Harmonic Curvatures of Null Curves and the Null Helix in R_1^{m+2}

Ayşe Altın

Hacettepe University
Science Faculty, Mathematics Department
06532 Beytepe Ankara, Turkey
ayse@hacettepe.edu.tr

Abstract

In this work we calculated the Frenet apparatus of a null curve C in R_1^{m+2} in terms of the Frenet apparatus of the curve C^* which is the orthogonal projection of C on R^{m+1} . The harmonic curvatures of C are defined. We also give the theorems which provides some information about a null helix. If C is a null helix then it must be contained in a four dimensional subspace in R_1^{m+2} .

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1 Preliminary

The smooth curve $C = \alpha(I)$ in a semi-Riemannian manifold (M^{m+2}, g) is said to be a null curve if the velocity vector to C at any point is a null vector.

Let TC be the tangent bundle of C and

$$TC^\perp = \cup_{t \in I} T_{\alpha(t)} C^\perp, \text{ where } T_{\alpha(t)} C^\perp = \{V_{\alpha(t)} \in T_{\alpha(t)} M : g(V_{\alpha(t)}, \alpha'(t)) = 0\}.$$

At each point $\alpha(t)$, we choose a complementary vector space to $T_{\alpha(t)} C$ in $T_{\alpha(t)} M$. Denote by $S(T_{\alpha(t)} C^\perp)$, this chosen subspace. Hence, we get a vector bundle $S(TC^\perp)$ on α . Since

$$TC^\perp = TC \perp S(TC^\perp),$$

$S(TC^\perp)$ is a vector bundle of rank m . The non-degenerate vector bundle $S(TC^\perp)$ is called a screen vector bundle of C . Therefore we have

$$TM|_C = S(TC^\perp) \perp S(TC^\perp)^\perp \quad (1)$$

where $S(TC^\perp)^\perp$ is a complementary orthogonal vector bundle to $S(TC^\perp)$ in $TM|_C$.

Theorem 1.1. Let C be a null curve of a proper semi-Riemannian manifold (M, g) and $S(TC^\perp)$ be a screen vector bundle of C . Then there exists a unique vector bundle $ntr(C)$ over C of rank 1, such that on each coordinate neighbourhood $U \subset C$ there is a unique $N \in \Gamma(ntr(C)|_U)$ satisfying

$$g(\alpha'(t), N) = 1, \quad g(N, N) = 0, \quad g(N, X) = 0, \quad \forall X \in \Gamma(S(TC^\perp)|_U). \quad (2)$$

Consider

$$tr(C) = ntr(C) \perp S(TC^\perp),$$

from (1), (2) then we have the following sum

$$TM|_C = TC \oplus tr(C) = (TC \oplus ntr(C)) \perp S(TC^\perp). \quad (3)$$

The vector field N , which was constructed in this theorem, is said to be the null transversal vector field of C with respect to α' [2.p.53].

A null curve C in R_1^{m+2} is given locally by the equation of the following form

$$\alpha(s) = (s, \int_0^s \alpha_1, \dots, \int_0^s \alpha_{m+1}), \quad (4)$$

where, $\alpha_1 = \cos b_1(s)ds + c_1$, $\alpha_a = \cos b_a(s) \prod_{k=1}^{a-1} \sin b_k(s)ds + c_a$,

$a \in \{2, \dots, m\}$, $\alpha_{m+1} = \prod_{k=1}^m \sin b_k(s)ds + c_{m+1}$, $c_k \in R$,

b_k are smooth functions for any $k \in \{1, \dots, m\}$, and s is the arc-length of the orthogonal projection $C^* = \alpha^*(I)$ of C on R^{m+1} give by, [2,p.73],

$$\alpha^*(t) = \{\alpha_1^*(t), \alpha_2^*(t), \dots, \alpha_{m+1}^*(t)\}.$$

By the ideas in [4,p.73] and [5,p.160], a helix is defined as a curve which has a constant scalar product of its tangent vector field and a constant vector field.

In this paper, we mean by α^l 's, $(1 \leq l \leq r)$ the derivatives of the curve α .

Let α^* be a regular curve in R^{m+1} and $\psi = \{(\alpha^*)'(t), (\alpha^*)''(t), \dots, \alpha^{*r}(t)\}$ be a maximal linearly independent set. The orthonormal system $\{V_1(t), V_2(t), \dots, V_r(t)\}$ can be obtained from ψ . This is called a Frenet frame at the point $\alpha^*(t)$, [3].

Definition 1.2. Let α^* be a regular curve in R^{m+1} and $\{V_1(t), V_2(t), \dots, V_r(t)\}$ be the Frenet frame at the point $\alpha^*(t)$.

The functions $k_i : I \longrightarrow R$ defined by

$$k_i(t) = g(V'_i(t), V_{i+1}(t)), \quad 1 \leq i \leq r-1 \quad (5)$$

are called curvature functions on α^* . Moreover, the real number $k_i(t)$ is called the i -th curvature on α^* at the point $\alpha^*(t)$.

Let α^* be a unit speed curve in R^{m+1} and the set $\{V_1(t), V_2(t), \dots, V_r(t)\}$ be the Frenet frame at the point $\alpha^*(t)$. Then, the followings hold, [6,p.194],

$$V'_1(t) = k_1(t)V_2(t), \quad (6)$$

$$V'_i(t) = -k_{i-1}(t)V_{i-1}(t) + k_i(t)V_{i+1}(t), \quad i < i < r, \quad (7)$$

$$V'_r(t) = -k_{r-1}(t)V_{r-1}(t). \quad (8)$$

2.HELIX AND HARMONIC CURVATURES

Theorem 2.1. Let α and α^* be the curves as in equation (4) and $\{V_1, V_2, \dots, V_r\}$, $\{k_1, k_2, \dots, k_{r-1}\}$ be Frenet fields and curvature functions of curve α^* respectively. Then, α is a null curve in R_1^{r+1} . More over if we choose $S(TC^\perp)$ spanned by $\{W_2, \dots, W_r\}$, then we have the null transversal vector field $N = \frac{1}{2}(-1, V_1)$, and the Frenet equations are

$$\left\{ \begin{array}{l} \alpha'' = k_1 W_2, \\ N' = \frac{1}{2} k_1 W_2, \\ (W_2)' = -\frac{1}{2} k_1 W_0 - k_1 N + k_2 W_3, \\ (W_3)' = -k_2 W_2 + k_3 W_4, \\ (W_4)' = -k_3 W_3 + k_4 W_5, \\ \dots\dots\dots \\ (W_{r-1})' = -k_{r-2} W_{r-2} + k_{r-1} W_r, \\ (W_r)' = -k_{r-1} W_{r-1}, \end{array} \right. \quad (9)$$

where

$$W_j = (0, V_j) \quad j \in \{2, \dots, r\} \quad (10)$$

and the Frenet frame is $F = \{\alpha', N, W_2, \dots, W_r\}$ on R_1^{r+1} along α .

Proof. From (4) we have

$$\alpha' = (1, (\alpha^*)'), \quad \alpha^j = (0, (\alpha^*)^j), \quad j \geq 2.$$

Therefore $\{\alpha', \alpha'', \dots, \alpha^r\}$, $r \leq m+1$ is the maximal linearly independent set. Since α^* has at most $(r-1)$ non zero curvatures, α^* is contained in R^r . Therefore α is also contained in R_1^r .

We choose $S(TC^\perp) = \text{span}\{W_2, \dots, W_r\}$ and a complementary vector bundle H to TC in $S(TC^\perp)$,

$Y = (0, (\alpha^*)') \in \Gamma(H|_U)$ and $g(\alpha', Y) \neq 0$ on U . We calculated the null transversal vector field and found that

$$N = \frac{1}{g(\alpha', Y)} \left\{ Y - \frac{g(Y, Y)}{2g(\alpha', Y)} \alpha' \right\} = \frac{1}{2}(-1, (\alpha^*)').$$

Since $(\alpha^*)' = V_1$, we have

$$\alpha' = (1, V_1), \quad N = \frac{1}{2}(-1, V_1). \quad (11)$$

We differentiate (10), (11) and by using (6), (7), (8) we obtain (9).

We now give the definition of a null helix in semi-Euclidean space R_1^{m+2} in a similar way to [1], as follows.

Definition 2.2. Let α be a null curve in R_1^{m+2} and X be a non zero constant vector field. If

$$g(\alpha'(t), X) = \text{constant} \neq 0, \quad \text{for all } t \in I,$$

then, α is said to be a null helix in R_1^{m+2} and $\text{span}\{X\}$ is said to be the inclination axes of α ,

Example 2.3. Let a, σ, ρ be non-zero constants, b be constant and let $\alpha : R \longrightarrow R_1^3$ be the curve difined by

$$\alpha(t) = (at + b, \quad \frac{1}{\rho}\sigma\cos\rho t, \quad \frac{1}{\rho}\sigma\sin\rho t), \quad X = (1, 0, 0)$$

where $a^2 = \sigma^2$.

Example 2.4. $\alpha : R \longrightarrow R_1^3$ be the curve difined by

$$\alpha(t) = (\frac{4}{3}t^3 + t, \quad 2t^2, \quad \frac{4}{3}t^3 - t), \quad X = (1, 0, 1)$$

Example 2.5. Let a, σ, ρ be non-zero constants, b, c, d be constants and let $\alpha : R \longrightarrow R_1^4$ be the curve difined by

$$\alpha(t) = (at + b, \quad ct + d, \quad \frac{1}{\rho}\sigma\cos\rho t, \quad \frac{1}{\rho}\sigma\sin\rho t), \quad X = (1, 0, 0, 0)$$

where $a^2 = c^2 + \sigma^2$.

Example 2.6. Let $a, \sigma, \rho, \omega, d$ be non-zero constants, b be constant and let $\alpha : R \longrightarrow R_1^5$ be the curve difened by

$$\alpha(t) = (at + b, \frac{1}{\rho}\sigma\cos\rho t, \frac{1}{\rho}\sigma\sin\rho t, \frac{1}{\omega}d\cos\omega t, \frac{1}{\omega}d\sin\omega t), \quad X = (1, 0, 0, 0, 0)$$

where $a^2 = \sigma^2 + d^2$.

Definition 2.7. Suppose that $k_1, k_2, \dots, k_{n-1}$ are curvature functions of a curve α . A function $H_i : I \longrightarrow R$ defined by

$$H_i(t) = \begin{cases} \frac{k_1(t)}{k_2(t)}, & \text{if } i = 1 \\ \frac{1}{k_{i+1}(t)}\{H'_{i-1}(t) + k_i(t)H_{i-2}(t)\}, & \text{if } 2 \leq i \leq n-2 \end{cases} \quad (12)$$

is called the i -th harmonic curvature function of α .

Theorem 2.8. Let α be a null helix in R_1^{m+2} , and $\text{span}\{X\}$ be the inclination axes of α . If $\{\alpha', N, W_2, \dots, W_r\}$ is the Frenet frame fields of α and $H_1, H_2, \dots, H_{r-2}$ are harmonic curvature functions of α , then

$$g(W_{i+2}, X) = H_i g\left(\frac{\alpha' + 2N}{2}, X\right), \quad 1 \leq i \leq r-2. \quad (13)$$

Proof. We use induction on i . Since α is null helix, then $g(\alpha', X) = \text{constant}$. If we take derivative, then we have $\langle \alpha'', X \rangle = 0$. Thus by using (6), we obtain

$$g(W_2, X) = 0. \quad (14)$$

Taking derivative in (14) and using (7), we get

$$g(W_3, X) = H_1\left(\frac{\alpha' + 2N}{2}, X\right).$$

Hence (13) is true for $i=1$. By the induction hypothesis, we have

$$g(W_{i+1}, X) = H_{i-1} g\left(\frac{\alpha' + 2N}{2}, X\right). \quad (15)$$

Using (15) and (7) we get

$$g(W_{i+2}, X) = \frac{1}{k_{i+1}}\{k_i g(W_i, X) + (H_{i-1})' g\left(\frac{\alpha' + 2N}{2}, X\right)\}. \quad (16)$$

By the induction hypothesis and by (16), we have

$$g(W_{i+2}, X) = \frac{1}{k_{i+1}} \{k_i H_{i-2} + (H_{i-1})'\} g\left(\frac{\alpha' + 2N}{2}, X\right).$$

Therefore, (13) follows from (12).

Lemma 2.9. Let α be a null helix in R_1^{m+2} , $\text{span}\{X\}$ be the inclination axes, $\{\alpha', N, W_2, \dots, W_r\}$ be the Frenet frame fields of α and let Π be the orthogonal projection of R_1^{m+2} onto the space $\text{span}\{\alpha', N, W_2, \dots, W_r\}$. If $r < m$, then $\text{span}\{\Pi(X)\}$ is the inclination axes in $\text{span}\{\alpha', N, W_2, \dots, W_r\}$.

Proof. We can choose $\{\alpha', N, W_2, \dots, W_r, w_1, \dots, w_{m-r}\}$ as an quasi-orthonormal basis of R_1^{m+2} . In this case

$$X = x_0 \alpha' + x_1 N + \sum_{i=2}^r x_i W_i + \sum_{j=1}^{m-r} b_j w_j, \quad \Pi(X) = x_0 \alpha' + x_1 N + \sum_{i=2}^r x_i W_i.$$

Since $g(\sum_{j=1}^{m-r} b_j w_j, \alpha') = 0$ and $g(X, \alpha') = \text{constant} \neq 0$, we have

$$g(x_0 \alpha' + x_1 N + \sum_{i=2}^r x_i W_i, \alpha') = \text{constant} \neq 0$$

Since $\text{span}\{X\}$ is inclination axes, then $\Pi(X)$ is also non zero and constant.

Theorem 2.10. Let α be a curve in R_1^{m+2} with the Frenet frame field $\{\alpha', N, W_2, \dots, W_r\}$ and with harmonic curvatures $H_1, H_2, \dots, H_{r-2}$, $r \leq m$. Then, α is a null helix in R_1^{m+2} if and only if H_i 's are constant and $x_1 \neq 0$.

Proof. Let $\text{span}\{X\}$ be the inclination axes of α . Thus

$$g(\alpha', X) = \text{constant} \neq 0 \quad \text{and} \quad g(\alpha', \Pi(X)) = \text{constant} \neq 0.$$

Using (14) and (15), we obtain

$$\Pi(X) = x_0 \alpha' + x_1 N + \sum_{i=3}^r H_{i-2} g\left(\frac{\alpha' + 2N}{2}, X\right) W_i.$$

Since $\Pi(X)$ is a non zero constant vector field, H_i 's must be constant and $x_1 \neq 0$.

Conversely, choose a constant vector field, $X = \alpha' + N + \sum_{i=3}^r H_{i-2} W_i$. Then $g(\alpha', X) = 2$. Therefore α is a helix.

Theorem 2.11. There is a relation between curvatures and harmonic curvatures of a curve in R_1^{m+2} as follows.

$$k_r = \frac{(\sum_{i=1}^{r-2} H_i^2)'}{2H_{r-1}H_{r-2}}, \quad 3 \leq r \leq m-1 \quad (17)$$

Proof. We use induction on r . By (12),

$$k_{i+1} = \frac{(H_{i-1})' + k_i H_{i-2}}{H_i}, \quad 2 \leq i \leq m-2. \quad (18)$$

For $i = 2$, $k_3 = \frac{(H_1^2)'}{2H_1H_2}$ and the statement (17) is true for $r = 3$.

The induction hypothesis states that

$$k_{p+1} = \frac{(\sum_{i=1}^{p-1} H_i^2)'}{2H_pH_{p-1}}, \quad p < m-1. \quad (19)$$

Using (18) and (19), we get

$$k_{p+2} = \frac{(\sum_{i=1}^p H_i^2)'}{2H_{p+1}H_p}$$

which completes the proof.

Consequently, combining (12), (17) and theorem 2.10 we can give our main theorem.

Theorem 2.12. The curve α is a null helix in R_1^{m+2} if and only if $n \geq 4$ and $k_j = 0$ for $j \geq 3$.

As a consequence of this theorem we obtain the following.

Corollary 2.13. if α is null helix then α is contained in a four dimensional subspaces in R_1^{m+2} .

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