Scale Size Based Ranking of Frontier DMUs in Data Envelopment Analysis

M. Zohrehbandian

Department of Mathematics
Islamic Azad University
Karaj Branch, P.O. Box 31485-313, Karaj, Iran
zohrehbandian@yahoo.com

Abstract

Data envelopment analysis (DEA) is basically a linear programming based technique used for measuring the relative performance of decision making units (DMUs), Where DMUs can be any organizational units (e.g. hospital, bank, etc.) transforming a number of inputs to a number of outputs. This paper proposes a new approach to ranking the efficient DMUs. This is a new perspective based on the concept of scale size and input consumption. The proposed approach distinguish between CCR and BCC efficient DMUs and so is applicable in both constant and variable returns to scale contexts. Likewise, this ranking method has the desirable feature of ranking not only the strongly efficient DMUs but the efficient ones as well, i.e. non-extreme efficient points.

Mathematics Subject Classification: Operations Research. No. 90

Keywords: DEA, Ranking, Returns to Scale, Scale Elasticity.

1 Introduction

Data envelopment analysis (DEA), Charnes et al. (1978), is basically a linear programming based technique used for estimating production frontiers and evaluating the relative efficiency of organizational units, referred to as decision-making units (DMUs). DEA by focusing on certain simple ratios, provides for

each DMU just a score and based on these scores a set of DMUs can be partitioned into two groups: frontier DMUs (efficient) and non-frontier DMUs (inefficient). Efficient DMUs take a score unity and inefficient DMUs a score below or above unity.

Over the past two decades the DEA methodology has been established as a robust and valuable methodology for frontier estimation. Various theoretical extensions have been developed in the field based on the original CCR model, Charnes et al. (1978), and many additional theoretical papers have adapted the models to deal with problems that have occurred in practice. One of the practical problems in the field is the ranking of DMUs.

Often decision makers (DM) are interested in a complete ranking, beyond the dichotomized classification. Therefore, subgroup of papers have been developed in this field in which many researchers have sought to improve the differential capabilities of DEA and to fully rank both efficient, as well as inefficient DMUs. See for instance; Andersen and Petersen (1993), Sexton et al. (1986), Torgersen et al. (1996).

Since these ranking methods has been developed based on some of the aspects of Production Possibility Set (PPS), in certain cases, different calculations are reached in applying the alternative ranking methods; see Torgersen et al. (1996). Furthermore, for each method, there are problematic areas e.g. infeasibility and instability of the proposed model. Hence, whilst each ranking technique is useful in a specialist area, no one methodology can be prescribed as the complete solution to the question of ranking.

In this paper, we develop a new ranking method based on the concept of scale size while this technique provide results in overcoming some problems encountered by the previous ranking methods. However, the suggested ranking procedure is not without problems of its own. Clearly, the logic behind the reason for ranking the DMUs will decide the ultimate ranking procedure chosen and consequently the results.

The remainder of this paper is organized as follows. In section 2, some necessary notations about the basic DEA models and conception of Returns to scale (RTS) will be introduced. In section 3, we introduce our ranking method. In section 4, computational experiments are reported. Finally, section 5 gives our conclusive remarks.

2 Preliminaries

Consider n DMUs to be evaluated indexed by $j=1, \dots, n$. Each DMU_j is assumed to use m different input x_{ij} ($i=1, \dots, m$) to produce s different output y_{rj} ($r=1, \dots, s$). In the literature, the traditional classification of efficient DMUs partitions the efficient units into three subsets.

F—the set of the weakly efficient DMUs,

E'—the set of the efficient DMUs, and

E—the set of the strongly efficient DMUs

DMUs belonging to F are the boundary points on the extended portion of the efficiency surface lying outside the convex hull of any subset of the strongly efficient DMUs. The efficient DMUs in E' (non-extreme efficient points) can be expressed as the linear combination of other efficient DMUs while the strongly efficient DMUs in E cannot be expressed in this manner.

Although there are a variety of ways of computing an efficiency frontier, it is important to identify the returns to scale (RTS) characteristics which are embedded in the choice of PPS.

2.1 Variable Returns to Scale Context

In the empirical economics of efficient production, RTS is commonly quantified as scale elasticities which are the proportionate increase in outputs resulting from the proportionate increase in inputs. A production correspondence is said to exhibit increasing returns to scale (IRS) if a radial increase in input levels leads to a more than proportionate radial increase in output levels. If a radial increase in input levels leads to a less than proportionate radial increase in output levels a production correspondence is said to exhibit decreasing returns to scale (DRS) and otherwise is said to exhibit constant returns to scale (CRS).

Efficiency frontiers which allow returns to scale to vary according to the scale of inputs are known as variable returns to scale (VRS) frontiers. The VRS version produces a frontier which has increasing returns to scale at low input levels and decreasing returns to scale at high input levels.

It is obvious that an efficient DMU which operate under IRS would gain in average productivity if it increases marginally its scale size. This is because if the unit remains efficient, proportional increases in its input levels will be followed by raises in its output levels by a larger percentage than was the

case for the input levels. Following a similar argument we can deduce that an efficient DMU operate under DRS would become more productive if it decrease marginally its scale size. Only for CRS DMUs, productivity will be unaffected by marginal changes in scale size.

Thus any efficient DMU not operating within local CRS will gain in productivity if it changes its scale size up to the point where returns to scale become constant. This intuitive argument leads to the conclusion that the optimal scale size to operate at is where local CRS hold. Therefore, DMUs operating within local CRS, Most Productive Scale Size (MPSS) region, have a greater priority than any efficient DMU not operating within this region and the results of any ranking method, in variable RTS context, should be satisfy this condition.

Here we note that we can access local CRS region by invoking the following projection formulas for DMU_o outside the MPSS region:

$$\begin{aligned}
 &Min \quad \frac{\alpha}{\beta} \\
 &s.t. \quad \sum_{j=1}^n \lambda_j x_j \leq \alpha x_o \\
 &\quad \sum_{j=1}^n \lambda_j y_j \geq \beta y_o \\
 &\quad \sum_{j=1}^n \lambda_j = 1 \\
 &\quad \lambda_j \geq 0 \quad j = 1, 2, \dots, n \\
 &\quad \alpha, \beta \geq 0
 \end{aligned} \tag{1}$$

This is a fractional programming problem from which input oriented CCR model can be derived by a simple variable substitution. Based on the optimal solution of above model, we can obtain the point $(\alpha^* x_o, \beta^* y_o)$ which is associated with the points in local CRS region.

Since β^* and α^* are the radial increase and decrease in output and input levels of DMU_o to change its scale size up to the point where returns to scale become constant, we can rank the efficient DMUs not operating within local CRS on the descending order of objective value in model 1, $\frac{\alpha^*}{\beta^*}$, or as its counterpart, on the descending order of objective value in CCR model. In other

words, the optimal value of model 1 is a distance function which estimate maximum radial input contraction of the evaluated DMU such that the projection of it is within the MPSS region. But, our motivation for ranking the efficient DMUs within local CRS is different.

2.2 The Laws of Diminishing Returns

Although the entrepreneur requires a certain minimum amount of each input to produce outputs and the greater sources of inputs will make possible a more capitalistic method of production and by expanding the scale of production certain economies can be obtained, but a number of things, however, limit the scale of production. This is our motivation for ranking the efficient DMUs within local CRS based on the input consumption of DMUs.

First among these things is increasing complexity of management which is in fact the main factor that sets a limit to the growth in the size of input consumption. Hence, Large management structures are less efficient generally. Secondly, as the scale of production increases, so do the risks of production. The greater the inputs, the greater therefore, will be the loss from an error of judgment. Therefore, unwillingness to bear greater risks may be another limitation on the growth of consuming more inputs. Furthermore, the law of diminishing returns is another limitation on the scale of production.

The law of diminishing returns states that if the amount of one or more factors used in a particular form of production is fixed, and increasing amounts of other factors are combined with the fixed factors, then both the average output in relation to the variable factors and successive additions to total output will eventually diminish. Therefore, at a certain size there will be no motive for further expansion of production, for at any larger size it would be less efficient. This is due to the fact that the law of diminishing returns will be to operate and expansion will cease at the point immediately before diminishing returns. Hence, let us to define:

Definition Suppose DMU_o and DMU_p operate within local CRS and hold in $E \cup E'$. DMU_o has a less ranking score against DMU_p if the inputs of DMU_o is dominated by the inputs of DMU_p . i.e. $x_{io} \leq x_{ip} \quad i = 1, \dots, m$

In other words, for DMUs operating within local CRS, the more consume

inputs and hold in local CRS region (the optimal scale size) the more the priority of DMU. So, we are in a position to propose our ranking method in the following manner.

3 Scale Size based Ranking Method

Step 1: Evaluate each DMU by the CCR and BCC models and classify the frontier DMUs into two categories. The efficient and strongly efficient DMUs operate within CRS region in category 1, say G , and the other frontier DMUs (weakly efficient DMUs and efficient DMUs not operating within local CRS) in category 2, say G' .

Step 2: Frontiers DMUs in G have a priority due to the fact that they hold into MPSS region. We rank them on the basis of the scale of inputs consumption and the law of diminishing. Therefore, we evaluate $DMU_o \in G$ by model 2 and rank these DMUs on the ascending order of its objective function. Where model 2 is similar to the output oriented BCC model (without input).

$$\begin{aligned}
 &Max \quad \varphi \\
 &s.t. \quad \sum_{j \in G} \lambda_j x_j \geq \varphi x_o \\
 &\quad \sum_{j \in G} \lambda_j = 1 \\
 &\quad \lambda_j \geq 0 \quad j \in G
 \end{aligned} \tag{2}$$

Step 3: Because of distinguished structure between DMUs in G with DMUs in G' and priority of DMUs belonging to G , we rank the efficient DMUs in G' , separately, on the descending order of objective value in model 1, $\frac{\alpha^*}{\beta^*}$, or as its counterpart, on the descending order of objective value in CCR model.

We can identify three mains reasons to use the above method for ranking the efficient DMUs. Firstly, most of the ranking methods published over the last decade, have problems of their own and some of these problems, in particular, are infeasibility and instability. But, the proposed method in this paper is able to remove these difficulties. Secondly, most of the proposed methods either fail in ranking of DMUs in variable RTS context or ranked the DMUs

operating in local CRS region together with the DMUs out of it. Therefore, this treatment may be caused to the contradiction with the priority of CRS DMUs against the other DMUs. thirdly, the method has the advantage that the ranking extends to the non-extreme efficient units, DMUs in E' , something that most of ranking methods do not permit.

4 Computational Results

In order to illustrate the results of using the proposed ranking method, we apply a set of performance and demographic data from the nursing home example developed in Sexton et al. (1986), in which six DMUs are compared over four variables. The raw data and the results of the CCR, BCC and scale size ranking method of efficient DMUs, in VRS context, are presented in Table 1 (A,B,C,D $\in G$, E $\in G'$ and F is inefficient).

	In1	In2	Out1	Out2	CCR	BCC	Model 2	Rank
A	150	0.2	14000	3500	1	1	7.067	4
B	400	0.7	14000	21000	1	1	1.325	2
C	320	1.2	42000	10500	1	1	1.656	3
D	520	2.0	28000	42000	1	1	1.000	1
E	350	1.2	19000	25000	0.98	1	-	5
F	320	0.7	14000	15000	0.87	0.9	-	-

Table 1. Raw data of nursing home example

5 Conclusion

Within DEA is a subgroup of papers in which many researchers have sought to improve the differential capabilities of DEA and to fully rank the DMUs. However, whilst each technique may be useful in a specific area, no one methodology can be prescribed here as the panacea of all ills. Hence, for each method there are problematic areas e.g. infeasibility and instability.

In this paper, we proposed a new approach for ranking the efficient DMUs which with due regard to the priority of CRS DMUs, attempts to visualize differences between the efficient DMUs operating within local CRS and the other efficient DMUs. Hence, it has several attractive features. In particular,

the procedure is applicable for variable RTS context while it ranked the DMUs by consideration of a priority for CRS DMUs against the other DMUs.

References

- [1] P. Andersen, N.C. Petersen, A procedure for ranking efficient units in data envelopment analysis, *Management Science*, 39, (1993), 1261-1264.
- [2] A. Charnes, W.W. Cooper, E. Rhodes, Measuring the Efficiency of Decision Making Units, *European Journal of Operational Research*, 2, (1978), 429 - 444.
- [3] T.R. Sexton, R.H. Silkman, A.J. Hogan, Data envelopment analysis: Critique and extensions, In: Silkman R.H.(Ed.), *Measuring Efficiency: An Assessment of Data Envelopment Analysis*. Jossey-Bass, San Francisco, CA, pp. 73-105.
- [4] A.M. Torgersen, F.R. Forsund, S.A.C. Kittelsen, Slack-adjusted efficiency measures and ranking of efficient units, *Journal of Productivity Analysis*, 7, (1996), 379-398.

Received: March 2, 2006