The Maps Preserving Approximation

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Abstract

The purpose of this paper is to introduce and discuss the concept of the maps which preserve approximation. We show that if an operator on a normed space is an isometry then preserves all approximation property, every linear operator preserving approximation is an isometry multiplied by a constant.

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1. Introduction

Suppose $X$ is a normed linear space and $x, y \in X$, $x$ is said to be orthogonal to $y$ and is denoted by $x \perp y$ if and only if $\|x\| \leq \|x + \alpha y\|$ for all scalar $\alpha$. If $G_1$ and $G_2$ are subsets of $X$, it is defined $G_1 \perp G_2$ if and only if for all $g_1 \in G_1, g_2 \in G_2, g_1 \perp g_2$. (see [1]). Let $G$ be a subspace of $X$, it is defined the set of the metric complemented

$\hat{G} = \{ x \in X : x \perp G \} = \{ x \in X : \|x\| = \|x + G\|$, 

and the set of the cometric complement

$\tilde{G} = \{ x \in X : G \perp x \}$. 

We know that a point $g_0 \in G$ is said to be a best approximation (resp. best coapproximation) for $x \in X$ if and only if $\|x - g_0\| = \|x + G\| = \text{dist}(x, G)$ (resp. $\|g_0 - g\| \leq \|x - g\| \ \forall \ g \in G$). It can be easily proved that $g_0$ is a best approximation (resp. best coapproximation) for $x \in X$ if and only if $x - g_0 \in \hat{G}$ (resp. $x - g_0 \in \tilde{G}$). The set of all best approximations (resp.
best coapproximations) of \( x \in X \) in \( G \) is shown by \( P_G(x) \) (resp. \( R_G(x) \)). In other words,

\[
P_G(x) = \{ g_0 \in G : x - g_0 \in \hat{G} \}
\]

and

\[
R_G(x) = \{ g_0 \in G : x - g_0 \in \tilde{G} \}.
\]

If \( P_G(x) \) (resp. \( R_G(x) \)) is non-empty for every \( x \in X \), then \( G \) is called an Proximinal (resp. coproximinal) set. The set \( M \) is Chebyshev (resp. cochebyshev) if \( P_G(x) \) (resp. \( R_G(x) \)) is a singleton set for every \( x \in X \). (see [5-13])

A proximal subspace \( G \) is called quasi-Chebyshev if and only if \( P_G(x) \) is compact, for all \( x \in X \) (see [3-4]).

Suppose \( G \) is a subspace of \( X \) then \( G \) is called orthogonal complemented subspace, if either \( G \) is Chebyshev and \( \hat{G} \) is a subspace of \( X \) or \( G \) is cochebyshev and \( \tilde{G} \) is a subspace of \( X \).

2. Preserves approximation

In this section we shall obtain characterization of preserving approximation maps.

**Lemma 2.1.** Let \( X \) be a normed linear space. If \( T : X \rightarrow X \) is an isometry operator, then for all subspace \( G \) of \( X \) and all \( x \in X \),

\[
T(P_G(x)) = P_{T(G)}(T(x)) \text{ and } T(R_G(x)) = R_{T(G)}(T(x)).
\]

**Proof.** We have for all \( x \in X \), \( \|T(x)\| = \|x\| \), therefore for all \( x \in X \) and all subspace \( G \) of \( X \) and \( g, g_0 \in G \),

\[
\|x - g_0\| \leq \|x - g\| \Leftrightarrow \|T(x) - T(g_0)\| \leq \|T(x) - T(g)\|. \quad \blacksquare
\]

**Definition 2.2.** Suppose \( X, Y \) are two linear normed spaces. A map \( T : X \rightarrow Y \) is called preserving approximation (resp. preserving coapproximation) if and only if for all subspace \( G \) of \( X \) and all \( x \in X \),

\[
T(P_G(x)) = P_{T(G)}(T(x)) \text{ (resp. } T(R_G(x)) = R_{T(G)}(T(x)).
\]

**Corollary 2.3.** Let \( X \) be a normed linear space. Every isometry operator \( T : X \rightarrow X \) is preserving approximation (resp. preserving coapproximation).

**Theorem 2.4.** Suppose \( X, Y \) are two linear normed spaces and \( T : X \rightarrow Y \) is a linear map which is preserves approximation (resp. preserves coapproximation).

a) Suppose \( G \) is a subspace of \( X \), then \( G \) is proximinal (resp. coproximinal) of \( X \), if and only if \( T(G) \) is proximinal (resp. coproximinal) of \( Y \).
b) Suppose $G$ is a subspace of $X$, then $G$ is Chebyshev (resp. cochebyshev) if and only if $T(G)$ is Chebyshev (resp. cochebyshev).

c) If $T$ is linear, then $\forall x, y \in X, x \perp y \implies T(x) \perp T(y)$. ($T$ is called preserving orthogonality).

d) For a subspace $G$ of $X$, $T(G) = \tilde{T}(G)$ (resp. $T(\tilde{G}) = \tilde{T}(G)$).

e) Suppose $G$ is a subspace of $X$, then $G$ is orthogonality complemented in $X$ if and only if $T(G)$ is orthogonality complemented in $Y$.

f) Suppose $G$ is a subspace of $X$, if $T$ is a continuous and onto preserves approximation map, then $G$ is quasi chebyshev if and only if $T(G)$ is quasi Chebyshev.

Proof. We can easily prove (a), (b), (d), (e). We prove the parts (c) and (f).

c) Suppose $T$ is a preserves approximation, if $x, y \in X$ and $x \perp y$, then $0 \in P_{<y>}(x)$. Therefore $0 = T(0) \in P_{T(<y>)}(T(x))$, since $T$ is linear, $T(<y>) = T(y)$. Hence $T(x) \perp T(y)$

Now suppose $T$ is a preserves approximation, if $x, y \in X$ and $x \perp y$, then $0 \in R_{<y>}(y)$. Therefore $0 = T(0) \in R_{T(<x>)}(T(y))$, since $T$ is linear, $T(<x>) = T(x)$.

f) Let $z \in Y$ and $\{u_n\} \subseteq P_{T(G)}(z)$. Since $T$ is onto, we have $z = T(x)$ for some $x \in X$, therefore $\{u_n\} \subseteq T(P_G(x))$. Then there exists a sequence $\{v_n\} \subseteq P_G(x)$ such that $u_n = T(v_n)$. Since $G$ is quasi Chebyshev, the set $P_G(x)$ is compact. Hence there exists subsequence $\{v_{n_k}\}_{k \geq 1}$ of $\{v_n\}$ and $v_0 \in X$ such that $v_{nk} \to v_0$. Since $T$ is continuous, $u_{nk} = T(v_{nk}) \to u_0 = T(v_0)$. Therefore $P_{T(G)}(z)$ is compact.

Theorem 2.5. Let $X$ be a normed linear space and the operator $T : X \to X$ be preserving approximation, then $T = kU$ such that $k \in R$ and $U$ is an isometry.

Proof. We know that $T$ is preserving approximation, from Theorem 2.4, $T$ is preserving orthogonality. From [1], $T$ is an isometry multiplied by a constant.

Example 2.6. There exists non-linear map which it is preserving orthogonality. We define $T : R^2 \to R^2$ by $T(x, y) = (x, y)$ if $x, y \neq 0$, $T(x, y) = (1, 1)$ if $y = 0, x \neq 0$, $T(x, y) = (-1, 1)$ if $x = 0, y \neq 0$ and $T(0, 0) = (0, 0)$.

3. $\epsilon$-Preserves approximation

In this section we shall obtain characterization of $\epsilon$-Preserves approximation maps. Let $X$ be a normed linear space $\epsilon > 0$ and $x, y \in X$. We call $x$ is $\epsilon$-orthogonal to $y$ and is denoted by $x \perp_{\epsilon} y$ if and only if $||x|| \leq ||x + \alpha y|| + \epsilon$ for all scalar $\alpha$. If $G_1$ and $G_2$ are subsets of $X$, we define $G_1 \perp_{\epsilon} G_2$ if and only if for all $g_1 \in G_1$ and $g_2 \in G_2$ we have, $g_1 \perp_{\epsilon} g_2$. 
For $\epsilon > 0$ we have,
\[ \hat{G}_\epsilon = \{ x \in X : x \perp \epsilon G \} . \]

For $\epsilon > 0$, a point $g_0 \in G$ is said to be a $\epsilon$-approximation for $x \in X$ if $x - g_0 \in \hat{G}_\epsilon$. The set of all $\epsilon$-approximation for $x \in X$ be denoted by $P_G,\epsilon(x)$. Also we have
\[ P_G,\epsilon(x) = \{ g_0 \in G : \| x - g_0 \| \leq \| x - g \| + \epsilon \text{ for all } g \in G \} . \]

For all $\epsilon > 0$, it is clear that the set $P_G,\epsilon(x)$ is a nonempty set. for more information see [6] and [11].

**Definition 3.1.** Suppose $X,Y$ are two linear normed spaces and $\epsilon > 0$. A map $T : X \rightarrow Y$ is called $\epsilon$-preserving approximation if and only if for all subspace $G$ of $X$ and all $x \in X$,
\[ T(P_G,\epsilon(x)) = P_{T(G),\epsilon}(T(x)) . \]

**Corollary 3.2.** Let $X$ be a normed linear space and $\epsilon > 0$. Then every isometry operator $T : X \rightarrow X$ be $\epsilon$-preserving approximation.

**Theorem 3.3.** Suppose $X,Y$ are two linear normed spaces, $\epsilon > 0$ and $T : X \rightarrow Y$ is an onto which is $\epsilon$-preserves approximation.

c) If $T$ is linear, then $\forall x, y \in X \ x \perp \epsilon y \implies T(x) \perp \epsilon T(y)$.

d) For a subspace $G$ of $X$, $T(\hat{G}_\epsilon) = \hat{T(G)_\epsilon}$.

**References**


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