Real Divisors of a Projective Variety Containing a Given Scheme

E. Ballico¹

Dept. of Mathematics
University of Trento
38050 Povo (TN), Italy
ballico@science.unitn.it

Abstract. Let $Z \subset X \subset \mathbf{P}^n$ be real projective schemes with X integral and Z of codimension at least 2 in X . Here we find a real hypersurface of X containing Z , with bounded degree and with other properties.

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1. INTRODUCTION

Let $Z \subset X \subset \mathbf{P}^n$ be real projective schemes with X integral and Z of codimension at least 2 in X . Here (under suitable assumptions) we find a real hypersurface $X(f)$ of X containing Z , with bounded degree and with other properties. In Theorem 1 we consider the case $X(f)(\mathcal{R}) = Z(\mathcal{R})$. In Theorem 2 we find $X(f)$ whose real locus intersects many connected components of $X_{reg}(\mathcal{R})$ for the semialgebraic topology.

Lemma 1. *Let $X \subset \mathbf{P}^n$ be an integral projective variety defined over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) = 0$ and $Z \subset X$ a closed subscheme with codimension $c \geq 2$. Let μ denote the minimal integer t such that Z_{red} is the set-theoretic base locus of $H^0(\mathbf{P}^n, \mathcal{I}_Z(t))$. Fix a finite $S \subseteq Z_{reg} \cap X_{reg}$ (we allow the case $S = \emptyset$). Assume also that the scheme-theoretic base locus of $H^0(\mathbf{P}^n, \mathcal{I}_Z(t))$ is equal to Z in a neighborhood of S . Fix an integer e such that $1 \leq e \leq c - 1$, an integer $d \geq \mu$ and a general e -dimensional linear subspace of $H^0(\mathbf{P}^n, \mathcal{I}_Z(d))$. Let $A(V, X)$ the scheme-theoretic intersection of X with the codimension e and degree d^e complete intersection $A(V) := \{f = 0\}_{f \in V \setminus \{0\}}$.*

- (i) *$A(V, X)$ has dimension $\dim(X) - e$ and it is smooth at each point of $X_{reg} \setminus Z$ and at each point of S .*

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- (ii) If $d > \mu$, then $A(V, X)$ is integral;
- (iii) Assume $d = \mu$; $A(V, X)$ is integral if for a general $P \in X$ the base locus of $H^0(\mathbf{P}^n, \mathcal{I}_{Z \cup \{P\}}(d))$ has dimension at most $\dim(X) - e - 1$.

Proof. Part (i) outside S follows from Bertini's theorem ([3], part 2) of Th. 6.3). Since S is finite, it is sufficient to notice that for general V and any fixed $P \in Z$, $T_P V$ is transversal to $T_P X$ by our assumption on S . Part (iii) for an arbitrary $d \geq \mu$ is true by another theorem of Bertini ([3], part 4) of Th. 6.3). It is easy to check that if $d > \mu$, then the condition in (iii) is satisfied (use reducible hypersurfaces, union of hypersurfaces of degree μ containing Z and arbitrary degree $d - \mu$ hypersurfaces. Part (ii) for $S = \emptyset$ was also checked in [1], proof of Th. 1.2). $\square$

Quite often, but not always (e.g. if $X = \mathbf{P}^n$ and Z is a point) in part (iii) the “only if” assertion holds.

Remark 1. Take X, Z, S, d as in Lemma 1. Assume that X, Z, S are defined over a real closed field $\mathcal{R}$ and that $\mathbb{K}$ is the algebraic closure of $\mathbb{R}$. Notice that $H^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = H^0(\mathbf{P}_{\mathcal{R}}^n, \mathcal{I}_Z(d)) \otimes_{\mathcal{R}} \mathbb{K}$, because any extension of fields is flat and cohomology commutes with base change ([2], Prop. III.9.3). Since $H^0(\mathbf{P}_{\mathcal{R}}^n, \mathcal{I}_Z(t))$ is Zariski dense in $H^0(\mathbf{P}^n, \mathcal{I}_Z(d))$, and smoothness is an open condition, we may take V as in the statement of Lemma 1 with V defined over $\mathcal{R}$ and (in parts (ii) and (iii)) with $A(V, X)$ geometrically irreducible.

Theorem 1. Let $\mathcal{R}$ be a real closed field and $\mathbb{K}$ its algebraic closure. Let $X \subset \mathbf{P}^n$ (resp. $Z \subset X$) a geometrically integral (resp. a closed subscheme of codimension at least 2) defined over $\mathbb{R}$. Let μ denote the minimal integer t such that Z_{red} is the set-theoretic base locus of $H^0(\mathbf{P}^n, \mathcal{I}_Z(t))$. Fix an even integer $k \geq 2\mu$. If $k = 2\mu$ assume that for a general $P \in X(\mathbb{K})$ the base locus of $H^0(\mathbf{P}^n, \mathcal{I}_{Z \cup \{P\}}(\mu))$ in $X(\mathbb{K})$ has dimension at most $\dim(X) - 2$. Then there is $f \in H^0(\mathbf{P}_{\mathcal{R}}^n, \mathcal{I}_Z(k))$ such that $X(f) := X \cap \{f = 0\}$ is a geometrically integral hypersurface of X , $(X(\mathbb{K})_{reg} \cap X(f)) \setminus Z(\mathbb{K}) \subseteq X(f)(\mathbb{K})_{reg}$ and $X(f)(\mathcal{R}) = Z(\mathcal{R})$.

Proof. Set $d := k/2$. Take a basis $g_1, \dots, g_s$ of $H^0(\mathbf{P}_{\mathcal{R}}^n, \mathcal{I}_Z(k))$. Hence $Z_{red} = \{g_1 = \dots = g_s = 0\}$. Apply the proofs of Lemma 1 and of Remark 1 for $e = 1$ to the $\mathbb{K}$ -vector space spanned by $g_1^2, \dots, g_s^2$. Since $(\mathcal{R}_{>0}^n)$ is Zariski dense in $\mathbb{K}^n$ we obtain the existence of $c_i \in \mathcal{R}$, $1 \leq i \leq s$, $c_i > 0$ for all i , such that, setting $f = \sum_{i=1}^s c_i g_i^2$, $X(f) := X \cap \{f = 0\}$ is a geometrically integral hypersurface of X and $(X(\mathbb{K})_{reg} \cap X(f)) \setminus Z(\mathbb{K}) \subseteq X(f)(\mathbb{K})_{reg}$. Since $c_i > 0$ for all i and each g_i is real, we have $\{f = 0\}(\mathcal{R}) = Z(\mathcal{R})$. Hence $X(f)(\mathcal{R}) = Z(\mathcal{R})$. $\square$

Notation 1. Let $X \subset \mathbf{P}^n$ be an integral projective variety defined over an algebraically closed field with $\text{char}(\mathbb{K}) = 0$ and $Z \subset X$ a closed subscheme with codimension $c \geq 2$. For every integer d let $W(X, d)$ denote the image of the restriction map $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) \rightarrow H^0(X, \mathcal{O}_X(d))$. Set $W(X, d, Z) := W(X, d) \cap H^0(X, \mathcal{I}_{Z, X}(d))$, $w(X, d) := \dim(W(X, d))$ and $w(X, d, Z) := \dim(W(X, d, Z))$.

Let $\alpha(X, d)$ (resp. $\alpha(X, d, Z)$) denote the maximal integer $t \geq 0$ such that for a general $S \subset X$ the base locus of $W(X, d) \cap H^0(X, \mathcal{I}_S(d))$ (resp. $W(X, d, Z) \cap H^0(X, \mathcal{I}_S(d))$) does not contain a codimension 1 subscheme of X with the convention $\alpha(X, d) = -\infty$ (resp. $\alpha(X, d, Z) = -\infty$) if no such integer $t \geq 0$ exists.

Remark 2. Let $\mathcal{R}$ be a real closed field with algebraic closure $\mathbb{K}$. In the set-up of Notation 1 assume that X, Z and the embedding $X \subset \mathbf{P}^n$ are defined over $\mathcal{R}$. Assume $X_{\text{reg}}(\mathcal{R}) \neq \emptyset$. The last assumption implies that $X_{\text{reg}}(\mathcal{R})$ is Zariski dense in $X(\mathbb{K})$. Hence if $\alpha(X, d)$ (resp. $\alpha(X, d, Z)$) is finite, then $\alpha(X, d)$ (resp. $\alpha(X, d, Z)$) is the maximal integer $t \geq 0$ such that there is $S \subset X_{\text{reg}}$ such that $\sharp(S) = t$, $\dim(W(X, d) \cap H^0(X, \mathcal{I}_S(d))) = w(X, d) - t$ (resp. $\dim(W(X, d, Z) \cap H^0(X, \mathcal{I}_S(d))) = w(X, d, Z) - t$) and the base locus of $W(X, d, Z) \cap H^0(X, \mathcal{I}_S(d))$ (resp. $W(X, d, Z) \cap H^0(X, \mathcal{I}_S(d))$) does not contain a codimension 1 subscheme of X . Let $T_1, \dots, T_s$ be the connected components of $X_{\text{reg}}(\mathcal{R})$ in the semialgebraic topology. If $\mathcal{R} = \mathbb{R}$, then $T_1, \dots, T_s$ are the connected components of $X_{\text{reg}}(\mathbb{R})$ in the euclidean topology. Fix integers $t_i \geq 0$, $1 \leq i \leq s$, such that $t_1 + \dots + t_s = t$. Since each T_i is Zariski dense in $X(\mathbb{K})$, we may find S as above and such that $\sharp(S \cap T_i) = t_i$ for all i .

Lemma 2. *Let $X \subset \mathbf{P}^n$ be an integral projective variety defined over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) = 0$ and $Z \subset X$ a closed subscheme with codimension $c \geq 2$. Fix an integer $d > 0$ such that $h^i(\mathbf{P}^n, \mathcal{I}_Z(d - i)) = 0$ for all $i > 0$. Then $\alpha(X, d + 1, Z) \geq w(X, d - 1, Z)$.*

Proof. Fix a general $S \subset X$ such that $\sharp(S) = w(X, d - 1, Z)$. The generality of S and the definition of the integer $w(X, d - 1, Z)$ implies $h^0(\mathbf{P}^n, \mathcal{I}_{Z \cup S}(d - 1)) = h^0(\mathbf{P}^n, \mathcal{I}_Z(d - i)) - w(X, d - 1, Z)$. Since $h^1(\mathbf{P}^n, \mathcal{I}_Z(d - 1)) = 0$, we get $h^1(\mathbf{P}^n, \mathcal{I}_{Z \cup S}(d - 1)) = 0$. Since S is finite, we have $h^i(\mathbf{P}^n, \mathcal{I}_{Z \cup S}(d - i)) = h^i(\mathbf{P}^n, \mathcal{I}_Z(d - i))$ for all $i \geq 2$. Hence $h^i(\mathbf{P}^n, \mathcal{I}_{Z \cup S}(d - i)) = 0$ for all $i > 0$. By Castelnuovo-Mumford's lemma the homogeneous ideal of $S \cup Z$ in $\mathbf{P}^n$ is generated by forms of degree at most d . Apply part (ii) of Lemma 1. $\square$

Remark 3. Take X, Z, d as in Lemma 2 and a general $S \subset X$ such that $\sharp(S) = w(X, d - 1, Z)$. We saw in the proof of Lemma 2 that $h^i(\mathbf{P}^n, \mathcal{I}_{Z \cup S}(d - i)) = 0$ for all $i > 0$. By Castelnuovo-Mumford's lemma the homogeneous ideal of $S \cup Z$ in $\mathbf{P}^n$ is generated by forms of degree at most d . Thus we may apply part (i) of Lemma 1 to the set $Z' := Z \cup S$. As remarked in Lemma 2 we apply part (ii) of Lemma 1 to the set Z' and the integer $d' := d + 1$ (and hence to all higher integers).

Lemma 3. *Fix an integral projective variety $X \subset \mathbf{P}^n$ defined over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) = 0$, an integer $d > 0$ and a vector space $V \subseteq H^0(X, \mathcal{O}_X(d))$ such that $\dim(V) \geq 2$. Then fix a general $P \in X$ and set $V(-P) := \{f \in V : f(P) = 0\}$. Let ϕ_V (resp. $\phi_{V(-P)}$) be the rational map on X induced by the linear system $|V|$ (resp. $|V(-P)|$). Hence*

$\dim(V(-P)) = \dim(V) - 1$. We have $\dim(\phi_{V(-P)}(X)) = \dim(\phi_V(X))$ if and only if the rational map ϕ_V from X into $\mathbf{P}(V^*)$ is not dominant.

Proof. The equality $\dim(V(-P)) = \dim(V) - 1$ is obvious, by the generality of P . Let $Y \subseteq \mathbf{P}(V^*)$ denote the closure of $\text{Im}(\phi_V)$ in $\mathbf{P}(V^*)$. The linear projection of Y from $Q \in Y$ maps Y into a lower dimensional variety if and only if Y is a cone with vertex containing Q . Since Y spans $\mathbf{P}(V^*)$, general $A \in Y$ is a vertex of Y if and only if $Y = \mathbf{P}(V^*)$. $\square$

Remark 4. In the set-up of Lemma 3 ϕ_V is not dominant if $\dim(V) \geq \dim(X) + 2$.

Theorem 2. Take the real closed field $\mathcal{R}$ as the base field. Let $X \subset \mathbf{P}^n$ be a geometrically integral variety and $Z \subset X$ a closed subscheme with codimension at least 2 in X . Let $T_1, \dots, T_s$ denote the connected components of $X_{\text{reg}}(\mathcal{R})$ in the semialgebraic topology. Fix integers d, k such that $h^i(\mathbf{P}^n, \mathcal{I}_Z(d-i)) = 0$ for all $i > 0$ and $d \geq k+1$. Let $w(X, k, Z)$ denote the dimension of the image of the restriction map $H^0(\mathbf{P}^n, \mathcal{I}_Z(k)) \rightarrow H^0(X, \mathcal{O}_X(d))$. If $d = k+1$, then assume $w(X, k, Z) \geq \dim(X) + 2$. Set $a := \min\{s, w(X, k, Z)\}$. Then there exists a degree d hypersurface $Y \subset \mathbf{P}^n$ defined over $\mathcal{R}$ such that $Z \subset Y$, $X \cap Y$ is geometrically integral, $X_{\text{reg}} \cap Y \setminus Z$ is smooth and Y intersects $T_1, \dots, T_a$.

Proof. Apply Lemma 3 and Remarks 2 and 4. $\square$

Remark 5. If $X_{\text{reg}}(\mathcal{R}) \cap Z_{\text{reg}}(\mathcal{R}) \neq \emptyset$, then we may apply part (i) of Lemma 1 and get $X(f)$ as in the statement of Theorem 2 and smooth at a finite (prescribed in advance) set of $X_{\text{reg}}(\mathcal{R}) \cap Z_{\text{reg}}(\mathcal{R})$.

Remark 6. In the statements of Theorems 1 and 2 assume that Z has codimension $c \geq 3$ in X . Fix an integer e such that $2 \leq e \leq c-1$. Iterating the proof of Theorem 2 we may also find a complete intersection Y of e hypersurfaces of degree d and with properties listed in the statement of Theorem 2. This extension of Theorem 1 is even easier.

REFERENCES

- [1] R. Ghiloni, Equations and complexity for the Dubois-Efroymsen dimension theorem, preprint.
- [2] R. Hartshorne, Algebraic Geometry, Springer, Berlin, 1977.
- [3] J.-P. Jouanolou, Théorèmes de Bertini et applications, Progress in Math. 42, Birkhäuser, Basel, 1983.

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