International Mathematical Forum, 2, 2007, no. 23, 1105 - 1110

Derivation which Acts as a Homomorphism or as an Anti-homomorphism in a Prime Ring

Asma Ali and Deepak Kumar

Department of Mathematics
Aligarh Muslim University
Aligarh - 202002, India
asma_ali2@rediffmail.com

Abstract

Let R be a prime ring and S a non-empty subset of R. Suppose that θ, ϕ are endomorphisms of R. An additive mapping $F : R \longrightarrow R$ is called a generalized (θ, ϕ)-derivation on S if there exists a (θ, ϕ)-derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$, holds for all $x, y \in S$. Suppose that U is a Lie ideal of R such that $u^2 \in U$, for all $u \in U$. The main result of the present paper states that if F is a generalized (θ, θ)-derivation on U which also acts as a homomorphism or as an anti-homomorphism on U, then either $d = 0$ or $U \subseteq Z(R)$.

Mathematics Subject Classification: 16W25,16N60, 16U80

Keywords: Prime rings, Lie ideals, derivations, generalized derivations

1. INTRODUCTION AND PRELIMINARIES

Throughout the paper R will denote an associative ring with centre $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$. A ring R is said to be a 2-torsion free if whenever $2x = 0$ with $x \in R$ implies that $x = 0$. A ring R is called a prime ring if for any $x, y \in R$, $xRy = \{0\}$ implies that either $x = 0$ or $y = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$ and $r \in R$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \longrightarrow R$ given by

$I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation.

An additive mapping $F_{a,b} : R \longrightarrow R$ is called a generalized inner derivation if $F_{a,b}(x) = ax + xb$ for some fixed $a, b \in R$. It is straight forward to note that if $F_{a,b}(x)$ is a generalized inner derivation, then for any $x, y \in R$

$$F_{a,b}(xy) = F_{a,b}(x)y + x[y, b] = F_{a,b}(x)y + xI_b(y)$$

where I_b is an inner dervation. In view of the above observation , the concept of generalized derivation is introduced as follows : An additive mapping $F : R \longrightarrow R$ is called a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Generally we do not mention the derivation d associated with a generalized derivation F rather prefer to call F simply a generalized dervation. One may observe that the concept of generalized derivation includes the concept of derivations and generalized inner derivations, also of the left multiplier when $d = 0$. Hence it should be interesting to extend some results concerning to these notions to generalized derivations. Recently some authors have also studied generalized derivation in theory of operator algebras and C^*-algebra (see for example [5]).

Inspired by the definition of (θ, ϕ)-derivation the notion of generalized derivation was extended as follows : Let θ, ϕ be endomorphisms of R and let S be a nonempty subset of R. An additive mapping $F : R \longrightarrow R$ is called a generalized (θ, ϕ)-derivation on S if there exists a (θ, ϕ)-derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)\theta(y) + \phi(x)d(y)$, holds for all $x, y \in S$.

2. MAIN RESULT

Bell and Kappe [2] proved that if R is a semiprime ring and d is a derivation of R which is either an endomorphism or an anti-endomorphism, then $d = 0$. of course derivations which are not endomorphisms or anti-endomorphisms on R may behave as such on certain subsets of R, for example, any derivation d behaves as the zero endomorphism on the subring C consisting of all constants (i.e. elements x for which $d(x) = 0$). In fact, in a semiprime ring R, d may behave as an endomorphism on a proper ideal of R. As an example of such R and d, let S be any semiprime ring with a nonzero derivation δ, take $R = S \oplus S$ and define d by $d(r_1, r_2) = (\delta(r_1), 0)$. However, Bell and Kappe in the mentioned paper remarked that the behavior of d is some what restricted in case of prime rings and showed that if R is a prime ring and d is a derivation on R which acts as a homomorphism or an anti-homomorphism on a nonzero right ideal U of R, then $d = 0$ on R.

Further, Yenigul and Argac [6] obtained the above result for α-derivations in prime rings. Recently Ashraf et al.[1] extended the result for (σ, τ)-derivations in prime and semiprime rings.

In the present paper we attempt to establish the above mentioned result for generalized (θ, ϕ)-derivations in prime rings.

We begin with the following :

Lemma 2.1. Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R. Let θ, ϕ be automorphisms of R. If R admits a (θ, ϕ)-derivation d such that $d(U) = 0$, then $d = 0$ or $U \subseteq Z(R)$.

Proof. We have $d(u) = 0$, for all $u \in U$. This yields that $d([u, r]) = 0$, for all $u \in U$ and $r \in R$. Now using the fact that $d(u) = 0$, the above expression yields that

$$\phi(u)d(r) - d(r)\theta(u) = 0, \quad \text{for all } u \in U \text{ and } r \in R. \tag{2.1}$$

Now for any $s \in R$, replace r by rs in (2.1) and use (2.1), to get

$$d(r)[\theta(s), \theta(u)] - [\phi(r), \phi(u)]d(s) = 0, \quad \text{for all } u \in U \text{ and } r, s \in R. \tag{2.2}$$

Again replacing s by sv in (2.2), our hypotheses yield that $d(r)\theta(s)[\theta(v), \theta(u)] = 0$, for all $u, v \in U$ and $r, s \in R$. Hence $\theta^{-1}(d(r))s[v, u] = 0$, for all $u, v \in U$ and $r, s \in R$. This implies that $\theta^{-1}(d(r))R[v, u] = \{0\}$, for all $u, v \in U$ and $r \in R$. Thus the primeness of R implies that either $[v, u] = 0$ or $d(r) = 0$. If $[v, u] = 0$, for all $u, v \in U$, then it follows that $[u, [u, ru]] = 0$ for all $u \in U, r \in R$. Since char$R \neq 2$, the above relation yields that $[u, r][u, s] = 0$ for all $r, s \in R$. Thus $[u, r]s[u, s] = [u, r][u, rs] = 0$ and hence $[u, s] = 0$ for all $u \in U, r \in R$ i.e., $U \subseteq Z(R)$.

We are now well-equipped to prove our main theorem.

Theorem 2.1. Let R be a 2-torsion-free prime ring and U be a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. Suppose θ is an automorphism of R and $F : R \longrightarrow R$ is a generalized (θ, θ)-derivation associated with a derivation d.

(*i*) If F acts as a homomorphism on U, then either $d = 0$ on R or $U \subseteq Z(R)$.

(*ii*) If F acts as an anti-homomorphism on U, then either $d = 0$ on R or $U \subseteq Z(R)$.

Proof of Theorem. Suppose that $U \not\subseteq Z(R)$.

(i) If F acts as a homomorphism on U, then we have

$$F(uv) = F(u)\theta(v) + \theta(u)d(v) = F(u)F(v), \text{for all } u, v \in U. \tag{2.3}$$

Replacing v by $2vw$ in (2.3) and using the fact that char$R \neq 2$, we get

$$F(u)\theta(v)\theta(w) + \theta(u)(d(v)\theta(w) + \theta(v)d(w)) = F(u)(F(v)\theta(w) + \theta(v)d(w)).$$

Using (2.3), the above relation yields that $(F(u) - \theta(u))\theta(v)d(w) = 0$, for all $u, v, w \in U$ i.e., $\theta^{-1}(F(u) - \theta(u))v\theta^{-1}(d(w)) = 0$, for all $u,\ v,\ w \in U$ and hence $\theta^{-1}(F(u) - \theta(u))U\theta^{-1}(d(w)) = \{0\}$, for all $u, w \in U$. Hence by Lemma 4 of [3] either $F(u) - \theta(u) = 0$ or $d(w) = 0$. If $F(u) - \theta(u) = 0$, for all $u \in U$, then the relation (2.3) implies that $\theta(u)d(v) = 0$, for all $u, v \in U$. Now replace u by $2uw$, to get $\theta(u)\theta(w)d(v) = 0$, for all $u, v, w \in U$. This implies that $uw\theta^{-1}(d(v)) = 0$ and hence $uU\theta^{-1}(d(v)) = \{0\}$, for all $u, v \in U$. Thus by Lemma 4 of [3], we get either $u = 0$ or $d(v) = 0$. But since U is nonzero, we find that $d(v) = 0$, for all $v \in U$ and hence by Lemma 2.1, we get the required result.

(ii) If F acts as an anti-homomorphism on U, then we have

$$F(uv) = F(u)\theta(v) + \theta(u)d(v) = F(v)F(u), \text{for all } u, v \in U. \tag{2.4}$$

Replacing u by $2uv$ in (2.4) and using the fact that *char*$R \neq 2$, we get

$$\theta(u)\theta(v)d(v) = F(v)\theta(u)d(v), \text{for all } u, v \in U. \tag{2.5}$$

Again replace u by $2wu$ in (2.5), to obtain

$$\theta(w)\theta(u)\theta(v)d(v) = F(v)\theta(w)\theta(u)d(v), \text{for all } u, v \in U. \tag{2.6}$$

In view of (2.5), the relation (2.6) yields that $[F(v), \theta(w)]\theta(u)d(v) = 0$, for all $u, v, w \in U$. This implies that $\theta^{-1}([F(v), \theta(w)])u\theta^{-1}(d(v)) = 0$, for all $u, v, w \in U$. Thus using Lemma 4 of [3], either $d(v) = 0$ or $[F(v), \theta(w)] = 0$. If $[F(v), \theta(w)] = 0$, for all $u, v \in U$, then replacing v by $2vw$ in the above relation, we get

$$\theta(v)[d(w),\theta(w)]+[\theta(v),\theta(w)]d(w)=0, \quad \text{for all } v,w\in U. \tag{2.7}$$

Now replace v by $2v_1v$ in (2.7) to get $[\theta(v_1),\theta(w)]\theta(v)d(w)=0$, for all $v,v_1,w\in U$. This gives that $[v_1,w]v\theta^{-1}(d(w))=0$, for all $v,v_1,w\in U$. Again by Lemma 4 of [3], for each fixed $w\in U$, either $[v_1,w]=0$ or $d(w)=0$. Hence by using Braur's trick, we find that either $[v_1,w]=0$, for all $v_1,w\in U$ or $d(w)=0$, for all $w\in U$. If $[v_1,w]=0$, for all $v_1,w\in U$, then U is central, a contradiction. On the other hand if $d(w)=0$, then by Lemma 2.1, we get the required result.

The immediate consequence of the above Theorem is the following:

Corollary 2.1. Let R be a prime ring and I be a nonzero ideal of R. Suppose θ is an automorphism of R and $F:R\longrightarrow R$ is a generalized (θ,θ)-derivation with associated derivation d.

(i) If F acts as a homomorphism on I, then $d=0$ on R or $U\subseteq Z(R)$.

(ii) If F acts as an anti-homomorphism on I, then $d=0$ on R or $U\subseteq Z(R)$.

Remark 2.1. Since every ideal in a ring R is a Lie ideal of R, conclusion of the above theorem holds even if U is assumed to be an ideal of R. Though the assumption that $u^2\in U$, for all $u\in U$ seems close to assuming that U is an ideal of the ring, but there exist Lie ideals with this property which are not ideals. For example, let $R=\left\{\begin{pmatrix} x & y\\ 0 & z\end{pmatrix} \mid x,y,z\in Z\right\}$. Then it can be easily seen that $U=\left\{\begin{pmatrix} x & y\\ 0 & x\end{pmatrix} \mid x,y\in Z\right\}$ is a Lie ideal of R satisfying $u^2\in U$, for all $u\in U$. However, U is not an ideal of R.

References

[1] M. Ashraf, R. Nadeem and M. A. Quadri, *On (σ,τ)-derivations in certain classes of rings*, Rad. Mat. 9 (2) (1999), 187-192.

[2] H. E. Bell, and L. C. Kappe, *Rings in which derivations satisfy certain algebraic conditions*, Acta. Math. Hungar. 53 (1989), 339-346.

[3] J. Bergen, I. N. Herstein and J. M. Kerr, *Lie ideals and derivations of prime ring*, J. Algebra 71 (1981), 254-267.

[4] I. N. Herstein, Topics in ring theory, *Univ. Chicago Press*, Chicago (1969).

[5] B. Hvala, *Generalized derivations in rings*, Comm. Algebra 26 (4) (1998), 1147-1166.

[6] Yenigul and Argac, *On prime and semiprime rings with α-derivations*, Turkish J. Math. 18(1994), 280-284.

Received: February 23, 2006