Real Divisors of a Projective Variety
Containing a Given Scheme

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Abstract. Let \( Z \subset X \subset \mathbb{P}^n \) be real projective schemes with \( X \) integral and \( Z \) of codimension at least 2 in \( X \). Here we find a real hypersurface of \( X \) containing \( Z \), with bounded degree and with other properties.

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1. Introduction

Let \( Z \subset X \subset \mathbb{P}^n \) be real projective schemes with \( X \) integral and \( Z \) of codimension at least 2 in \( X \). Here (under suitable assumptions) we find a real hypersurface \( X(f) \) of \( X \) containing \( Z \), with bounded degree and with other properties. In Theorem 1 we consider the case \( X(f)(\mathbb{R}) = Z(\mathbb{R}) \). In Theorem 2 we find \( X(f) \) whose real locus intersects many connected components of \( X_{reg}(\mathbb{R}) \) for the semialgebraic topology.

Lemma 1. Let \( X \subset \mathbb{P}^n \) be an integral projective variety defined over an algebraically closed field \( \mathbb{K} \) with \( \text{char}(\mathbb{K}) = 0 \) and \( Z \subset X \) a closed subscheme with codimension \( c \geq 2 \). Let \( \mu \) denote the minimal integer \( t \) such that \( Z_{\text{red}} \) is the set-theoretic base locus of \( H^0(\mathbb{P}^n, \mathcal{I}_Z(t)) \). Fix a finite \( S \subset Z_{\text{reg}} \cap X_{\text{reg}} \) (we allow the case \( S = \emptyset \)). Assume also that the scheme-theoretic base locus of \( H^0(\mathbb{P}^n, \mathcal{I}_Z(t)) \) is equal to \( Z \) in a neighborhood of \( S \). Fix an integer \( e \) such that \( 1 \leq e \leq c - 1 \), an integer \( d \geq \mu \) and a general \( e \)-dimensional linear subspace of \( H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) \). Let \( A(V, X) \) the scheme-theoretic intersection of \( X \) with the codimension \( e \) and degree \( d^e \) complete intersection \( A(V) := \{ f = 0 \}_{f \in V\setminus\{0\}} \).

(i) \( A(V, X) \) has dimension \( \dim(X) - e \) and it is smooth at each point of \( X_{\text{reg}} \setminus Z \) and at each point of \( S \).

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(ii) If \( d > \mu \), then \( A(V,X) \) is integral;
(iii) Assume \( d = \mu \); \( A(V,X) \) is integral if for a general \( P \in X \) the base locus of \( H^0(\mathcal{I}_Z \cap \mathcal{I}(P)) \) has dimension at most \( \dim(X) - e - 1 \).

Proof. Part (i) outside \( S \) follows from Bertini’s theorem ([3], part 2) of Th. 6.3. Since \( S \) is finite, it is sufficient to notice that for general \( V \) and any fixed \( P \in Z \), \( T_P V \) is transversal to \( T_P X \) by our assumption on \( S \). Part (iii) for an arbitrary \( d \geq \mu \) is true by another theorem of Bertini ([3], part 4) of Th. 6.3). It is easy to check that if \( d > \mu \), then the condition in (iii) is satisfied (use reducible hypersurfaces, union of hypersurfaces of degree \( \mu \) containing \( Z \) and arbitrary degree \( d - \mu \) hypersurfaces. Part (ii) for \( S = \emptyset \) was also checked in [1], proof of Th. 1.2).

Quite often, but not always (e.g. if \( X = \mathbb{P}^n \) and \( Z \) is a point) in part (iii) the “only if” assertion holds.

Remark 1. Take \( X, Z, S, d \) as in Lemma 1. Assume that \( X, Z, S \) are defined over a real closed field \( \mathcal{R} \) and that \( \mathbb{K} \) is the algebraic closure of \( \mathcal{R} \). Notice that \( H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) = H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{I}_Z(d)) \otimes_{\mathbb{R}} \mathbb{K} \), because any extension of fields is flat and cohomology commutes with base change ([2], Prop. III.9.3). Since \( H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{I}_Z(t)) \) is Zariski dense in \( H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) \), and smoothness is an open condition, we may take \( V \) as in the statement of Lemma 1 with \( V \) defined over \( \mathcal{R} \) and (in parts (ii) and (iii)) with \( A(V,X) \) geometrically irreducible.

Theorem 1. Let \( \mathcal{R} \) be a real closed field and \( \mathbb{K} \) its algebraic closure. Let \( X \subset \mathbb{P}^n \) (resp. \( Z \subset X \)) a geometrically integral (resp. a closed subscheme of codimension at least 2) defined over \( \mathcal{R} \). Let \( \mu \) denote the minimal integer \( t \) such that \( Z_{\text{red}} \) is the set-theoretic base locus of \( H^0(\mathbb{P}^n, \mathcal{I}_Z(t)) \). Fix an even integer \( k \geq 2\mu \). If \( k = 2\mu \) assume that for a general \( P \in X(\mathbb{K}) \) the base locus of \( H^0(\mathbb{P}^n, \mathcal{I}_Z \cap \mathcal{I}(P)) \) in \( X(\mathbb{K}) \) has dimension at most \( \dim(X) - 2 \). Then there is \( f \in H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{I}_Z(k)) \) such that \( X(f) := X \cap \{ f = 0 \} \) is a geometrically integral hypersurface of \( X \), \( (X(\mathbb{K})_{\text{reg}} \cap X(f)) / Z(\mathbb{K}) \subseteq X(f)(\mathbb{K})_{\text{reg}} \) and \( X(f)(\mathcal{R}) = Z(\mathcal{R}) \).

Proof. Set \( d := k/2 \). Take a basis \( g_1, \ldots, g_s \) of \( H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{I}_Z(k)) \). Hence \( Z_{\text{red}} = \{ g_1 = \cdots = g_s = 0 \} \). Apply the proofs of Lemma 1 and of Remark 1 for \( e = 1 \) to the \( \mathbb{K} \)-vector space spanned by \( g_1^2, \ldots, g_s^2 \). Since \( (\mathcal{R}^n_{\mathbb{R}})_{\text{Zar}} \) is Zariski dense in \( \mathbb{K}^n \) we obtain the existence of \( c_1 \in \mathcal{R} \), \( 1 \leq i \leq s \), \( c_i > 0 \) for all \( i \), such that, setting \( f = \sum_{i=1}^s c_i g_i^2 \), \( X(f) := X \cap \{ f = 0 \} \) is a geometrically integral hypersurface of \( X \) and \( (X(\mathbb{K})_{\text{reg}} \cap X(f)) / Z(\mathbb{K}) \subseteq X(f)(\mathbb{K})_{\text{reg}} \). Since \( c_i > 0 \) for all \( i \) and each \( g_i \) is real, we have \( \{ f = 0 \}(\mathcal{R}) = Z(\mathcal{R}) \). Hence \( X(f)(\mathcal{R}) = Z(\mathcal{R}) \).

Notation 1. Let \( X \subset \mathbb{P}^n \) be an integral projective variety defined over an algebraically closed field with \( \text{char}(\mathbb{K}) = 0 \) and \( Z \subset X \) a closed subscheme with codimension \( c \geq 2 \). For every integer \( d \) let \( W(X,d) \) denote the image of the restriction map \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathbb{P}^n, \mathcal{O}_X(d)) \). Set \( W(X,d,Z) := W(X,d) \cap H^0(X, \mathcal{I}_Z, dx)(d) \), \( w(X,d) := \dim(W(X,d)) \) and \( w(X,d,Z) := \dim(W(X,d,Z)) \).
Let $\alpha(X,d)$ (resp. $\alpha(X,d,Z)$) denote the maximal integer $t \geq 0$ such that for a general $S \subset X$ the base locus of $W(X,d) \cap H^0(X,\mathcal{I}_S(d))$ (resp. $W(X,d,Z) \cap H^0(X,\mathcal{I}_S(d))$) does not contain a codimension 1 subscheme of $X$ with the convention $\alpha(X,d) = -\infty$ (resp. $\alpha(X,d,Z) = -\infty$) if no such integer $t \geq 0$ exists.

**Remark 2.** Let $\mathcal{R}$ be a real closed field with algebraic closure $\mathbb{K}$. In the setup of Notation 1 assume that $X,Z$ and the embedding $X \subset \mathbb{P}^n$ are defined over $\mathcal{R}$. Assume $X_{\text{reg}}(\mathcal{R}) \neq \emptyset$. The last assumption implies that $X_{\text{reg}}(\mathcal{R})$ is Zariski dense in $X(\mathbb{K})$. Hence if $\alpha(X,d)$ (resp. $\alpha(X,d,Z)$) is finite, then $\alpha(X,d)$ (resp. $\alpha(X,d,Z)$) is the maximal integer $t \geq 0$ such that there is $S \subset X_{\text{reg}}$ such that $\sharp(S) = t$, $\dim(W(X,d) \cap H^0(X,\mathcal{I}_S(d))) = w(X,d) - t$ (resp. $\dim(W(X,d,Z) \cap H^0(X,\mathcal{I}_S(d))) = w(X,d,Z) - t$ and the base locus of $W(X,d,Z) \cap H^0(X,\mathcal{I}_S(d))$ (resp. $W(X,d,Z) \cap H^0(X,\mathcal{I}_S(d))$ does not contain a codimension 1 subscheme of $X$. Let $T_1, \ldots, T_s$ be the connected components of $X_{\text{reg}}(\mathcal{R})$ in the semialgebraic topology. If $\mathcal{R} = \mathbb{R}$, then $T_1, \ldots, T_s$ are the connected components of $X_{\text{reg}}(\mathcal{R})$ in the euclidean topology. Fix integers $t_i \geq 0$, $1 \leq i \leq s$, such that $t_1 + \cdots + t_s = t$. Since each $T_i$ is Zariski dense in $X(\mathbb{K})$, we may find $S$ as above and such that $\sharp(S \cap T_i) = t_i$ for all $i$.

**Lemma 2.** Let $X \subset \mathbb{P}^n$ be an integral projective variety defined over an algebraically closed field $\mathbb{K}$ with char($\mathbb{K}$) = 0 and $Z \subset X$ a closed subscheme with codimension $c \geq 2$. Fix an integer $d > 0$ such that $h^i(\mathbb{P}^n, \mathcal{I}_Z(d-i)) = 0$ for all $i > 0$. Then $\alpha(X,d+1,Z) \geq w(X,d-1,Z).

**Proof.** Fix a general $S \subset X$ such that $\sharp(S) = w(X,d-1,Z)$. The generality of $S$ and the definition of the integer $w(X,d-1,Z)$ implies $h^0(\mathbb{P}^n, \mathcal{I}_{Z \cup S}(d-1)) = h^0(\mathbb{P}^n, \mathcal{I}_Z(d-i)) - w(X,d-1,Z)$. Since $h^1(\mathbb{P}^n, \mathcal{I}_Z(d-1)) = 0$, we get $h^1(\mathbb{P}^n, \mathcal{I}_{Z \cup S}(d-1)) = 0$. Since $S$ is finite, we have $h^i(\mathbb{P}^n, \mathcal{I}_{Z \cup S}(d-i)) = h^i(\mathbb{P}^n, \mathcal{I}_Z(d-i))$ for all $i \geq 2$. Hence $h^i(\mathbb{P}^n, \mathcal{I}_{Z \cup S}(d-i)) = 0$ for all $i > 0$. By Castelnuovo-Mumford’s lemma the homogeneous ideal of $S \cup Z$ in $\mathbb{P}^n$ is generated by forms of degree at most $d$. Apply part (ii) of Lemma 1.

**Remark 3.** Take $X,Z,d$ as in Lemma 2 and a general $S \subset X$ such that $\sharp(S) = w(X,d-1,Z)$. We saw in the proof of Lemma 2 that $h^i(\mathbb{P}^n, \mathcal{I}_{Z \cup S}(d-i)) = 0$ for all $i > 0$. By Castelnuovo-Mumford’s lemma the homogeneous ideal of $S \cup Z$ in $\mathbb{P}^n$ is generated by forms of degree at most $d$. Thus we may apply part (i) of Lemma 1 to the set $Z' := Z \cup S$. As remarked in Lemma 2 we apply part (ii) of Lemma 1 to the set $Z'$ and the integer $d' := d + 1$ (and hence to all higher integers).

**Lemma 3.** Fix an integral projective variety $X \subset \mathbb{P}^n$ defined over an algebraically closed field $\mathbb{K}$ with char($\mathbb{K}$) = 0, an integer $d > 0$ and a vector space $V \subset H^0(X,\mathcal{O}_X(d))$ such that $\dim(V) \geq 2$. Then fix a general $P \in X$ and set $V(-P) := \{f \in V : f(P) = 0\}$. Let $\phi_V$ (resp. $\phi_{V(-P)}$) be the rational map on $X$ induced by the linear system $|V|$ (resp. $|V(-P)|$). Hence
\[ \dim(V(-P)) = \dim(V) - 1. \] We have \[ \dim(\phi_{V(-P)}(X) = \dim(\phi_{V}(X)) \] if and only if the rational map \( \phi_{V} \) from \( X \) into \( P(V^*) \) is not dominant.

**Proof.** The equality \( \dim(V(-P)) = \dim(V) - 1 \) is obvious, by the generality of \( P \). Let \( Y \subseteq P(V^*) \) denote the closure of \( \text{Im}(\phi_{V}) \) in \( P(V^*) \). The linear projection of \( Y \) from \( Q \in Y \) maps \( Y \) into a lower dimensional variety if and only \( Y \) is a cone with vertex containing \( Q \). Since \( Y \) spans \( P(V^*) \), general \( A \in Y \) is a vertex of \( Y \) if and only if \( Y = P(V^*) \).

**Remark 4.** In the set-up of Lemma 3 \( \phi_{V} \) is not dominant if \( \dim(V) \geq \dim(X) + 2 \).

**Theorem 2.** Take the real closed field \( R \) as the base field. Let \( X \subset P^n \) be a geometrically integral variety and \( Z \subset X \) a closed subscheme with codimension at least 2 in \( X \). Let \( T_1, \ldots, T_s \) denote the connected components of \( X_{\text{reg}}(R) \) in the semialgebraic topology. Fix integers \( d, k \) such that \( h^i(P^n, I_Z(d-i)) = 0 \) for all \( i > 0 \) and \( d \geq k + 1 \). Let \( w(X, k, Z) \) denote the dimension of the image of the restriction map \( H^0(P^n, I_Z(k)) \to H^0(X, \mathcal{O}_X(d)) \). If \( d = k + 1 \), then assume \( w(X, k, Z) \geq \dim(X) + 2 \). Set \( a := \min\{s, w(X, k, Z)\} \). Then there exists a degree \( d \) hypersurface \( Y \subset P^n \) defined over \( R \) such that \( Z \subset Y, X \cap Y \) is geometrically integral, \( X_{\text{reg}} \cap Y \setminus Z \) is smooth and \( Y \) intersects \( T_1, \ldots, T_a \).

**Proof.** Apply Lemma 3 and Remarks 2 and 4.

**Remark 5.** If \( X_{\text{reg}}(R) \cap Z_{\text{reg}}(R) \neq \emptyset \), then we may apply part (i) of Lemma 1 and get \( X(f) \) as in the statement of Theorem 2 and smooth at a finite (prescribed in advance) set of \( X_{\text{reg}}(R) \cap Z_{\text{reg}}(R) \).

**Remark 6.** In the statements of Theorems 1 and 2 assume that \( Z \) has codimension \( c \geq 3 \) in \( X \). Fix an integer \( e \) such that \( 2 \leq e \leq c - 1 \). Iterating the proof of Theorem 2 we may also find a complete intersection \( Y \) of \( e \) hypersurfaces of degree \( d \) and with properties listed in the statement of Theorem 2. This extension of Theorem 1 is even easier.

**References**


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