii). Let A_i , $I \in I$, with I a finite set, the irreducible components of the general fiber of η_{s-1} over any fixed irreducible component of $\text{Im}(\eta_{s-1})$. We have fixed $Q_1, \dots, Q_{s-1}$ and then move Q_s . For every $i \in I$ we get $A_i \cap \Gamma(Q_1, \dots, Q_s) = \emptyset$ if $\dim(A_i) = 0$ and $\dim(A_i \cap \Gamma(Q_1, \dots, Q_s)) \leq \dim(A_i) - 1$ if $\dim(A) > 0$. We get $\dim(\text{Im}(\eta_s)) \geq \min\{3g - 3 + s, \dim(\Gamma) - s\}$ if $\dim(\Gamma) \geq s$. Since the reverse inequality is obvious, we are done. $\square$

Proof of Theorem 1. If $s = 0$, then M is irreducible ([2]). Hence the case $s = 0$ is part (3) of [3], Th. 4.2. If $s > 0$, then apply Proposition 1. $\square$

Proof of Theorem 2. Let $u : A \rightarrow \mathbf{P}^2$ be the blowing-up of Q_0 . Hence $A \cong F_1$ and $\text{Pic}(A) \cong \mathbb{Z}^2$. Set $Q'_i := u^{-1}(Q_i)$ for $1 \leq i \leq s$. We will take $h := u^{-1}(Q_0)$ and a fiber f of a ruling of A as a basis of $\text{Pic}(A)$. Hence $h^2 = -1$, $h \cdot f = 1$ and $f^2 = 0$. For any integral curve $T \in \mathbf{P}^2$ let T' denote the strict transform of T in u . $T \in M_0$ if and only if T' is a nodal integral element of $|kh + (d+k)f|$ with geometric genus g and intersecting transversally h . Conversely, if $E \in |kh + (d+k)f|$ is a nodal integral curve with geometric genus g and intersecting transversally h , then $u(E) \in M_0$. Let W_g denote the set of all integral nodal curves $E \in |kh + (d+k)f|$ with geometric genus g . Apply [4], Cor 2.14, to a general union of k general smooth (and hence rational) members of $h + f$ and $d - k$ general elements of $|f|$. You get $W_g \neq \emptyset$ and $\dim(W_g) = 4d - k + g - 1$. W_g is irreducible ([5]). Hence M_0 is irreducible and $\dim(M_0) = 4d - k + g - 1$. By Proposition 1 it is sufficient to prove the case $s = 0$. Since $d \geq (g + k + 2)/2$, Theorem 2 is true by [1], Th. 0.2. $\square$

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