Rough Marcinkiewicz Integrals On Product Spaces¹

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Abstract

In this paper, we establish an L^p boundedness result of a class of Marcinkiewicz integral operators on product domains with rough kernels.

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1 Introduction

Let $\mathbf{R}^d$ ($d = n$ or $d = m$), $d \geq 2$ be the d -dimensional Euclidean space and $\mathbf{S}^{d-1}$ be the unit sphere in $\mathbf{R}^d$ equipped with the induced Lebesgue measure $d\sigma$. Suppose that Ω be a homogeneous function of degree zero on $\mathbf{R}^n$ which is integrable on $\mathbf{S}^{n-1}$ and $\int_{\mathbf{S}^{n-1}} \Omega(y) d\sigma(y) = 0$. Then the Marcinkiewicz integral operator μ_Ω which was introduced by E. M. Stein in ([20]) is defined by

$$\mu_\Omega(f)(x) = \left(\int_{-\infty}^{\infty} \left| \int_{|y| \leq 2^t} f(x-y) \Omega(y) |y|^{1-n} dy \right|^2 2^{-2t} dt \right)^{\frac{1}{2}}. \quad (1.1)$$

E. M. Stein proved that if $\Omega \in Lip_\alpha(\mathbf{S}^{n-1})$, ($0 < \alpha \leq 1$), then μ_Ω is bounded on L^p for all $1 < p \leq 2$ ([20]). Subsequently, the study of the L^p boundedness of μ_Ω under various conditions on the function Ω has been studied by many authors ([2], [3], [4], [5], [7], [8], [9], [17], among others). A particular result that is closely related to our work is the boundedness result of μ_Ω obtained by

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Chen-Fan-Pan in ([9]). In fact, the authors of ([9]) showed that μ_Ω is bounded on L^p for $p \in ((2+2\alpha)/(1+2\alpha), 2+2\alpha)$ provided that Ω satisfies the following condition

$$\sup_{\xi \in \mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \left| \Omega(y') \right| \left(\log \frac{1}{\xi \cdot y'} \right)^{1+\alpha} d\sigma(y') < \infty, \quad (1.2)$$

for some $\alpha > 0$. The conditions (1.2) were introduced by Grafakos and Stefanov in their study of singular integral operators ([14]). It should be pointed out here that Grafakos and Stefanov showed that for any $\alpha > 0$, the following relations hold:

$$F(\alpha, \mathbf{S}^{n-1}) \not\subseteq L(\log^+ L)(\mathbf{S}^{n-1}) \text{ and } L(\log^+ L)(\mathbf{S}^{n-1}) \not\subseteq F(\alpha, \mathbf{S}^{n-1}), \quad (1.3)$$

where $F(\alpha, \mathbf{S}^{n-1})$ is the space of all integrable functions on $\mathbf{S}^{n-1}$ which satisfy (1.2).

The main purpose of this paper is studying the L^p boundedness of a class Marcinkiewicz integral operators on product domains. Namely, for suitable mappings $\varphi, \psi : \mathbf{R}^+ \rightarrow \mathbf{R}$ and $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ that satisfy the conditions

$$\Omega(tx, sy) = \Omega(x, y) \text{ for any } t, s > 0 \quad \text{and} \quad (1.4)$$

$$\int_{\mathbf{S}^{n-1}} \Omega(u, \cdot) d\sigma(u) = \int_{\mathbf{S}^{m-1}} \Omega(\cdot, v) d\sigma(v) = 0, \quad (1.5)$$

consider the operator $\mathcal{M}_{\Omega, \varphi, \psi}$ given by

$$\mathcal{M}_{\Omega, \varphi, \psi} f(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| F_{t,s}^{(\varphi, \psi)}(f)(x, y) \right|^2 2^{-2(t+s)} dt ds \right)^{\frac{1}{2}}, \quad (1.6)$$

where

$$F_{t,s}^{(\varphi, \psi)}(f)(x, y) = \iint_{\Lambda(t,s)} f(x - \varphi(|u|)u', y - \psi(|v|)v') |u|^{1-n} |v|^{1-m} \Omega(u, v) dudv \quad (1.7)$$

and $\Lambda(t, s) = \{(u, v) \in \mathbf{R}^n \times \mathbf{R}^m : |u| \leq 2^t \text{ and } |v| \leq 2^s\}$.

Clearly, when $\varphi(r) = \psi(r) = r$, the operator $\mathcal{M}_{\Omega, \varphi, \psi}$ is the classical Marcinkiewicz integral operator on product domains which we shall denote by $\mathcal{M}_{\Omega, c}$. In ([16]), Ding showed that $\mathcal{M}_{\Omega, c}$ is bounded on L^2 provided that $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Subsequently, the authors of ([11]) proved the L^p boundedness of $\mathcal{M}_{\Omega, c}$ for all $1 < p < \infty$ under the condition $\Omega \in L(\log^+ L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Recently, the L^p boundedness of $\mathcal{M}_{\Omega, c}$ was established under the weaker condition $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$; see Choi ([15]) for $p = 2$ and Al-Qassem, Al-Salman, Chang, and Pan ([3]) for all $1 < p < \infty$.

The primary focus of this paper is establishing L^p estimates of $\mathcal{M}_{\Omega, \varphi, \psi}$ for various functions φ and ψ , and for functions Ω satisfying conditions similar to

(1.2). Our conditions on the function Ω are stated as follows:

$$\sup_{(\xi', \eta') \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} \iint_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} |\Omega(u', v')| \{G(\xi', \eta')\}^{1+\alpha} d\sigma(u') d\sigma(v') < \infty, \quad (1.8)$$

where $G(\xi', \eta') = \log^+(|\xi' \cdot u'|^{-1}) + \log^+(|\eta' \cdot v'|^{-1}) + \log^+(|\xi' \cdot u'|^{-1}) \log^+(|\eta' \cdot v'|^{-1})$ for some $\alpha > 0$.

For $\alpha > 0$, we let $F(\alpha, \mathbf{S}^{n-1}, \mathbf{S}^{m-1})$ be the space of all integrable functions on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ which satisfy (1.8).

Clearly, $\bigcup_{q>1} L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \subset F(\alpha, \mathbf{S}^{n-1}, \mathbf{S}^{m-1})$ for any $\alpha > 0$ and the inclusion is proper. Moreover, by (1.3) it can be easily seen that

$$F(\alpha, \mathbf{S}^{n-1}, \mathbf{S}^{m-1}) \not\subset L(\log^+ L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \text{ and} \\ L(\log^+ L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \not\subset F(\alpha, \mathbf{S}^{n-1}, \mathbf{S}^{m-1}). \quad (1.9)$$

Our results are the following:

Theorem 1.1. *Suppose that $\varphi, \psi : \mathbf{R}^+ \rightarrow \mathbf{R}$ are $\mathcal{C}^2$ increasing convex functions with $\varphi(0) = \psi(0) = 0$. If $\Omega \in F(\alpha, \mathbf{S}^{n-1}, \mathbf{S}^{m-1})$ for some $\alpha > 0$ and satisfies (1.3)-(1.4), then $\mathcal{M}_{\Omega, \varphi, \psi}$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ for all $p \in ((2 + 2\alpha)/(1 + 2\alpha), 2 + 2\alpha)$.*

Theorem 1.2. *Suppose that $\varphi, \psi : \mathbf{R}^+ \rightarrow \mathbf{R}$ are $\mathcal{C}^2$ functions that satisfy*

$$|\phi_l(t)| \leq C_{1,l} t^{d_l}, \quad |\phi_l''(t)| \leq C_{2,l} t^{d_l-2}, \quad (1.10)$$

$$C_{3,l} t^{d_l-1} \leq |\phi_l'(t)| \leq C_{4,l} t^{d_l-1} \quad (1.11)$$

for some $d_l \neq 0$ and $t \in (0, \infty)$, where $\phi_1 = \varphi$, $\phi_2 = \psi$, $C_{1,l}$, $C_{2,l}$, $C_{3,l}$, and $C_{4,l}$ are positive constants independent of t . If $\Omega \in F(\alpha, \mathbf{S}^{n-1}, \mathbf{S}^{m-1})$ for some $\alpha > 0$ and satisfies (1.3)-(1.4), then $\mathcal{M}_{\Omega, \varphi, \psi}$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ for all $p \in ((2 + 2\alpha)/(1 + 2\alpha), 2 + 2\alpha)$.

2 Preparation

We start this section by establishing certain Fourier transform estimates. For suitable mappings $\varphi, \psi : \mathbf{R}^+ \rightarrow \mathbf{R}$, $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, and positive real numbers r_1 and r_2 , let $K_{\Omega, r_1, r_2} : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}$ be given by

$$K_{\Omega, r_1, r_2}(\xi, \eta) = \iint_{\Gamma(r_1, r_2)} e^{-i\{\varphi(|x|)\xi \cdot x' + \psi(|y|)\eta \cdot y'\}} |x|^{1-n} |y|^{1-m} \Omega(x, y) dx dy, \quad (2.1)$$

where $\Gamma(r_1, r_2) = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m : \frac{r_1}{2} \leq |x| < r_1, \frac{r_2}{2} \leq |y| < r_2\}$.

We have the following:

Lemma 2.1. Suppose that $\Omega \in F(\alpha, \mathbf{S}^{n-1}, \mathbf{S}^{m-1})$ for some $\alpha > 0$ and satisfying (1.3)-(1.4).

(a) If $\varphi, \psi : \mathbf{R}^+ \rightarrow \mathbf{R}$ are $\mathcal{C}^2$ increasing convex functions with $\varphi(0) = \psi(0) = 0$, then

$$|K_{\Omega, r_1, r_2}(\xi, \eta)| \leq r_1 r_2 C |\varphi(r_1)\xi| |\psi(r_2)\eta|; \quad (2.2)$$

$$|K_{\Omega, r_1, r_2}(\xi, \eta)| \leq r_1 r_2 C (\log \left| \varphi\left(\frac{r_1}{2}\right)\xi \right|)^{-1-\alpha} (\log \left| \psi\left(\frac{r_2}{2}\right)\eta \right|)^{-1-\alpha}; \quad (2.3)$$

$$|K_{\Omega, r_1, r_2}(\xi, \eta)| \leq r_1 r_2 C |\varphi(r_1)\xi| (\log \left| \psi\left(\frac{r_2}{2}\right)\eta \right|)^{-1-\alpha}; \quad (2.4)$$

$$|K_{\Omega, r_1, r_2}(\xi, \eta)| \leq r_1 r_2 C (\log \left| \varphi\left(\frac{r_1}{2}\right)\xi \right|)^{-1-\alpha} |\psi(r_2)\eta|. \quad (2.5)$$

(b) If $\varphi, \psi : \mathbf{R}^+ \rightarrow \mathbf{R}$ are $\mathcal{C}^2$ functions that satisfy (1.10)-(1.11), then K_{Ω, r_1, r_2} satisfies (2.2)-(2.5) with φ and ψ in the right hand sides of (2.1)-(2.4) are replaced by the power functions t^{d_1} and t^{d_2} respectively where d_1 and d_2 are as in the statement of Theorem 1.2.

Proof. We shall start by the proof of (a). Let $I_{\Omega, r_1} : \mathbf{R}^n \times \mathbf{S}^{n-1} \rightarrow \mathbf{R}$ and $J_{\Omega, r_2} : \mathbf{R}^m \times \mathbf{S}^{m-1} \rightarrow \mathbf{R}$ be given by

$$I_{\Omega, \varphi, r_1}(\xi, x') = \int_1^2 e^{-i(\xi \cdot x')\varphi(\frac{r_1 r}{2})} dr \text{ and } J_{\Omega, \psi, r_2}(\eta, y') = \int_1^2 e^{-i(\eta \cdot y')\psi(\frac{r_2 r}{2})} dr. \quad (2.6)$$

Then since φ and ψ are increasing functions, we immediately get

$$|I_{\Omega, \varphi, r_1}(\xi, x') - 1| \leq |\varphi(r_1)\xi| \quad (2.7)$$

$$|J_{\Omega, \psi, r_2}(\eta, y') - 1| \leq |\psi(r_2)\eta|. \quad (2.8)$$

On the other hand since φ and ψ are convex increasing functions, by integration by parts we have

$$|I_{\Omega, \varphi, r_1}(\xi, x')| \leq 2 \left| \varphi\left(\frac{r_1}{2}\right)\xi \cdot x' \right|^{-1} \quad (2.9)$$

$$|J_{\Omega, \psi, r_2}(\eta, y')| \leq 2 \left| \psi\left(\frac{r_2}{2}\right)\eta \cdot y' \right|^{-1}. \quad (2.10)$$

By combining the estimates (2.9), (2.10), and the trivial estimates $|I_{\Omega, \varphi, r_1}(\xi, x')| \leq 1$ and $|J_{\Omega, \psi, r_2}(\eta, y')| \leq 1$, we get

$$|I_{\Omega, \varphi, r_1}(\xi, x')| \leq C (\log \left| \varphi\left(\frac{r_1}{2}\right)\xi \right|)^{-1-\alpha} \{\log |\xi' \cdot x'|^{-1}\}^{1+\alpha} \quad (2.11)$$

$$|J_{\Omega, \psi, r_2}(\eta, y')| \leq C (\log \left| \psi\left(\frac{r_2}{2}\right)\eta \right|)^{-1-\alpha} \{\log |\eta' \cdot y'|^{-1}\}^{1+\alpha}. \quad (2.12)$$

Now, we are ready to prove (2.2)-(2.5). By the cancelation property of Ω and the estimates (2.7) and (2.8), it is straightforward to see that (2.2) holds. To see that (2.3), notice that

$$|K_{\Omega, r_1, r_2}(\xi, \eta)|$$

$$\leq r_1 r_2 C \iint_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} |\Omega(x', y')| |I_{\Omega, \varphi, r_1}(\xi, x')| |J_{\Omega, \psi, r_2}(\eta, y')| d\sigma(x') d\sigma(y'); \quad (2.13)$$

when combined with the estimates (2.11)-(2.12) and the fact that $\Omega \in F(\alpha, \mathbf{S}^{n-1}, \mathbf{S}^{m-1})$ imply (2.3).

To see (2.4), notice that by the cancelation property of Ω in the y' -variable, we have

$$\begin{aligned} & |K_{\Omega, r_1, r_2}(\xi, \eta)| \\ & \leq r_1 r_2 C \iint_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} |\Omega(x', y')| |I_{\Omega, \varphi, r_1}(\xi, x')| |J_{\Omega, \psi, r_2}(\eta, y') - 1| d\sigma(x') d\sigma(y'). \end{aligned} \quad (2.14)$$

Therefore, (2.4) follows by (2.14), (2.12), (2.7), and the fact that $\Omega \in F(\alpha, \mathbf{S}^{n-1}, \mathbf{S}^{m-1})$. Finally, (2.5) follows by symmetry. This completes the proof of part (a).

To proof part (b), we claim that

$$|I_{\Omega, \varphi, r_1}(\xi, x') - 1| \leq C |(r_1)^{d_1} \xi| \quad (2.15)$$

$$|J_{\Omega, \psi, r_2}(\eta, y') - 1| \leq C |(r_2)^{d_2} \eta|. \quad (2.16)$$

$$|I_{\Omega, \varphi, r_1}(\xi, x')| \leq C |(r_1)^{d_1} \xi \cdot x'|^{-1} \quad (2.17)$$

$$|J_{\Omega, \psi, r_2}(\eta, y')| \leq C |(r_2)^{d_2} \eta \cdot y'|^{-1}. \quad (2.18)$$

Clearly, the estimates (2.15)-(2.16) hold. The estimates (2.17)-(2.18) follow from the properties of the functions φ and ψ and an integration by parts (see). Hence, the proof of part (b) is complete by (2.15)-(2.18) and repeating exactly the same argument in the proof of part (a). This completes the proof of part (b) and hence the lemma.

Now, we state the following lemma on maximal functions which can be prove by a well known bootstrapping argument ([13], [9]):

Lemma 2.2. *For $\xi \in \mathbf{R}^d (d \geq 1)$ and a suitable function $\phi : \mathbf{R}^+ \rightarrow \mathbf{R}$, let $M_{\xi, \phi}$ be the maximal function defined on $\mathbf{R}^d$ by*

$$M_{\phi, \xi}(f)(x) = \sup_{t \in \mathbf{R}} \left| 2^{-t} \int_0^{2^t} f(x - \varphi(r)\xi) dr \right|. \quad (2.19)$$

If ϕ is $\mathcal{C}^2$ increasing convex function with $\phi(0) = 0$ or ϕ is $\mathcal{C}^2$ function satisfying the conditions (1.10)-(1.11), then

$$\|M_{\phi, \xi}(f)\|_p \leq C \|f\|_p$$

for all $1 < p < \infty$ where C independent of $\xi \in \mathbf{R}^d$.

Now, we have the following:

Lemma 2.3. Suppose that $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and $\varphi, \psi : \mathbf{R}^+ \rightarrow \mathbf{R}$ are suitable functions. Let $M_{\Omega, \varphi, \psi}$ be the maximal function defined on $\mathbf{R}^n \times \mathbf{R}^m$ by

$$M_{\Omega, \varphi, \psi} f(x, y) = \sup_{t \in \mathbf{R}} \frac{1}{2^{t+s}} \left| \iint_{\Lambda(t, s)} f(x - \varphi(|u|)u', y - \psi(|v|)v') |u|^{1-n} |v|^{1-m} \Omega(u, v) du dv \right|.$$

If φ and ψ are $\mathcal{C}^2$ increasing convex functions with $\varphi(0) = \psi(0) = 0$ or φ and ψ are $\mathcal{C}^2$ functions satisfying the conditions (1.10)-(1.11), then

$$\|M_{\Omega, \varphi, \psi} f\|_p \leq C \|f\|_p$$

for all $1 < p < \infty$.

Proof. Notice that by Hölder's inequality, we have

$$\|M_{\Omega, \varphi, \psi} f\|_p \leq \|\Omega\|_1 \iint_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} \|M_{\varphi, u'}^1 \circ M_{\psi, v'}^2 f\|_p d\sigma(u') d\sigma(v') \quad (2.20)$$

where $M_{\varphi, u'}^1 f(x, y) = M_{\varphi, u'} f(\cdot, y)(x)$, $M_{\psi, v'}^2 f(x, y) = M_{\psi, v'} f(x, \cdot)(y)$, and $\circ$ denotes the composition of operators. Hence the result follows by (2.20) and Lemma 2.2. This completes the proof.

3 Proof of Main Results

Proof of Theorem 1.1. For $t, s \in \mathbf{R}$, let $\sigma_{t, s}$ be the measure defined on the Fourier transform side by

$$\hat{\sigma}_{t, s}(\xi, \eta) = 2^{-(t+s)} K_{\Omega, 2^t, 2^s}(\xi, \eta), \quad (3.1)$$

where $K_{\Omega, 2^t, 2^s}$ is given by (2.1). Then it follows that

$$\mathcal{M}_{\Omega, \varphi, \psi}(f)(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\sigma_{t, s} * f(x, y)|^2 dt ds \right)^{\frac{1}{2}}. \quad (3.2)$$

By an elementary procedure, choose two collections of $\mathcal{C}^\infty$ functions $\{\omega_k^{(1)}\}_{k \in \mathbf{Z}}$ and $\{\omega_k^{(2)}\}_{k \in \mathbf{Z}}$ on $(0, \infty)$ with the following properties:

$$\text{supp}(\omega_k^{(1)}) \subseteq \left[\frac{1}{\varphi(2^{k+1})}, \frac{1}{\varphi(2^{k-1})} \right] \text{ and } \text{supp}(\omega_k^{(2)}) \subseteq \left[\frac{1}{\psi(2^{k+1})}, \frac{1}{\psi(2^{k-1})} \right]; \quad (3.3)$$

$$0 \leq \omega_k^{(1)}, \omega_k^{(2)} \leq 1; \quad (3.4)$$

$$\sum_{k \in \mathbf{Z}} \omega_k^{(1)}(u) = \sum_{k \in \mathbf{Z}} \omega_k^{(2)}(u) = 1; \quad (3.5)$$

$$\left| \frac{d^l \omega_k^{(1)}}{du^l}(u) \right|, \left| \frac{d^l \omega_k^{(2)}}{du^l}(u) \right| \leq C_l u^{-l}. \quad (3.6)$$

Define the functions $\{v_k^{(1)} : k \in \mathbf{Z}\}$ on $\mathbf{R}^n$ and $\{v_k^{(2)} : k \in \mathbf{Z}\}$ on $\mathbf{R}^m$ by

$$(v_k^{(1)})^\wedge(x) = \omega_k^{(1)}(|x|^2) \text{ and } (v_k^{(2)})^\wedge(y) = \omega_k^{(2)}(|y|^2).$$

Therefore, by (3.2) and (3.5), we get

$$\mathcal{M}_{\Omega, \varphi, \psi}(f)(x, y) \leq \sum_{k \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} I_{k,j}(f)(x, y), \quad (3.7)$$

where

$$I_{k,j}(f)(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| (v_{[t]+k}^{(1)} \otimes (v_{[s]+j}^{(2)})) * \sigma_{t,s} * f(x, y) \right|^2 dt ds \right)^{\frac{1}{2}} \quad (3.8)$$

Here, $[x]$ is the greatest integer function less than or equal to x . For $k, j \in \mathbf{Z}$, let $S_{k,j}$ be the operator given by

$$S_{k,j}f(x, y) = \left(\int_{-\infty}^{\infty} \left| (v_{[t]+k}^{(1)} \otimes (v_{[s]+j}^{(2)})) * f(x, y) \right|^2 dt ds \right)^{\frac{1}{2}}. \quad (3.9)$$

Then, by an argument of Fefferman and Stein ([18]), it can be easily shown that

$$\|S_{k,j}(f)\|_p \leq C \|f\|_p \quad (3.10)$$

for all $p \in (1, \infty)$ with constant C depends only on p and the dimension of the underlying space $\mathbf{R}^n \times \mathbf{R}^m$.

By (3.10), Lemma 2.3, the fact that $\|\sigma_{t,s}\| \leq 1$, and the proof of Lemma 1 in ([13]) with minor modification, we obtain the crude estimate

$$\|I_{k,j}(f)\|_p \leq C \|f\|_p, \quad (3.11)$$

for all $p \in (1, \infty)$.

Now, we estimate $\|I_{k,j}(f)\|_2$. For $k, j \in \mathbf{Z}$, nonzero $\xi \in \mathbf{R}^n$, and nonzero $\eta \in \mathbf{R}^m$, let $E_k^1(\xi)$ and $E_j^2(\eta)$ be the intervals in $\mathbf{R}$ given by

$$\begin{aligned} E_k^1(\xi) &= [\log_2(2^{k-1}\varphi^{-1}(|\xi|^{-1})), \log_2(2^{k+2}\varphi^{-1}(|\xi|^{-1}))]; \\ E_j^2(\eta) &= [\log_2(2^{j-1}\psi^{-1}(|\eta|^{-1})), \log_2(2^{j+2}\psi^{-1}(|\eta|^{-1}))]. \end{aligned}$$

Clearly, we have the following

$$|E_k^1(\xi)| = |E_j^2(\eta)| = 3; \quad (3.12)$$

$$\varphi(2^{-k-1}\varphi^{-1}(|\xi|^{-1})) \leq \varphi(2^t) \leq \varphi(2^{-k+2}\varphi^{-1}(|\xi|^{-1})); \quad (3.13)$$

$$\psi(2^{-j-1}\psi^{-1}(|\eta|^{-1})) \leq \psi(2^s) \leq \psi(2^{-j+2}\psi^{-1}(|\eta|^{-1})), \quad (3.14)$$

for $k, j \in \mathbf{Z}$, nonzero $\xi \in \mathbf{R}^n$, nonzero $\eta \in \mathbf{R}^m$, and $(t, s) \in E_k^1(\xi) \times E_j^2(\eta)$.

Therefore, Since the functions φ and ψ are convex increasing, the inequalities (3.13)-(3.14) imply that

$$\varphi(2^t) \leq 2^{-k+2} |\xi|^{-1} \text{ for } k \geq 3 \text{ and } t \in E_k^1(\xi); \quad (3.15)$$

$$\varphi(2^{t-1}) \geq 2^{-k+2} |\xi|^{-1} \text{ for } k \leq -2 \text{ and } t \in E_k^1(\xi); \quad (3.16)$$

$$\psi(2^s) \leq 2^{-j+2} |\eta|^{-1} \text{ for } j \geq 3 \text{ and } s \in E_j^2(\eta); \quad (3.17)$$

$$\psi(2^{s-1}) \geq 2^{-j+2} |\eta|^{-1} \text{ for } j \leq -2 \text{ and } s \in E_j^2(\eta). \quad (3.18)$$

Thus, by (2.2)-(2.5), (3.15)-(3.18), the trivial estimate $\|\sigma_{t,s}\| \leq 1$ if necessary, and Plancherel's theorem, we obtain

$$\|I_{k,j}(f)\|_2 \leq C_{k,j} \|f\|_2, \quad (3.19)$$

where

$$C_{k,j} = \begin{cases} 2^{-k-j}, & \text{if } k, j \geq 3 \\ 2^{-k} |j|^{-(1+\alpha)}, & \text{if } k \geq 3 \text{ and } j \leq -2 \\ 2^{-j} |k|^{-(1+\alpha)}, & \text{if } j \geq 3 \text{ and } k \leq -2 \\ |kj|^{-(1+\alpha)}, & \text{if } k, j \leq -2 \\ 1, & \text{if } -2 \leq k, j \leq 3. \end{cases}$$

Hence, the result follows by (3.7) and an interpolation between (3.19) and (3.11) with any p .

Proof of Theorem 1.2. It can be easily seen that the proof of Theorem 1.2 follows by repeating exactly the same argument in the proof of Theorem 1.1 with minor modifications. We omit the details.

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