International Mathematical Forum, 2, 2007, no. 21, 1025 - 1032

On Weak Central Extensions and Perfect Topological Groups

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Abstract

Let G be a topologically perfect group (i.e. $G = \overline{[G, G]}$) and $M(G)$ the topological Schur multiplier of G. In this paper we show that G has a weak universal central extension whose center is the Schur multiplier.

Mathematics Subject Classification: 22A05

Keywords: Free topological group; Schur multiplier; universal extension

Introduction

Perfect groups are important object of study in group theory. For a topologist , perfect groups tend to be those that arise in certain interesting situations. For example via the study of fundamental groups. For any discrete group G ,$H_1(G) = G_{ab}$. For any topological space X ,$H_1(X) = \pi_1(X)_{ab}$.The case where X is the classifying space of G ,the Eilenberg-Maclane space $K(G, 1)$, reduces to

$$H_1(G) = H_1(K(G, 1)) = \pi_1(K(G, 1))_{ab} = G_{ab}$$

Hence for classes of space X for which homology groups are easily calculated , perfect subgroups of the fundamental group have a role to play in calculation of $\pi_1(X)$. For example if $H_1(X) = 0$, then $\pi_1(X)$ is perfect.

A good invariant of any group G is its abelianization G_{ab}. The G_{ab} is characterized by the universal property that every continuous group homomorphism factors uniquely through $G \to G_{ab}$. The abelianizations are indeed easer object to deal with than arbitrary groups. One way to get more information about a group is the case where G has trivial abelinization. In general case ,the information lost in abelianization is measured by the kernel of $G \to G_{ab}$ (for more on perfect groups see [2]). In this paper we consider topologically perfect groups and will show that such a group has a universal topological extension whose kernel is the Schur multiplier.

All spaces are assumed to be Tychanov (completely regular , Hausdorff).A topological extension of G by N , denoted by (G, π) ,is a short exact sequence $0 \to N \xrightarrow{i} Q \xrightarrow{\pi} G \to 0$,where i is a topological embedding onto a closed subgroup ,ϕ a topological isomorphism of G/N onto G.The extension is central if N is in the center of G. We consider extensions with a continuous section i.e. $u : G \to Q$ such that $\pi u = Id$. For example ,if G is a connected locally compact group ,then any topological extension of G by a connected simply connected Lie group has a continuous section [S,theorem 2].

1 Perfect groups and the Schur multiplier

In this section we consider the Schur multiplier in the topological context. We recall an example which shows that a perfect topological group need not be perfect.

The *free topological group* is in Markov sense[5]: If X is completely regular ,then the (Markov) free topological group on X is the group $F(X)$ equipped with the finest group topology inducing the given topology on X as a subspace. Such a topology always axists [5], and has the universal property of the following kind:every continuous mapping f from X to an arbitrary topological group G lifts to a unique continuous homomorphism

$$g : F(X) \to G$$

i.e.the restriction of g to X is f. For information on free topological groups see [5,4].

Let G be a topological group . A *free topological presentation* of G is a short exact sequence $0 \to R \to F \xrightarrow{\pi} G \to 0$ with a continuous section $u : G \to F$ such that $\pi u = Id_G$ where F is a free topological group in Morkov[6] sense. Consider the abelian group $R \cap [F, F]/\overline{[R, F]}$. We call this group *the Schur multiplier* of G denoted by

$$M(G) = R \cap [F, F]/\overline{[R, F]}$$

By Alperin et al[1], $M(G)$ is independent of the choice of the presentation.

The following example shows that a topologically perfect group need not be perfect.
Recall that a topological group G is perfect if $[G,G]$ is dense in G.

Example [3]. Let $\widetilde{SL_2(R)}$ be the universal covering of SL_2R ,special real linear group. Let $\alpha \in SL_2(R)$ be central element of infinite order. Let α be irrational rotation in the circle group T. Consider the group

$$G = \widetilde{SL_2(R)} / < \alpha, \alpha >$$

then G is topologically perfect[see 3].

2 Central extensions and perfect groups

If G is a perfect group , then there exists a universal extension $e = (U, \pi)$ of G which has the property that for any other extension $e^{'} = (G_1, \pi^{'})$ there exists a unique homomorphism $\phi : U \to G_1$ with $\pi^{'}\phi = \pi$. The kernel of π is the second homology group of G ([8],[6]). In this section we consider the idea in a topological context and will show that for a topological perfect group i.e. $G = \overline{[G,G]}$ the weak universal extension exists and its kernel is the Schur multiplier.

Definition 2.1 . A *weak central extension* of N by Q is an exact sequence $0 \to N \to G \to Q \to 0$ with N a central subgroup and π an onto continuous homomorphism(not necessarily open). A (weak) central extension of N by Q is (weak) universal if for any other (weak) central extension , $0 \to N^{'} \to G^{'} \xrightarrow{\pi^{'}} Q \to 0$ there is one and only one continuous homomorphism from G to $G^{'}$ over Q such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \to & N & \to & G & \xrightarrow{\pi} & Q & \to & 0 \\
 & & \downarrow & & \beta\downarrow & & \| & & \\
0 & \to & N^{'} & \to & G^{'} & \xrightarrow{\pi^{'}} & O & \to & 0
\end{array}$$

$\pi^{'}\beta = \pi$

Lemma 2.2 *Let $e = (G, \pi)$ and $e^{'} = (G^{'}, \pi^{'})$ be two central extensions of Q. If G is topologically perfect i.e. $G = \overline{[G,G]}$, then there is at most on continuous homomorphismfrom $G^{'}$ over Q.*

Proof. Let f_1 ,f_2 be two continuous homomorphisms from G to $G^{'}$ over Q i.e.$\pi f_1 = \pi$,$\pi^{'} f_2 = \pi$. Let $y, z \in G$. Now $\pi^{'} f_1(y) = \pi$ and $\pi^{'} f_2(y) = \pi$. Hence

, there are c,$c^{'}$ in the kernel of Q such that $f_1(y) = f_2(y)c$, $f_1(z) = f_2(z)c^{'}$. Since f_1 and f_2 are homomorphisms and c,$c^{'}$ are in the center of $G^{'}$ we have,

$$\begin{aligned} f_1[y,z] &= f_1[yzy^{-1}z^{-1}] \\ &= f_1(y)f_1(z)f_1(y^{-1})f_1(z^{-1}) \\ &= f_2(y)cf_2(z)c^{'}c^{-1}f_2(y)^{-1}c^{'-1}f_2(z)^{-1} \\ &= f_2(y)f_2(z)f_2(y^{-1})f_2(z^{-1}) \\ &= f_2(yzy^{-1}z^{-1}) = f_2[y,z] \end{aligned}$$

Hence f_1,f_2 agree on the commutator subgroup of G .Since $[G,G]$ is dense in G so f_1,f_2 are the same on G.

Lemma 2.3 *Let $e = (G,\pi)$ be a central extension of Q. If G is not topologically perfect then for a suitably chosen $e^{'} = (G^{'},\pi^{'})$ a central extension of Q ,there exists more than one continuous homomorphism from G to $G^{'}$ over Q.*

Proof. If G is not topologically perfect then there is a non zero continuous homomorphism from G to some abelian topological group (for example $\phi : G \to G/\overline{[G,G]}$). Let $e^{'} : 0 \to N^{'} \to G^{'} \to Q \to 0$ be the split extension of $N^{'}$ by Q with $\pi^{'}(q,n^{'}) = q$, $q \in Q, n^{'} \in N^{'}$. Clearly e is central extension. Setting $f_1(g) = (\pi(g),1)$ and $f_2(g) = (\pi(g),\phi(g))$, we obtain two distinct continuous homomorphisms from G to $G^{'}$ over Q because for any $g \in G$,

$$\pi^{'}f_1(g) = \pi^{'}((\pi(g),1) = \pi(g)$$

$$\pi^{'}f_2(g) = \pi^{'}(\pi(g),\phi(g)) = \pi(g)$$

Since π and ϕ are continuous then f_1 ,f_2 are continuous. These two maps are distinct because ϕ is a non zero homomorphism

$$\begin{array}{ccccc} N & \to & G & \xrightarrow{\pi} & Q \\ \downarrow & \overset{\phi}{\swarrow} & f_1\downarrow\downarrow f_2 & & \| \\ N^{'} & \to & G & \xrightarrow{\pi^{'}} & Q \end{array}$$

Lemma 2.4 *If $e = (G,\pi)$ is a central extension of a topologically perfect group Q i.e.$Q = \overline{[Q,Q]}$, then $G = \overline{[G,G]}$ is topologically perfect and maps onto Q.In particular, $(G^{'},\pi|_{G^{'}})$ is a weak central extension.*

Proof. Let $e = (G,\pi)$ be a central extension of Q. since π is a homomorphism ,it maps $[G,G]$ onto $[Q,Q]$. Let $z \in G$, $\pi(z) = y$ and V be a neighborhood of z . So $y \in \overline{[Q,Q]}$. Hence there is a sequence (y_n) in $[Q,Q]$ such that y_n converges to y. Sinec π is open ,$\pi(V)$ is an open set containing y . So $\pi(V) \cap [Q,Q] \neq \emptyset$.

Therefore, $V \cap [G,G] \neq \emptyset$. It follows that z is a limit point of $[G,G]$ i.e. $z \in \overline{[G,G]}$. Thus every $g \in G$ can be written as a product $x^{'}c$ with $x^{'} \in \overline{[G,G]}$ and $c \in Ker\pi$. Therefore, if $g_1, g_2 \in G$, then $g_1 = x_1^{'}c_1, g_2 = x_2^{'}c_2$

$$[g_1, g_2] = [x_1^{'}c_1, x_2^{'}c_2] = [x_1^{'}, x_2^{'}]$$

i.e. $[G,G] = [G^{'}, G^{'}]$. Hence $G^{'} = \overline{[G,G]} = \overline{[G^{'},G^{'}]}$.

Now we define the *backward induced extension* in the category of topological groups.

Let $\nu : Q_1 \to Q$ and $\pi : G \to Q$ be continuous . The *backward extension* of $\{\nu, \pi\}$ is a group $G^{(2)}$ and two continuous homomorphisms $\pi_0 : G^{(2)} \to Q$ and $\beta : G \to Q$ such that $\pi_0\nu = \beta\pi$ and with the property that given any topological group Γ and $\alpha : \Gamma \to G^{(2)}$,$\delta : \Gamma \to G$ such that $\nu\alpha = \pi\delta$ then there exists a unique continuous homomorphism $\gamma : \Gamma \to G^{(2)}$ so that $\delta = \beta\gamma$ and $\alpha = \pi_0\gamma$. By [C] backward extension exists. Note that $G^{(2)} = \{(g,u); \nu(u) = \pi(g)\}$ and it is a subgroup of $G \times Q_1$ together with the restriction of cononical projection from $G \times Q_1$ onto G and Q_1 respectively. Remark. If the topological extension $e = (G, \pi)$ of Q has a continuous section $u : Q \to G$ i.e. $\pi u = Id$, then $\sigma : Q \to G^{(2)}$ defined by $\sigma(q) = (u\nu(q), q)$ is a continuous section of $G^{(2)}$.

Lemma 2.5 *A central extension (U, ν) of Q is universal if and only if U is topologically perfect and every central extension of U splits.*

Proof. Suppose that every central topological extension of U splits and U is topologically perfect. Let $e : N \to G \to Q$ be a central exrension of N by G.Now we form the backward extension of e as follows:

$$(*) \qquad \begin{array}{ccccccccc} 0 & \to & N & \to & G^{(2)} & \xrightarrow{\pi_0} & U & \to & 0 \\ & & \| & & \beta\downarrow & & \downarrow\nu & & \\ 0 & \to & N & \to & G & \xrightarrow{\pi} & Q & \to & 0 \end{array}$$

$\nu\pi_0 = \pi\beta$. By[4,lemma 3.3] ,(*) is computative with the topological exact rows. The top row of $(*)$ is central since N is a U-module via π_0 and a trivial Q-module. Hence there is a continuous homomorphism $s : U \to G^{(2)}$ with $\pi s = 1_U$. Now $(\pi(\beta s))(u) = \nu(\pi_0 s(u)) = \nu(u)$, $u \in U$ and βs is a continuous homomorphism. By lemma 2.2 , βs is unique. Therefore (U, ν) is the universal central extension of Q.

Now let (U, ν) be the universal central extension of Q. By lemma 2.3 , U is topologically perfect. We show that every central extension of U splits. Given a central extension (G, π) of U , we will show that $(G, \nu\pi)$ is a central extension of Q. For if $\nu(\pi(g_0)) = 1$, then $\pi(g_0)$ is in the center of U. Hence, the the map $f : G \to G$, $f(g) = g_0 g g_0^{-1}, g \in G$ is a continuous homomorphism because

$$\pi(f(g)) = \pi(g_0 g g_0^{-1}) = \pi(g_0)\pi(g)\pi(g_0^{-1}) = \pi(g), g \in G$$

Restricting to $G' = \overline{[G, G]}$ by lemmas 2.4 and 2.2 the resulting homomorphism from G' to G over U is the identity. Thus g_0 commutes with elements of G'. But every $g \in G$ can be expressed as $g = g'c, g' \in G', c \in ker f$ For if $g \in G$,since π maps $G' = \overline{[G, G]}$ onto U there exists $g' \in G'$ with $\pi(g) = \pi(g')$. Hence $g = g'c$ for $c \in ker\pi$. So g_0 commutes with all elements of G. Since π,ν are open onto continuous homomorphisms so is $\nu\pi$. Thus $(G, \nu\pi)$ is a central extension of Q.Since (U, π) is universal , there is a continuous homomorphism $S : U \to G$ over Q. The map πS is a continuous map from U to U over Q; hence equals to the identity. Thus S is a splitting. Therefore , every central extension of U splits.

The next theorem gives the sufficient and necessary condition for a topological group to have the weak universal extension.

Theorem 2.6 *A topological group Q admits the weak universal central extension if and only if Q is topologically perfect.*

Proof. Let (U, ν) be the weak universal central extension of Q . By lemma 2.3 ,U is topologically perfect. By the continuity of ν

$$\nu\overline{[U, U]} \subset \overline{\nu[U, U]} = \overline{[Q, Q]}$$

On the other hand $\nu\overline{[U, U]} = Q$. Hence $Q = \overline{[Q, Q]}$.

Conversely , if Q is perfect choose a free topological presentation of Q,

$$e : 0 \to R \to F \to Q \to 0$$

and consider

$$0 \to R/\overline{[R, F]} \to F/\overline{[R, F]} \to Q \to 0$$

the centralizer of (e). By lemma 2.4, $\overline{[F/\overline{[R, F]}, F/\overline{[R, F]}]} = \overline{[F, F]/\overline{[R, F]}} \to Q$ is a weak central extension . By [6], $[F, F]$ is closed in F, so $[F, F]/\overline{[R, F]}$ is closed. Hence

$$[F, F]/\overline{[R, F]} = \overline{[F, F]/\overline{[R, F]}}$$

We claim that $[F,F]/\overline{[R,F]} \to Q$ is the weak universal central extension of Q. Given any weak central extension (G,π) of Q with a continuous section $S : Q \to G$,since F is a free topological group there is a continuous homomorphism $\phi : F \to G$ over Q. Also $\phi[R,F] = 1$ since (G,π) is central. Now by the continuity of ϕ , $\phi\overline{[R,F]} = 1$. Thus ϕ induces a continuous homomorphism from $F/\overline{[R,F]}$ to G over Q. Restricting this map to $[F,F]/\overline{[R,F]}$,we obtain the required homomorphism from $[F,F]/\overline{[R,F]}$ to G over Q which is unique by lemma 2.2.

Proposition 2.7 *Let Q be a perfect topological group and the weak universal central extension (U,π) of Q exists and has a continuous section. Then the kernel of π is isomorphic to $M(Q)$, the Schur multiplier of Q.*

Proof. Let$N \to U \xrightarrow{\pi} Q$ be the universal central extension of Q with a section $S : Q \to U$, $\pi S = Id$. By lemma 2.5, U is perfect and every central extension of U splits. By [1,theorem 2.3] $Hom_C(M(U), M(U)) = 0$. This set contains the identity map, hence $M(U) = 0$. By [1, proposition 1.8] there is an exact sequence

$$M(U) \to M(Q) \to N \to U_{ab} \to Q_{ab} \to 0$$

But $M(U) = 0$ and,$U_{ab} = 0$,$Q_{ab} = 0$ because they are perfect. Therefore, $M(Q)$ is isomorphic to N.

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Received: August 7, 2006