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The Maps Preserving Approximation

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Abstract

The purpose of this paper is to introduce and discuss the concept of the maps which preserve approximation. We show that if a operator on a normed space is an isometry then preserves all approximation property, every linear operator preserving approximation is an isometry multiplied by a constant.

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1. Introduction

Suppose X is a normed linear space and $x, y \in X$, x is said to be orthogonal to y and is denoted by $x \perp y$ if and only if $\|x\| \leq \|x + \alpha y\|$ for all scalar α. If G_1 and G_2 are subsets of X, it is defined $G_1 \perp G_2$ if and only if for all $g_1 \in G_1, g_2 \in G_2$, $g_1 \perp g_2$. (see [1]). Let G be a subspace of X, it is defined the set of the metric complemented

$$\hat{G} = \{x \in X : \quad x \perp G\} = \{x \in X : \ \|x\| = \|x + G\|,$$

and the set of the cometric complement

$$\breve{G} = \{x \in X : \quad G \perp x\}.$$

We know that a point $g_0 \in G$ is said to be a best approximation (resp. best coapproximation) for $x \in X$ if and only if $\|x - g_0\| = \|x + G\| = dist(x, G)$ $(resp.\ \|g_0 - g\| \leq \|x - g\| \ \forall \ g \in G)$. It can be easily proved that g_0 is a best approximation (resp. best coapproximation) for $x \in X$ if and only if $x - g_0 \in \hat{G}$ $(resp. x - g_0 \in \breve{G})$. The set of all best approximations (resp.

best coapproximations) of $x \in X$ in G is shown by $P_G(x)$ $(resp. R_G(x))$. In other words,

$$P_G(x) = \{g_0 \in G \;:\; x - g_0 \in \hat{G}\}$$

and

$$R_G(x) = \{g_0 \in G \;:\; x - g_0 \in \breve{G}\}.$$

If $P_G(x)$ $(resp.\ R_G(x))$ is non-empty for every $x \in X$, then G is called an Proximinal (resp. coproximinal) set. The set M is Chebyshev (resp. cochebyshev) if $P_G(x)(resp.\ R_G(x))$ is a singleton set for every $x \in X$. (see [5-13])

A proximinal subspace G is called quasi-Chebyshev if and only if $P_G(x)$ is compact, for all $x \in X$ (see [3-4]).

Suppose G is a subspace of X then G is called orthogonal complemented subspace, if either G is Chebyshev and $\hat{G}$ is a subspace of X or G is cochebyshev and $\breve{G}$ is a subspace of X.

2. Preserves approximation

In this section we shall obtain characterization of preserving approximation maps.

Lemma 2.1. *Let X be a normed linear space. If $T : X \to X$ is an isometry operator, then for all subspace G of X and all $x \in X$,*

$$T(P_G(x)) = P_{T(G)}(T(x)) \text{ and } T(R_G(x)) = R_{T(G)}(T(x)).$$

Proof. We have for all $x \in X$, $\|T(x)\| = \|x\|$, therefore for all $x \in X$ and all subspace G of X and $g, g_0 \in G$,

$$\|x - g_0\| \leq \|x - g\| \Leftrightarrow \|T(x) - T(g_0)\| \leq \|T(x) - T(g)\|. \blacksquare$$

Definition 2.2. *Suppose X, Y are two linear normed spaces. A map $T : X \to Y$ is called preserving approximation (resp. preserving coapproximation) if and only if for all subspace G of X and all $x \in X$,*

$$T(P_G(x)) = P_{T(G)}(T(x)) \ (resp.\ T(R_G(x)) = R_{T(G)}(T(x)).$$

Corollary 2.3. *Let X be a normed linear space. Every isometry operator $T : X \to X$ is preserving approximation (resp. preserving coapproximation).*

Theorem 2.4. *Suppose X, Y are two linear normed spaces and $T : X \to Y$ is a linear map which is preserves approximation (resp. preserves coapproximation).*

a)*Suppose G is a subspace of X, then G is proximinal (resp. coproximinal)of X, if and only if $T(G)$ is proximinal (resp. coproximinal) of Y.*

b) *Suppose G is a subspace of X, then G is Chebyshev (resp. cochebyshev) if and only if $T(G)$ is Chebyshev (resp. cochebyshev.*

c) *If T is linear, then $\forall x, y \in X \quad x \perp y \Longrightarrow T(x) \perp T(y)$. ($T$ is called preserving orthogonality).*

d) *For a subspace G of X, $T(\widehat{G}) = \widehat{T(G)}$ (resp. $T(\breve{G}) = T(\breve{G})$).*

e) *Suppose G is a subspace of X, then G is orthogonality complemented in X if and only if $T(G)$ is orthogonality complemented in Y.*

f) *Suppose G is a subspace of X, if T is a cotinuous and onto preserves approximation map, then G is quasi chebyshev if and only if $T(G)$ is quasi Chebyshev.*

Proof. We can easily prove (a), (b), (d), (e). We prove the parts (c) and (f).

c) Suppose T is a preserves approximation, if $x, y \in X$ and $x \perp y$, then $0 \in P_{<y>}(x)$. Therefore $0 = T(0) \in P_{T(<y>)}(T(x))$, since T is linear, $T(< y >) =< T(y) >$. Hence $T(x) \perp T(y)$
Now suppose T is a preserves approximation, if $x, y \in X$ and $x \perp y$, then $0 \in R_{<x>}(y)$. Therefore $0 = T(0) \in R_{T(<x>)}(T(y))$, since T is linear, $T(< x >) =< T(x) >$. Hence $T(x) \perp T(y)$.

f) Let $z \in Y$ and $\{u_n\} \subseteq P_{T(G)}(z)$. Since T is onto, we have $z = T(x)$ for some $x \in X$, therefore $\{u_n\} \subseteq T(P_G(x))$. Then there exists a sequence $\{v_n\} \subseteq P_G(x)$ such that $u_n = T(v_n)$. Since G is quasi Chebyshev, the set $P_G(x)$ is compact. Hence there exists subsequence $\{v_{n_k}\}_{k \geq 1}$ of $\{v_n\}$ and $v_0 \in X$ such that $v_{n_k} \to v_0$. Sinve T is continuous, $u_{n_k} = T(v_{n_k}) \to u_0 = T(v_0)$. Therefore $P_{T(G)}(z))$ is compact. ■

Theorem 2.5. *Let X be a normed linear space and the operator $T : X \to X$ be preserving approximation, then $T = kU$ such that $k \in R$ and U is an isometry.*

Proof. We know that T is preserving approximation, from Theorem 2.4, T is preserving orthogonality. From [1], T is an isometry multiplied by a constant. ■

Example 2.6. There exists non-linear map which it is preserving orthogonality. We define $T : R^2 \longrightarrow R^2$ by $T(x, y) = (x, y)$ if $x, y \neq 0$, $T(x, y) = (1, 1)$ if $y = 0, x \neq 0$, $T(x, y) = (-1, 1)$ if $x = 0, y \neq 0$ and $T(0, 0) = (0, 0)$.

3. ϵ-Preserves approximation

In this section we shall obtain characterization of ϵ-Preserves approximation maps. Let X be a normed linear space $\epsilon > 0$ and $x, y \in X$. We call x is ϵ-orthogonal to y and is denoted by $x \perp_\epsilon y$ if and only if $\|x\| \leq \|x + \alpha y\| + \epsilon$ for all scalar α. If G_1 and G_2 are subsets of X, we define $G_1 \perp_\epsilon G_2$ if and only if for all $g_1 \in G_1$ and $g_2 \in G_2$ we have, $g_1 \perp_\epsilon g_2$.

For $\epsilon > 0$ we have,

$$\widehat{G_\epsilon} = \{x \in X : \ \ x \perp_\epsilon G\}.$$

For $\epsilon > 0$, a point $g_0 \in G$ is said to be a ϵ-approximation for $x \in X$ if $x - g_0 \in \widehat{G_\epsilon}$. The set of all ϵ-approximation for $x \in X$ be denoted by $P_{G,\epsilon}(x)$. Also we have

$$P_{G,\epsilon}(x) = \{g_0 \in G \ : \ \|x - g_0\| \leq \|x - g\| + \epsilon \ for \ all \ g \in G\}.$$

For all $\epsilon > 0$, it is clear that the set $P_{G,\epsilon}(x)$ is a nonempty set. for more information see [6] and [11].

Definition 3.1. *Suppose X, Y are two linear normed spaces and $\epsilon > 0$. A map $T : X \to Y$ is called ϵ-preserving approximation if and only if for all subspace G of X and all $x \in X$,*

$$T(P_{G,\epsilon}(x)) = P_{T(G),\epsilon}(T(x)).$$

Corollary 3.2. *Let X be a normed linear space and $\epsilon > 0$. Then every isometry operator $T : X \to X$ be ϵ-preserving approximation.*

Theorem 3.3. *Suppose X, Y are two linear normed spaces, $\epsilon > 0$ and $T : X \to Y$ is an onto which is ϵ-preserves approximation.*

c) *If T is linear, then $\forall x, y \in X \ \ x \perp_\epsilon y \Longrightarrow T(x) \perp_\epsilon T(y)$.*

d) *For a subspace G of X, $T(\widehat{G_\epsilon}) = \widehat{T(G)}_\epsilon$.*

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