A Topological Structure on the Dual Space of Fundamental Locally Multiplicative Topological Algebras

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Abstract

We have already introduced a topological structure on the dual spaces of fundamental locally multiplicative topological algebras. In this note we compare this structure with the other well known topological structures on the dual spaces.

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1 Introduction

The fundamental topological algebras have already been introduced in [1] to extend the meaning of locally bounded ness and local convexity.

The fundamental locally multiplicative topological algebras (abbreviated by FLM) with a property very similar to the normed algebras is also introduced in [3]. In [2] a topological structure is defined on the algebraic dual space of an FLM algebra to make it a normed space, and some of the famous theorems of Banach algebras are extended for complete metrizable FLM algebras.

Here, in this note we compare the topological dual space of a complete metrizable FLM algebra with its algebraic dual space when it is topologized by this new structure. Since every normed algebra is in fact an FLM algebra we also compare especially this structure with the normed dual space of a normed algebra. At first we begin with the pervious definitions and related results.
2 Definitions and related results

Definition 2.1. Let $A$ be a topological vector space. We say $A$ is a fundamental topological vector space, if there exists $b > 1$ such that for every sequence $(a_n)$ of $A$, the convergence of $b^n(a_n - a_{n-1})$ to zero in $A$ implies that $(a_n)$ is Cauchy sequence.

Proposition 2.2. ([1; Proposition 2.4]) Let $A$ be a fundamental topological vector space. Then, for every $c > 1$ and every sequence $(a_n)$ of $A$, the convergence of $c^n(a_n - a_{n-1})$ to zero in $A$ implies that $(a_n)$ is a Cauchy sequence.

Definition 2.3. A fundamental topological algebra is an algebra whose underlying topological vector space is fundamental.

Theorem 2.4. Let $A$ be a complete metrizable fundamental topological algebra and $x \in A$. If for some $b > 1, b^nx^n \to 0$ in $A$;

i) $x$ is quasi-invertible and $x^0 = -\sum_{n=1}^{\infty} x^n$,

ii) If $A$ possesses a unit element, then $1 - x$ is invertible and

$$(1 - x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n.$$ 

Definition 2.5. A fundamental topological algebra is called to be locally multiplicative, if there exists a neighborhood $U_0$ of zero such that for every neighborhood $V$ of zero, the sufficiently large powers of $U_0$ lie in $V$. We call such an algebra, an FLM algebra.

Definition 2.6. Let $X$ be a metrizable FLM algebra. By the algebraic dual of $X$ we mean the linear space of all linear functionals on $X$ and we denote it by $\hat{X}$; and by the topological dual space of $X$ we mean the linear space of all continuous linear functionals on $X$ and we denote it by $X^*$.

Definition 2.7. Let $X$ be a metrizable FLM algebra. Define:

$$S(X) = \{x \in X : \exists b > 1 \text{ such that } b^nx^n \to 0\},$$

$$v(f) = \sup\{|f(x)| : x \in S(X)\}$$

for all $f \in \hat{X}$ and

$$X' = \{f \in \hat{X} : v(f) < \infty\}.$$
Theorem 2.8. with the above notations:

i) $S(X)$ is a balanced absorbing set,

ii) $0 \leq v(f) \leq \infty$ and $v(f) = 0 \iff f \equiv 0$,

iii) $X'$ is a Banach space,

iv) $X' \subseteq X^*$

v) for multiplicative linear functional $f$, $v(f) \leq 1$, and moreover if $X$ has a unit element $1$ such that $1 \in S(X)$, then $v(f) = 1$.

Proof. A proof is given in [2].

3 new results

In a normed space $(X \mid \cdot \mid)$, the normed dual space $X^*$, of course, does not change if we use any equivalent norm $p \in En(X)$, but the norm of the elements of $X^*$ depends deeply on the choice of the equivalent norms on $X$, where it is easy to see that $S(X)$ and, as a result $v(f)$ for every $f \in \hat{X}$, are independent of the choice of the equivalent norms on $X$ and only depends on the algebraic structure of $X$.

Theorem 3.1. Suppose $X$ is an FLM algebra and $S(X)$ is the same that is defined in [2.7], then:

i) $\bigcup_{b > 1} b^{-1}U_0 \subseteq S(X)$ where $U_0$ is defined in [2.5].

ii) If $X$ is normed algebra with norm $\mid \cdot \mid$, then:

$U_0 = \{x \in X : \mid x \mid < 1\} \subseteq S(X)$.

iii) If $1 \in X$, there exists $p \in En(X)$ such that $U_p \neq S(X)$.

iv) If $X$ is a commutative normed algebra and $S(X)$ is bounded then, there exists $p \in En(X)$ such that $U_p = S(X)$.

Proof. For (iii) we see that if $\parallel 1 \parallel > 1$ and we choose $c$ and $b$ such that $\parallel 1 \parallel > c > b > 1$ then $x = c^{-1}.1 \notin U_0 \parallel 1 \parallel$ where, $b^n x^n = (\frac{b}{c})^n . 1$ and $\parallel b^n x^n \parallel = (\frac{b}{c})^n \parallel 1 \parallel \rightarrow 0$ which implies that $x \in S(X)$.

If $X$ is a commutative normed algebra and $S(X)$ is bounded, then $S(X)$ is a bounded sub semigroup and the result follows from [4; theorem 4.1].

Example 3.2 Let $X = \{ (\begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array}) : \alpha, \beta \in C \}$. Then $X$ is a commutative Banach algebra with $\parallel (\begin{array}{cc} \alpha & 0 \\ 0 & \beta \end{array}) \parallel = \sqrt{\alpha^2 + \beta^2}$. Now we see that $x = (\begin{array}{cc} \frac{2}{1 + \sqrt{2}} & 0 \\ 0 & \frac{2}{1 + \sqrt{2}} \end{array}) \in S(X)$, where $\parallel x \parallel > 1$. 
Proposition 3.3. Suppose $X$ is an FLM algebra and $S(X)$ is the same that is defined in [2.7], then:

i) $S(X) \subseteq q - \text{Inv}(X)$

ii) If $1 \in X$; $S(X) \subseteq 1 + \text{Inv}(X)$

Proof. This is a consequence of theorem 2.4.

Proposition 3.4. Let $X$ be a Banach algebra and $\Phi_X$ be the carrier space of $X$.

i) If $f \in X'$, then $\| f \| \leq v(f)$,

ii) If $1 \in X$ with $\| 1 \|= 1$ and $f \in \Phi_X$ then $\| f \|= v(f) = 1$.

Proof. For (i), suppose $x \in X$ with $\| x \| \leq 1$. Choose $x_n$ in $X$ with $\| x_n \| < 1$ such that $x_n \to x$, then $x_n \in S(X)$ and for $f \in X'$ we have $| f(x_n) | \leq v(f)$ where by the continuity of $f$, $| f(x) | \leq v(f)$ and hence $\| f \| \leq v(f)$.

For (ii) we see that $1 \in S(X)$, and then it follows from [2.8: v].

Theorem 3.5. Suppose $X$ is an FLM algebra and $S(X)$ is the same that is defined in [2.7], then:

i) If $q - \text{Inv}(X)$ is bounded, or $1 \in X$ and $\text{Inv}(X)$ is bounded and, or $S(X)$ is bounded; then $X' = X^*$.

ii) there are examples for which $X' \not\subseteq X^*$.

Proof. Let $f \in X^*$, and $V$ be a neighborhood of zero such that $f$ is bounded on $V$. Now, if $S(X)$ be bounded then $S(X) \subseteq \lambda V$ for sufficiently large $\lambda$ and therefore $f$ is bounded on $S(X)$ which implies that $v(f) < \infty$.

If $X$ is an FLM algebra for which every element $x \in X$ is nilpotent, then $S(X) = X$ and for non-zero linear functional $f : X \to C$, $\sup\{|f(x)| : x \in S(X)\} = \infty$, and therefore $X' = \{0\}$.

REFERENCES


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