Isoperimetric Inequalities, the Torsion Problem, and the First Eigenvalue of Laplacian of Multiply-connected, Compact Surfaces with Boundary

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Abstract

In Part 1 of this work, we shall resort to the method of interior parallels and give a proof of an isoperimetric inequality of the type of Bol-Fiala-Huber for a multiply-connected, compact surface with boundary whose Gauss curvature is bounded above by a number $K_0$. Also, we shall estimate the upper bound of the radius of the largest circle inscribed in such a surface in terms of its area and $K_0$. In Section 1.3, we shall improve this estimate by introducing a quantity $\ell_{\tau}(\rho)$, which depends on the perimeter of $M$, the area of $M$ and also the shape of $M$. In terms of this quantity, together with the perimeter of $M$, area of $M$, and the lower bound of the Gauss curvature of $M$, we estimate the lower bound of the largest radius of inscribed circle in Section 1.4. In Part 2, using the results in Part 1, we obtain some estimates in the torsion problem. In Part 3, we use the warping function as the trial function in the Rayleigh quotient for the first eigenvalue $\lambda_1$ of the fixed membranes in $M$, and obtain upper bound of $\lambda_1$ in terms of the area of $M$, the perimeter of $M$, together with the lower and upper bounds of the Gauss curvature of $M$.

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Introduction

In the beginning of Part 1 of this work, we shall resort to the method of interior parallels and give a proof of an isoperimetric inequality of the type of Bol-Fiala-Huber[3],[5],[1] for a multiply-connected compact surface with boundary whose
Gauss curvature is bounded above by a number $K_0$. Also, in Section 1.2, we shall estimate the upper bound of the largest circle inscribed in such a surface in terms of its area and $K_0$. Put precisely, given a compact surface $(D, d\sigma)$ with the metric of the form $d\sigma^2 = p(x)(dx_1^2 + dx_2^2)$, let $M \subseteq D$ be any fixed domain whose boundary consists of piecewise $C^2$ simple closed curves $\Gamma_1, \ldots, \Gamma_m$. We set $\ell(\Gamma_1), i = 1, \ldots, m$, and $a(M)$ to be respectively the length of $\Gamma_i$ and the area of $M$. Moreover, we denote the radius of the largest circle inscribes in $M$ as $\tilde{\rho}(M)$. The following two theorems will be proved respectively in Section 1.1 and Section 1.2.

**Theorem 0.1** Suppose that the Gauss curvature of $M$ is bounded above by the number $K_0$ and, moreover, that

$$K_0(a(M)) \leq 2\pi. \quad (1)$$

Then there holds

$$\left(\ell(\Gamma_1) + \cdots + \ell(\Gamma_m)\right)^2 \geq 4\pi(a(M)) - K_0(a(M))^2. \quad (2)$$

We note that the validity of Theorem 0.1 has been indicated in the footnote on page 38 of [2]. It, however, seems to the author that no formal proof has ever been given.

**Theorem 0.2** If $M$ (not necessarily simply-connected) satisfies the hypothesis of Theorem 1, then there holds

$$\begin{cases}
\tilde{\rho}(M) \leq \sqrt{\frac{a(M)}{\pi}}, & \text{if } K_0 = 0 \\
\frac{2}{\sqrt{-K_0}} \arctanh\left(\frac{\sqrt{-K_0}}{2} \sqrt{\frac{a(M)}{\pi - \left(\frac{K_0}{4}\right)a(M)}}\right), & \text{if } K_0 < 0 \\
\frac{2}{\sqrt{K_0}} \arctan\left(\frac{\sqrt{K_0}}{2} \sqrt{\frac{a(M)}{\pi - \left(\frac{K_0}{4}\right)a(M)}}\right), & \text{if } K_0 > 0 \text{ and } (1) \text{ holds.}
\end{cases}$$

In Section 1.3, this estimate will be improved by introducing a quantity $\ell^-_z(\tilde{\rho})$, which depends on $\ell(\Gamma_1) + \cdots + \ell(\Gamma_m)$, $a(M)$ and also the shape of $M$. In terms of this quantity, together with $\ell(\Gamma_1) + \cdots + \ell(\Gamma_m)$, $a(M)$, the shape of $M$ and the lower bound of the Gauss curvature of $M$, we estimate the lower bound $\tilde{\rho}(M)$ in Section 1.4.

In Part 2, using the results obtained in Part 1 and adapting the procedure used by Payne in [8], we obtain some estimates in the torsion problem for a
compact surface $M$ with boundary. That is, we shall consider, for $M^\delta \subset M$, the problem

$$
\begin{align*}
\Delta \eta &= -2, \quad \text{in } M - M^\delta \\
\eta &= 0, \quad \text{on } \partial M \\
\eta &= c, \quad \text{on } \partial M^\delta,
\end{align*}
$$

where $c$ is not known apriori but is determined from the condition

$$
\oint_{\partial M^\delta} \partial \eta \partial n H_1(\mathbf{x}) = 2 H_2(M^\delta) = 2(\text{area of } M^\delta)
$$

with $n$ being the outward unit normal of $M^\delta$, $H_1$ and $H_2$ being respectively the one-dimensional and two-dimensional Hausdorff measures. Thus, $\eta$ is the so called warping function and the Dirichlet integral

$$
S = \int_{M \setminus M^\delta} |\nabla \eta|^2 H_2(\mathbf{x})
$$

is the so called torsion rigidity. We shall estimate $c$ and $\eta_{\max} = \sup_{x \in M \setminus M^\delta} \eta(x)$ in terms of the shape of $M$, the area of $M \setminus M^\delta$, the perimeter of $M$, the lower and upper bounds $K_0$, $K_1$ of the Gauss curvature of $M$.

In Part 3, we consider the solution of the problem

$$
\begin{align*}
\Delta \eta &= -2, \quad \text{in } M, \\
\eta &= 0, \quad \text{on } \partial M,
\end{align*}
$$

and use it as the trial function in the Rayleigh quotient for the first eigenvalue $\lambda_1$ of the fixed membrane on $M$. Hence

$$
\lambda_1 \leq \frac{\int_M |\nabla \eta|^2 H_2(\mathbf{x})}{\int_M \eta^2(\mathbf{x}) H_2(\mathbf{x})} = \frac{S}{\int_M \eta^2(\mathbf{x}) H_2(\mathbf{x})}.
$$

We shall thus obtain an upper bound of $\lambda_1$ in terms of the perimeter of $M$, the area of $M$, the shape of $M$, together with the lower and upper bounds $K_0$, $K_1$ of the Gauss curvature of $M$.

1 Isoperimetric Inequalities. Estimates of Radius of Largest Inscribed Circle
1.1 (A Proof of Theorem 0.1)

Let $M$ be an oriented two-dimensional surface whose boundary is made up of piecewise $C^2$ simple closed curves $\Gamma_1, \cdots, \Gamma_m$. Let

$$x : M \to \mathbb{R}^3$$

be an isometric immersion of $M$ into $\mathbb{R}^3$. We introduce isothermal coordinates on $M$,

$$d\sigma^2 = e^{\nu(\xi_1, \xi_2)}(d\xi_1^2 + d\xi_2^2),$$

$\xi_1, \xi_2 \in D$, $D$ being a domain in the $(\xi_1, \xi_2)$-plane with $\partial D$ homeomorphic to a union of $m$ circles. Let $\Gamma_1$ corresponds to the outer boundary of $D$ under this correspondence. With $\Gamma_1^{-1}(\rho)$, we denote the set of points inside $M$ and at a distance $\rho$ from $\Gamma_1$ (with respect to the metric $d\sigma$).

In case $M$ is multiply-connected, i.e. $m \geq 2$, we may let $\rho_0$ be the smallest possible value of $\rho$ for which $\Gamma_1^{-1}(\rho)$ contacts some $\Gamma_i$, $2 \leq i \leq m$, while in case $M$ is simply-connected, let $\rho_0 = \sup \rho$, where $\rho$ ranges among all values for which $\Gamma_1^{-1}(\rho)$ is defined. In spite of the possible existence of cut loci of points of $\partial M$, it is shown in [6] that, under the hypothesis that $\Gamma_1$ is piecewise $C^2$, $\Gamma_1^{-1}(\rho)$ is made up of a finite number of piecewise $C^2$ simple closed curves, except for values of $\rho$ in a closed set $F$ of Lebesgue measure zero.

Let $\ell^-(\rho)$ be the length of $\Gamma_1^{-1}(\rho)$. It is shown in [6] that, for $0 < \rho < \rho_0$, $\rho \notin F$, the function $\ell^-(\rho)$ is of class $C^1$. With $M^{-}(\rho)$, we denote the region inside $M$ and enclosed by $\Gamma_1^{-1}(\rho) \cup (\cup_{i=2}^m \Gamma_i)$. Then, representing $\Gamma_1^{-1}(\rho)$ as $x_\rho(s) = (x_{\rho_1}(s), x_{\rho_2}(s))$, it follows from the argument on pages 38 and 39 of [2] that in case $\Gamma_1^{-1}(\rho)$ has no convex angles

$$\frac{d\ell^-}{d\rho}(\rho) = -\int_0^{\ell^-(\rho)} \frac{1}{2} \frac{\partial \nu}{\partial n}(x_\rho(s)) ds - 2\pi, \quad (ds = \sqrt{d\xi_1^2 + d\xi_2^2}),$$

where $n$ is the unit outward normal of $\Gamma_1^{-1}(\rho)$, whereas in the presence of convex angles on $\Gamma_1^{-1}(\rho) \cup (\cup_{i=2}^m \Gamma_i),$

$$\frac{d\ell^-}{d\rho}(\rho) \leq -\int_0^{\ell^-(\rho)} \frac{1}{2} \frac{\partial \nu}{\partial n}(x_\rho(s)) ds - 2\pi.$$

Moreover, setting

$$\frac{d\ell^-}{d\rho}(0) = \lim_{\rho \to 0^+} \frac{\ell^-(\rho) - \ell^-(0)}{\rho},$$

either (1) or (2) holds for $\rho = 0$.

Let $a^-\rho$ be the area of $M^{-}(\rho)$ with respect to the metric $d\sigma$. The argument on page 39 of [2] yields

$$\frac{da^-}{d\rho}(\rho) = -\ell^-(\rho),$$
for almost all $\rho$, $0 < \rho < \rho_0$; this is also shown to hold for $\rho = 0$ if we set

$$
\frac{da^-}{d\rho}(0) = \lim_{\rho \to 0^+} \frac{a^-(\rho) - a^-(0)}{\rho}.
$$

1.1.1. We now consider the following three cases separately, namely

Case 1

$$
\int_0^{\ell(\rho)} \frac{\partial \nu}{\partial n}(x_{\rho_0}(s))ds \geq 0,
$$

Case 2

$$
\int_0^{\ell(\rho)} \frac{\partial \nu}{\partial n}(x_0(s))ds \leq 0,
$$

Case 3

$$
\int_0^{\ell(\rho)} \frac{\partial \nu}{\partial n}(x_{\rho_0}(s))ds < 0, \quad \text{and} \quad \int_0^{\ell(\rho)} \frac{\partial \nu}{\partial n}(x_0(s))ds > 0.
$$

We shall prove that in all these three cases, there holds, for all $0 < \rho < \rho_0$,

$$
(\ell(\Gamma_1))^2 - (\ell^- (\rho))^2 \geq 4\pi[a(\rho) - a^- (\rho)] - K_0[(a(\rho))^2 - (a^- (\rho))^2]. \tag{6}
$$

1.1.1.1. First consider Case 1, in which

$$
\int_0^{\ell(\rho)} \frac{\partial \nu}{\partial n}(x_{\rho_0}(s))ds \geq 0.
$$

This, together with (4.1) and (4.2), yields, for all $0 < \rho < \rho_0$, $\rho \notin F$,

$$
\frac{d\ell^-}{d\rho}(\rho) \leq \frac{1}{2} \int_{\partial(M(\rho)\setminus M^- (\rho))} \frac{\partial \nu}{\partial n}(x_{\rho}(s))ds - 2\pi,
$$

since the outward normal of $\Gamma^- (\rho_0)$ with respect to $M \setminus M^- (\rho_0)$ points opposite to that of $\Gamma^- (\rho_0)$ with respect to $M^- (\rho_0)$. Hence, since (2) yields

$$
\Delta \nu + 2Ke^\nu = 0
$$

in $D$, we have, by Green’s identity

$$
\frac{d\ell^-}{d\rho}(\rho) \leq \frac{1}{2} \int_{M^- (\rho) \setminus M^- (\rho_0)} \Delta \nu(e^\nu)d\tau - 2\pi \quad \text{($d\tau = e^\nu d\xi_1 d\xi_2$)} \tag{7}
$$

$$
= \int_{M^- (\rho) \setminus M^- (\rho_0)} K(x)d\tau - 2\pi. \tag{8}
$$

Let us set

$$
\delta(\rho) = (\ell^- (\rho))^2 - 4\pi a^- (\rho) + K_0(a^- (\rho))^2.
$$

From (5) and (8), it then follows that

$$
\frac{d\delta}{d\rho}(\rho) \leq 2\ell^- (\rho) \left[ \int_{M^- (\rho)} K(x)d\tau - 2\pi \right] + 4\pi \ell^- (\rho) - 2K_0 \ell^- (\rho)a^- (\rho) \leq 0.
$$
for \( \rho = 0 \) and for almost all \( \rho, 0 < \rho < \rho_0 \). Hence, (1.47) on page 40 of [2] (which has been verified in [6]) yields, for \( 0 < \rho \leq \rho_0 \),

\[
\delta(0) \geq \delta(\rho),
\]

which amounts to (6).

1.1.1.2. Next we consider Case 2, in which

\[
\oint_{\Gamma} \ell^{-}(\rho) \frac{\partial \nu}{\partial n}(x_{\rho_0}(s))ds \geq 0.
\]  

(9)

In this case, we set, for \( \rho, 0 < \rho < \rho_0 \),

\[
\hat{\Gamma}(\varepsilon) = \Gamma(\rho - \varepsilon),
\]

and

\[
\hat{M}(\varepsilon) = M \setminus M^{-}(\rho - \varepsilon).
\]

Let \( \hat{L}(\varepsilon) \) be the length of \( \hat{\Gamma}(\varepsilon) \) and \( \hat{A}(\varepsilon) \) be the area of \( \hat{M}(\varepsilon) \). Then (1), (2) and (5) yield, for almost all \( \rho \),

\[
\frac{d\hat{A}}{d\varepsilon}(\varepsilon) = -\hat{L}(\varepsilon)
\]  

(10)

and

\[
\frac{d\hat{L}}{d\varepsilon}(\varepsilon) = \frac{d\ell^{-}}{d\rho}(\rho - \varepsilon)
\]

\[
\geq 2\pi + \int_{0}^{\ell^{-}(\rho_0 - \varepsilon)} \frac{1}{2} \frac{\partial \nu}{\partial n}(x_{\rho_0 - \varepsilon}(s))ds.
\]

Hence, if (9) holds,

\[
\frac{d\hat{L}}{d\varepsilon}(\varepsilon) \geq 2\pi + \int_{\partial(\hat{M}(\varepsilon))} \frac{1}{2} \frac{\partial \nu}{\partial n}(x)ds
\]

\[
= 2\pi - \int_{\hat{M}(\varepsilon)} K(x)dx.
\]  

(11)

Thus, setting

\[
\hat{\delta}(\varepsilon) = (\hat{L}(\varepsilon))^2 + 4\pi(\hat{A}(\varepsilon)) - K_0(\hat{A}(\varepsilon))^2.
\]

we have, by (10) and (11),

\[
\frac{d\hat{\delta}}{d\varepsilon}(\varepsilon) \geq 0,
\]

whence

\[
\hat{\delta}(\varepsilon) \geq \hat{\delta}(0).
\]
This amounts to (6) upon setting \( \varepsilon = \rho \).

1.1.1.3. It remains to consider \textbf{Case 3}, in which we can conclude the existence of a number \( \rho^* \), \( 0 < \rho^* < \rho_0 \), such that

\[
\oint_0 \ell^-(\rho^*) \frac{\partial \nu}{\partial n}(x_{\rho^*}(s))ds = 0.
\]

Then the argument in 1.1.1 for \textbf{Case 1} yields (6) holds for \( \rho \) with \( 0 < \rho \leq \rho^* \), while the argument in 1.1.2 for \textbf{Case 2} yields that (6) holds for \( \rho \) with \( \rho^* < \rho \leq \rho_0 \). The proof of (6) is therefore completed.

1.1.1.4. At this point, we note that, if the inequality (1) holds, then

\[
K_0[(a(M))^2 - (a^-(\rho))^2] \leq 2K_0(a(M))[a(M) - a^-(\rho)] \leq 4\pi[a(M) - a^-(\rho)],
\]

whence (6) gives us that, for \( 0 \leq \rho < \rho_0 \),

\[
\ell^-(\rho) \leq \ell(\Gamma_1).
\]

1.1.2 Suppose that \( M \) is \textit{multiply-connected} so that \( \Gamma^{-1}(\rho_0) \) contacts at least one of those simple closed curves \( \Gamma_2, \ldots, \Gamma_m \), say \( \Gamma_j \), \( j = 2, \ldots, m_1 \), \( m_1 \geq 2 \). Then \( \Gamma^{-1}(\rho_0) \cup (\bigcup_{j=2}^{m_1} \Gamma_j) \) is a piecewise \( C^2 \) simple closed curve, whose length is

\[
L_1 = \ell^-(\rho_0) + \sum_{j=2}^{m_1} \ell(\Gamma_j),
\]

and whose enclosed area is \( a^-(\rho_0) \).

1.1.2.1. Assume first that \( m_1 = m \). Then \( \Gamma^{-1}(\rho_0) \cup (\bigcup_{j=2}^{m_1} \Gamma_j) \) encloses one or several simply-connected regions in \( M \), (cf. Figure 1 and Figure 2).

In case that \( \Gamma^{-1}(\rho_0) \cup (\bigcup_{j=2}^{m_1} \Gamma_j) \) encloses \textit{one single} simply-connected region in \( M \), then we have, from Bol-Fiala-Huber’s isoperimetric inequality for simply-connected surfaces,

\[
(L_1)^2 \geq 4\pi(a^-(\rho_0)) - K_0(a^-(\rho_0))^2,
\]

under the hypothesis that (1) holds. Summing up (14) and (6) for \( \rho = \rho_0 \), we obtain

\[
4\pi(a(M)) - K_0(a(M))^2 \leq (\ell(\Gamma_1))^2 - (\ell^-(\rho_0))^2 + (L_1)^2 \leq (\ell(\Gamma_1))^2 - (\ell^-(\rho_0))^2 + \left( (\ell^-(\rho_0)) + \sum_{j=2}^{m} \ell(\Gamma_j) \right)^2.
\]
\[
\begin{align*}
&= \sum_{j=1}^{m} (\ell(\Gamma_j))^2 + 2 \sum_{2 < j_1 < j_2 < m} (\ell^- (\Gamma_{j_1})) (\ell(\Gamma_{j_2})) + 2 (\ell^- (\rho_0)) \sum_{j=2}^{m} \ell(\Gamma_j) \\
&\leq \sum_{j=1}^{m} (\ell(\Gamma_j))^2 + 2 \sum_{1 < j_1 < j_2 < m} (\ell(\Gamma_{j_1})) (\ell(\Gamma_{j_2})) \\
&\leq \left( \sum_{j=1}^{m} \ell(\Gamma_j) \right)^2,
\end{align*}
\]

where the second equality is obtained from (13) and the fourth inequality is obtained from (12). This is precisely (2).

In case that \( \Gamma^-_{\rho_0} \cup (\bigcup_{j=2}^{m} \Gamma_j) \) encloses \( q \), \( q \geq 2 \), simply-connected regions in \( M \), with perimeters \( \ell_1, \cdots, \ell_q \) and area \( a_1, \cdots, a_q \), respectively. Then, again by Bol-Fiala-Huber’s isoperimetric inequality for simply-connected surfaces, there holds, under the assumption that (1) occurs,

\[(\ell_i)^2 \geq 4\pi a_i - K_0(a_i)^2,\]

for \( i = 1, \cdots, q \). Hence, in view of (13),

\[
(L_1)^2 = (\ell_1 + \cdots + \ell_q)^2 \geq ((\ell_1)^2 + \cdots + (\ell_q)^2) \\
\geq 4\pi (a_1 + \cdots + a_q) - K_0((a_1)^2 + \cdots + (a_q)^2) \\
\geq 4\pi (a_1 + \cdots + a_q) - K_0(a_1 + \cdots + a_q)^2 \quad (16)
\]

Thus, we again arrive at (14) and hence (15), which is precisely (2). The proof of Theorem 1 in the case of \( m = m_1 \) is thus completed. In particular, we complete the proof of Theorem 0.1 in the case that \( m = 2 \), i.e. \( M \) is doubled-connected.

1.1.2.2. We now proceed to prove Theorem 0.1 by mathematical induction on the number of connectivity \( m \). Having proved Theorem 1 in the case that \( m = 2 \), we assume that Theorem 1 holds true for \( 2 \leq m \leq k-1 \). Then, consider a compact, multiply-connected surface with boundary and of connectivity \( m = k \). For the number \( \rho_0 \) specified in Section 1.1 and those simple closed curves \( \Gamma_j, j = 2, \cdots, m_1 \), which contacts \( \Gamma^-_{\rho_0} \), the piecewise \( C^2 \) simple closed curve \( \Gamma^-_{\rho_0} \cup (\bigcup_{j=2}^{m_1} \Gamma_j) \) encloses one or several, say \( p \), connected regions, the connectivity of each of which is \( \leq k \). Suppose their perimeters and area are respectively \( \ell_1, \cdots, \ell_p \) and \( a_1, \cdots, a_p \). Then, by the induction hypothesis, we have,

\[(\ell_i)^2 \geq 4\pi a_i - K_0(a_i)^2,\]

for \( 1 \leq i \leq p \). Thus a manipulation analogous to (16) yields again

\[
(L_1)^2 \geq 4\pi (a^- (\rho_0)) - K_0(a^- (\rho_0))^2.
\]
Then, as in 1.2.1, we obtain (15), which is precisely (2). Our proof of Theorem 0.1 is this completed.

1.2 A Proof of Theorem 0.2

We now proceed to give a proof of Theorem 2. To do so, for $B \subseteq M, B$: measurable, we define the surface radius $r(B)$ as on page 42 of [2] to be

$$(r(B))^2 = \frac{a(B)}{\pi - (\frac{K_0}{4})a(B)},$$

where $a(B)$ denotes the area of $B$. Then, in particular for $\rho$, $0 < \rho \leq \rho_0$, $\rho$ being specified in Section 1.1, setting

$$r(\rho) = r(M^-(\rho)),$$

i.e.

$$a^-(\rho) = \frac{\pi(r(\rho))^2}{1 + (\frac{K_0}{4})(r(\rho))^2},$$

we have, by (5) and (2),

$$\frac{-\pi(r(\rho))^2}{1 + (\frac{K_0}{4})(r(\rho))^2} \frac{2}{r(\rho)(1 + (\frac{K_0}{4})(r(\rho))^2)} \left(\frac{dr}{d\rho}(\rho)\right) = \frac{2a^-(\rho)}{r(\rho)} = \frac{2\pi r(\rho)}{1 + (\frac{K_0}{4})(r(\rho))^2},$$

that is,

$$\left(\frac{2\pi r(\rho)}{1 + (\frac{K_0}{4})(r(\rho))^2}\right)\left(\frac{dr}{d\rho}(\rho)\right) \geq \frac{2\pi r(\rho)}{1 + (\frac{K_0}{4})(r(\rho))^2},$$

or equivalently,

$$-\frac{dr}{d\rho}(\rho) \geq 1 + \frac{K_0}{4} (r(\rho))^2.$$ 

Hence

$$\rho_0 \leq \left\{ \begin{array}{ll}
          r(0) - r(\rho), & \text{if } K_0 = 0, \\
          \frac{2}{\sqrt{-K_0}} \left(\frac{\text{arctanh} \frac{r(0)\sqrt{-K_0}}{2}}{2}\right) - \frac{2}{\sqrt{-K_0}} \left(\frac{\text{arctanh} \frac{r(\rho)\sqrt{-K_0}}{2}}{2}\right), & \text{if } K_0 < 0, \\
          \frac{2}{\sqrt{K_0}} \left(\frac{\text{arctan} \frac{r(0)\sqrt{K_0}}{2}}{2}\right) - \frac{2}{\sqrt{K_0}} \left(\frac{\text{arctan} \frac{r(\rho)\sqrt{K_0}}{2}}{2}\right), & \text{if } K_0 > 0
        \end{array} \right\}.$$
and (1) holds; that is,

\[
\rho_0 \leq \begin{cases} 
\sqrt{\frac{a(M)}{\pi}} - \sqrt{\frac{a^- (\rho_0)}{\pi}}, & \text{if } K_0 = 0 \\
\frac{2}{\sqrt{-K_0}} \left( \arctanh \frac{\sqrt{2 - K_0}}{\sqrt{\pi - (\frac{K_0}{4}) a^- (\rho_0)}} - \arctan \frac{\sqrt{K_0}}{\sqrt{\pi - (\frac{K_0}{4}) a^- (\rho_0)}} \right), & \text{if } K_0 < 0, \\
\frac{2}{\sqrt{K_0}} \left( \arctan \frac{\sqrt{K_0}}{\sqrt{\pi - (\frac{K_0}{4}) a^- (\rho_0)}} - \arctan \frac{\sqrt{K_0}}{\sqrt{\pi - (\frac{K_0}{4}) a^- (\rho_0)}} \right), & \text{if } K_0 > 0 \text{ and (1) holds.}
\end{cases}
\]  

(18)

1.2.1. We now proceed to prove Theorem 0.2 by mathematical induction on the number of connectivity \( m \). First we suppose that \( m = 1 \), i.e. \( M \) is simply-connected. Thus, we have to set, as in Section 1.1, \( \rho_0 = \sup \rho \), where \( \rho \) ranges among all values for which \( \Gamma^- (\rho) \) is defined. Thus \( a^- (\rho_0) = 0 \). Then, since \( \bar{\rho}(M) \) is obviously equal to \( \rho_0 \), Theorem 0.2 amounts to (18).

1.2.2. Next, we suppose \( m = 2 \), i.e. \( M \) is doubly-connected. Then, for the value \( \rho_0 \) specified in Section 1.1, the region \( M^- (\rho_0) \) enclosed by \( \Gamma^- (\rho) \cup \Gamma_2 \) is made up of \( q \), \( q \geq 1 \) simply-connected regions with respective area \( a_1, \ldots, a_q \), as indicated in Section 1.1.1 for the proof of Theorem 0.1. Let us denote \( \bar{\rho}_1, \ldots, \bar{\rho}_m \) as the corresponding largest radius of inscribed circles in these regions. We may assume, without loss of generality that \( \bar{\rho}_1 \leq \bar{\rho}_i, \ 2 \leq i \leq q \), and then

\[
\bar{\rho}(M) \leq \rho_0 + \bar{\rho}_1.
\]

On the other hand, by the result in 1.2.1, we have

\[
\bar{\rho}_1 \leq \begin{cases} 
\sqrt{\frac{a_1}{\pi}}, & \text{if } K_0 = 0 \\
\frac{2}{\sqrt{-K_0}} \left( \arctanh \frac{\sqrt{2 - K_0}}{\sqrt{\pi - (\frac{K_0}{4}) a^- (\rho_0)}} \right), & \text{if } K_0 < 0, \\
\frac{2}{\sqrt{K_0}} \left( \arctan \frac{\sqrt{K_0}}{\sqrt{\pi - (\frac{K_0}{4}) a^- (\rho_0)}} \right), & \text{if } K_0 > 0 \text{ and (1) holds.}
\end{cases}
\]

Inserting this and (18) into (19) yields Theorem 0.2 in the case where \( M \) is doubly-connected.

1.2.3. Now we assume that Theorem 0.2 holds true for \( 2 \leq m \leq k - 1 \). Then the truth of Theorem for a compact, multiply-connected surface with boundary and of connectivity \( m = k \) follows in essentially the same way with
that in which the truth of Theorem 0.2 for $m = 2$ follows from that of $m = 1$ and we omit it.

### 1.3 Improving of Theorem 0.2

To improve the estimates made in Theorem 0.2, we observe that by introducing $m - 1$ suitably chosen crosscuts we can transform a surface $M$ of connectivity $m$ into a simply-connected surface $M_*$ with new boundary, which and $M$ have the same largest radius of inscribed circles; (here, by a crosscut we mean a simple Jordan arc, which apart from its endpoints, lies in $M$). Let us set

$$\rho_{0k} \leq \cdots \leq \rho_{02} \leq \rho_{01} = \tilde{\rho}(M)$$

(20)

to be the values of $\rho$ such that, using the notations introduced in Section 1.1, at least one component of $M^-(\rho_{0i})$, $1 \leq i \leq k$ is either an isolates point or else a Jordan arc (with two distinct endpoints). By the reasoning leading to (12), we know that there holds, under the hypothesis of (1),

$$\ell_s^- (\tilde{\rho}) = \ell_s^-(\rho_{01}) \leq \ell_s^-(\rho_{02}) \leq \cdots \leq \ell_s^-(\rho_{0k}) \leq \frac{1}{2} (\text{the perimeter of } M_*),$$

(21)

where we set $\ell_s^- (\rho)$ to be the perimeter of $M^-(\rho)$ for $\rho \neq \rho_{0i}$ and set

$$(\rho_{0i}) = \frac{1}{2} (\lim_{\rho \to \rho_{0i}^-} \ell_s^-(\rho)).$$

Setting $a_s^- (\rho)$ to be the area of $M^-(\rho)$, we obtain from (6) that

$$(\ell_s^- (\rho))^2 - (2\ell_s^- (\tilde{\rho}))^2 \geq 4\pi(a_s^- (\rho)) - K_0(a_s^- (\rho))^2.$$  

Thus, setting $r_s (\rho)$ to be the surface radius of $M^-(\rho)$, we have, instead of (17),

$$\left( \frac{d}{d\rho} \left( \frac{\pi(r_s(\rho))^2}{1 + (\frac{K_0}{4})(r_s(\rho))^2} \right) \right)^2 - (2\ell_s^- (\rho))^2$$

$$= \left( \frac{da_s^-}{d\rho} (\rho) \right)^2 - (2\ell_s^- (\tilde{\rho}))^2$$

$$= (\ell_s^- (\rho))^2 - (2\ell_s^- (\tilde{\rho}))^2$$

$$\geq (4\pi - K_0(a_s^- (\rho)))a_s^- (\rho)$$

$$= \left( \frac{2\pi r_s (\rho)}{1 + (\frac{K_0}{4})(r_s(\rho))^2} \right)^2 ;$$

i.e.

$$\left( \frac{d}{d\rho} \left( \frac{\pi(r_s(\rho))^2}{1 + (\frac{K_0}{4})(r_s(\rho))^2} \right) \right)^2 \geq \left( \frac{2\pi r_s (\rho)}{1 + (\frac{K_0}{4})(r_s(\rho))^2} \right)^2 + (2\ell_s^- (\tilde{\rho}))^2,$$  

(22)
or
\[
\left( \frac{dr_*}{d\rho}(\rho) \right)^2 \geq \left( 1 + \left( \frac{K_0}{4} \right)(r_*(\rho))^2 \right)^2 + \left( 2 \ell_*^{-}(\tilde{\rho}) \right)^2 \left( \frac{1 + \left( \frac{K_0}{4} \right)(r_*(\rho))^2}{2\pi r_*(\rho)} \right)^2,
\]
for almost all \( \rho, 0 < \rho \leq \rho_0 \). Thus, we have, for almost all \( \rho, 0 < \rho < \rho_0 \),
\[
-\frac{dr_*}{d\rho}(\rho) \geq 1 + \left( \frac{K_0}{4} \right)(r_*(\rho))^2, \text{ if } \ell_*^{-}(\tilde{\rho}) = 0
\]
and
\[
-\frac{dr_*}{d\rho}(\rho) \geq \frac{\sqrt{2}}{2} \left( 1 + \left( \frac{K_0}{4} \right)(r_*(\rho))^2 \right) + \frac{\sqrt{2}}{2} \left( \ell_*^{-}(\tilde{\rho}) \right) \left( 1 \right)_{\pi r_*(\rho) = 0} \left( \frac{1 + \left( \frac{K_0}{4} \right)(r_*(\rho))^2}{\pi r_*(\rho)} \right) \left[ \pi r_*(\rho) + \ell_*^{-}(\tilde{\rho}) \left( 1 \right)_{\pi r_*(\rho) = 0} \right],
\]
if \( \ell_*^{-}(\tilde{\rho}) \neq 0 \). Hence
\[
\frac{1}{-\left( \frac{dr_*}{d\rho} \right)} \leq \begin{cases} 
\frac{1}{\sqrt{2} \left( 1 + \left( \frac{K_0}{4} \right)(r_*(\rho))^2 \right)^2} \left( \frac{\ell_*^{-}(\tilde{\rho})}{\pi r_*(\rho) + \ell_*^{-}(\tilde{\rho}) \beta K_0} \right), & \text{if } K_0 \leq 0, \\
\frac{1}{\sqrt{2} \left( 1 + \left( \frac{K_0}{4} \right)(r_*(\rho))^2 \right)^2} \left( \frac{\ell_*^{-}(\tilde{\rho})}{\pi r_*(\rho) + \ell_*^{-}(\tilde{\rho}) \beta K_0} \right), & \text{if } K_0 > 0 \text{ and } (1) \text{ holds}.
\end{cases}
\]
where
\[
\beta_{K_0} = \beta_{K_0}(M) = \begin{cases} 
1, & \text{if } K_0 \leq 0, \\
\frac{a(M)}{\pi - \left( \frac{a}{4} \right) a(M)}, & \text{if } K_0 > 0 \text{ and } (1) \text{ holds}.
\end{cases}
\]
An integration of these yields the following.

**Theorem 1.1** If \( \ell_*^{-}(\tilde{\rho}) > 0 \), then there holds
\[
\frac{\tilde{\rho}(M)}{\sqrt{2}} \leq \begin{cases} 
\sqrt{\frac{a(M)}{\pi} - \frac{\ell_*^{-}(\tilde{\rho})}{\pi} \log \left( 1 + \frac{\pi}{\ell_*^{-}(\tilde{\rho})} \sqrt{\frac{a(M)}{\pi}} \right)}, & \text{if } K_0 = 0, \\
\frac{2}{\sqrt{-K_0}} \arctanh \left( \sqrt{-\frac{K_0}{2}} \sqrt{\frac{a(M)}{\pi - \left( \frac{a}{4} \right) a(M)}} \right), & \text{if } K_0 < 0, \\
\frac{2}{\sqrt{K_0}} \arctan \left( \sqrt{\frac{K_0}{2}} \sqrt{\frac{a(M)}{\pi - \left( \frac{a}{4} \right) a(M)}} \right) - \frac{\ell_*^{-}(\tilde{\rho})}{\pi} \log \left( 1 + \frac{\pi}{\ell_*^{-}(\tilde{\rho})} \sqrt{\frac{a(M)}{\pi - \left( \frac{a}{4} \right) a(M)}} \right), & \text{if } K_0 > 0 \text{ and } (1) \text{ holds}.
\end{cases}
\]
Furthermore, we may observe that, for \( \rho_{0(i+1)} < \rho < \rho_{0i}, 1 \leq i \leq k-1, (\rho_{0i}, 1 \leq i \leq k, being specified in (20)), we have

\[
- \frac{dr_*(\rho)}{d\rho} \leq \begin{cases} 
1 + (\frac{K_0}{4})(r_*(\rho))^2, & \text{if } \ell_*^{-}\beta_0 = 0, \\
\sqrt{2} \left[ 1 + (\frac{K_0}{4})(r_*(\rho))^2 + \ell_*^{-}\beta_0 \left( \frac{1+(\frac{K_0}{4})(r_*(\rho))^2}{\pi r_*(\rho)} \right) \right], & \text{if } \ell_*^{-}\beta_0 \neq 0.
\end{cases}
\]

Thus, if \( \ell_*^{-}\beta_0 > 0 \) for \( i = i_1, \ldots, i_p, 1 \leq i_1 \leq \cdots \leq i_p \leq k, \) then setting

\[
\ell_0 = \sum_{i=i_1,\ldots,i_p} \sqrt{2} \beta_0 \ell_*^{-}\beta_0 \left( \frac{\pi}{\pi r_*(\rho)} \right) - \log \left( 1 + \frac{\pi}{\ell_*^{-}\beta_0} \right),
\]

there holds,

\[
\hat{\rho}(M) \leq \begin{cases} 
\sqrt{2} \left[ \frac{a(M)}{\pi} - \ell_0, \right] & \text{if } K_0 = 0, \\
2\sqrt{\frac{2}{K_0}} \arctanh \left( \frac{\sqrt{\frac{\pi}{\pi r_*(\rho)}}}{\pi - \left( \frac{K_0}{4} \right) a_-(\rho)} \right) - \ell_0, & \text{if } K_0 < 0, \\
2\sqrt{\frac{2}{K_0}} \arctan \left( \frac{\sqrt{\frac{K_0}{4}}}{2} \frac{\pi - \left( \frac{K_0}{4} \right) a_-(\rho)}{\pi - \left( \frac{K_0}{4} \right) a_-(\rho)} \right) - \ell_0, & \text{if } K_0 > 0 \text{ and (1) holds.}
\end{cases}
\]

Furthermore, setting

\[
\hat{\beta}_0 = \hat{\beta}_0(M) = \begin{cases} 
1 + (\frac{K_0}{4}) \frac{a(M)}{\pi - \left( \frac{K_0}{4} \right) a(M)} = \frac{\pi}{\pi - \left( \frac{K_0}{4} \right) a(M)}, & \text{if } K_0 = 0, \\
1, & \text{if } K_0 \geq 0 \text{ and (1) holds.}
\end{cases}
\]

and using (23), we have, for almost all \( \rho, 0 < \rho < \hat{\rho}, \)

\[
\left( - \frac{dr_*(\rho)}{d\rho} \right)^2 \geq (\hat{\beta}_0(M))^2 \left[ 1 + \left( \frac{\ell_*^{-}\beta_0}{\pi r_*(\rho)} \right) \right],
\]

and hence

\[
- \frac{dr_*(\rho)}{d\rho} \geq \hat{\beta}_0(M) \sqrt{1 + \left( \frac{\ell_*^{-}\beta_0}{\pi r_*(\rho)} \right)},
\]

or

\[
- \frac{dr_*(\rho)}{d\rho} \geq \hat{\beta}_0(M) \sqrt{\left( \frac{\pi r_*(\rho)}{(\pi - \left( \frac{K_0}{4} \right) a(M))} \right)^2 + \left( \frac{\ell_*^{-}\beta_0}{\pi r_*(\rho)} \right)^2}.
\]

Thus, we obtain the following after an integration.

**Theorem 1.2** If \( \ell_*^{-}\beta_0 > 0, \) then

\[
\hat{\rho}(M) \leq \frac{1}{2\pi \hat{\beta}_0(M)} \left[ \pi^2 \left( \frac{a(M)}{\pi - \left( \frac{K_0}{4} \right) a(M)} \right) + \left( \frac{\ell_*^{-}\beta_0}{\pi r_*(\rho)} \right)^2 \right].
\]
Moreover, if \( \ell^- (\tilde{\rho}) > 0 \), we obtain from (24) that
\[
-\frac{dr_*}{d\rho}(\rho) \leq \sqrt{2} \frac{\hat{\beta} K_0}{2 \pi r_*(\rho)} [\pi r_*(\rho) + \ell^- (\tilde{\rho}) \hat{\beta} K_0].
\]
Moreover, if \( \ell^- (\tilde{\rho}) > 0 \) and \( \frac{\pi}{\ell^- (\tilde{\rho}) \beta K_0} \sqrt{\frac{a(M)}{\pi - (\frac{K_0}{4}) a(M)}} \leq \epsilon \leq \frac{1}{2} \), for some positive constant \( \epsilon \), then
\[
-\frac{dr_*}{d\rho}(\rho) \geq \frac{1}{\sqrt{1 + 2\epsilon}} \frac{\hat{\beta} K_0}{2 \pi r_*(\rho)} [\pi r_*(\rho) + \ell^- (\tilde{\rho}) \hat{\beta} K_0],
\]
which yields the following.

**Theorem 1.3** Suppose \( \ell^- (\tilde{\rho}) > 0 \). Setting
\[
h_0 = h_0(M) = \sqrt{\frac{a(M)}{\pi - (\frac{K_0}{4}) a(M)}} / \ell^- (\tilde{\rho}) \hat{\beta} K_0,
\]
we have
\[
\tilde{\rho}(M) \leq \sqrt{\frac{2}{\beta K_0}} \sqrt{\frac{a(M)}{\pi - (\frac{K_0}{4}) a(M)}} \left(1 - \frac{1}{\pi h_0} \log(1 + \pi h_0)\right),
\]
and, if we have additionally
\[
\pi h_0 \leq \epsilon \leq \frac{1}{2},
\]
then
\[
\tilde{\rho}(M) \leq \sqrt{\frac{1 + 2\epsilon}{\beta K_0}} \sqrt{\frac{a(M)}{\pi - (\frac{K_0}{4}) a(M)}} \left(1 - \frac{1}{\pi h_0} \log(1 + \pi h_0)\right).
\]

## 2 The Lower Bound of \( \tilde{\rho}(M) \).

To estimate the lower bound of \( \tilde{\rho}(M) \), we adopt the notations introduced in Section 1.3. In case that for some small positive number \( \epsilon_0 \), \( \Gamma^- (\tilde{\rho} - \epsilon_0) = \partial M_* (\tilde{\rho} - \epsilon_0) \) has convex angles, we set \( \Gamma^0(\epsilon_0) \) to be the set obtained from \( \Gamma^- (\tilde{\rho} - \epsilon_0) \) by replacing each convex corner of \( \Gamma^- (\tilde{\rho} - \epsilon_0) \) with a small geodesic circular arc (exceeding a half circle) such that \( \Gamma^0(\epsilon_0) \) has only convex angles and the area enclosed by \( \Gamma^0(\epsilon_0) \) exceeds that of \( M_* (\tilde{\rho} - \epsilon_0) \) by a number less than \( \frac{(\epsilon_0)^2}{(\tilde{\rho} - \rho_{02})^2} \) (cf. Figure 3). If, on the other hand, \( \Gamma^0(\epsilon_0) \) has no convex angles, then we set \( \Gamma^- (\tilde{\rho} - \epsilon_0) \) to be \( \Gamma^0(\epsilon_0) \). Moreover, for \( \epsilon, \alpha < \epsilon < \rho_{02} - \tilde{\rho} \), we let
\( \Gamma^0(\epsilon) \) be the set of points outside \( M_*(\tilde{\rho} - \epsilon_0) \) and at a distance \( \epsilon - \epsilon_0 \) from \( \Gamma^0(\epsilon_0) \).

Denote the length of \( \Gamma^0(\epsilon) \) as \( L^0(\epsilon) \) and the area enclosed by \( \Gamma^0(\epsilon) \) as \( A^0(\epsilon) \). The, by the reasoning which leads to (4.1) and (4.2), we obtain, for almost all \( \epsilon, 0 < \epsilon < \rho_{02} - \tilde{\rho} \),

\[
\frac{dA^0}{d\epsilon}(\epsilon) = L^0(\epsilon) \quad \text{and} \quad \frac{dL^0}{d\epsilon}(\epsilon) = 2Q_1\pi - \int_{M^0(\epsilon)} K(x) d\tau,
\]

where

\[
Q_1 = \max\{1, \text{the number of convex angles of } \Gamma^-(\tilde{\rho} - \epsilon_0)\}
\]

and \( M^0(\epsilon) \) is the region enclosed by \( \Gamma^0(\epsilon) \), (cf. the bottom of page 4 of [1] or page 40 of [2]). Thus, if \( K(x) \leq K_1 \), we have

\[
\frac{dL^0}{d\epsilon}(\epsilon) \leq 2Q_1\pi - K_1A^0(\epsilon).
\]

Multiplying the left hand side by \( L^0(\epsilon) \) and the left hand side by \( \frac{dA^0}{d\epsilon}(\epsilon) \), we obtain after an integration that

\[
(L^0(\epsilon))^2 - (L^0(\epsilon_0))^2 \leq [4Q_1 - K_1A^0(\epsilon)]A^0(\epsilon).
\]

As \( \epsilon_0 \) can be arbitrarily small, we obtain, for \( \rho_{02} < \rho < \tilde{\rho} \),

\[
(L^0(\epsilon))^2 - (2\ell^-(\tilde{\rho}))^2 \leq [4Q_1 - K_1A^0(\epsilon)]A^0(\epsilon).
\]

Thus, setting, for \( B \subseteq M^* \), \( B \):measurable,

\[
(r_1(B))^2 = \frac{a(B)}{\pi - (\frac{K_1}{4Q_1})a(B)},
\]

where \( a(B) \) is the area of \( B \), and then, setting, in particular, for \( \rho_{02} < \rho \tilde{\rho} \),

\[
(r_1(\rho))^2 = r_1(M^-_* (\rho))^2,
\]

we have, analogously to (22),

\[
\frac{d}{d\rho} \left( \frac{\pi(r_1(\rho))^2}{1 + (\frac{K_1}{4Q_1})(r_1(\rho))^2} \right)^2 - (2\ell^-(\tilde{\rho}))^2 \leq \left( \frac{2Q_1\pi r_1(\rho)}{1 + (\frac{K_1}{4Q_1})(r_1(\rho))^2} \right)^2.
\]

Thus, analogously to (23), we have

\[
\frac{1}{(Q_1)^2} \left( -\frac{dr_1}{d\rho}(\rho) \right)^2 \leq \begin{cases} 
1 + (\frac{K_1}{4Q_1})(r_1(\rho))^2, & \text{if } \ell^-(\tilde{\rho}) = 0, \\
1 + (\frac{K_1}{4Q_1})(r_1(\rho))^2 + 4(\ell^-(\tilde{\rho}))^2 \left( \frac{1 + (\frac{K_1}{4Q_1})(r_1(\rho))^2}{2\pi r_1(\rho)} \right)^2, & \text{if } \ell^-(\tilde{\rho}) \neq 0.
\end{cases}
\]
and hence

\[-\frac{dr_1}{d\rho}(\rho) \leq Q_1 \frac{1 + \left(\frac{K_1}{4Q_1}\right)r_1(\rho)}{2\pi r_1(\rho)} \left[2\pi r_1(\rho) + 2\ell_+^-(\tilde{\rho})\left(1 + \left(\frac{K_1}{4Q_1}\right)(r_1(\rho))^2\right)\right],\]

i.e.

\[-\frac{1}{\rho}\frac{dr_1}{d\rho}(\rho) \geq \frac{1}{Q_1} \left(\frac{\ell_+^-(\tilde{\rho})}{\left(1 + \left(\frac{K_1}{4Q_1}\right)(r_1(\rho))^2\right)} - \frac{\ell_+^-(\tilde{\rho})}{\pi r_1(\rho) + 2\ell_+^-(\tilde{\rho})(1 + \left(\frac{K_1}{4Q_1}\right)(r_1(\rho))^2)}\right).

Thus, we obtain

\[
\dot{\rho} - \rho_{o2} \geq \frac{1}{Q_1} \left[\sqrt{\frac{a_+(\rho_{o2})}{\pi}} - \frac{\ell_+^-(\tilde{\rho})}{\pi} \log\left(1 + \frac{\ell_+^-(\tilde{\rho})}{\pi} \sqrt{\frac{a_+(\rho_{o2})}{\pi}}\right)\right], \quad \text{if } K_1 = 0,
\]

\[
\dot{\rho} - \rho_{o2} \geq \frac{1}{Q_1} \left[\frac{2\sqrt{Q_1}}{\sqrt{K_1}} \arctanh\left(\frac{\sqrt{|K_1|}}{2\sqrt{Q_1}} \left[\sqrt{\frac{a_+(\rho_{o2})}{\pi}} - \left(\frac{K_1}{4Q_1}\right)a_+(\rho_{o2})\right]\right)\right. \\
\left. - \frac{\beta_{K_1}^1}{\pi} \ell_+^-(\tilde{\rho}) \log\left(1 + \frac{\pi}{\ell_+^-(\tilde{\rho})} \sqrt{\frac{a_+(\rho_{o2})}{\pi}} - \left(\frac{K_1}{4Q_1}\right)a_+(\rho_{o2})\right)\right], \quad \text{if } K_1 < 0
\]

and

\[
\dot{\rho} - \rho_{o2} \geq \frac{1}{Q_1} \left[\frac{2\sqrt{Q_1}}{\sqrt{K_1}} \arctan\left(\frac{\sqrt{K_1}}{2\sqrt{Q_1}} \left[\sqrt{\frac{a_+(\rho_{o2})}{\pi}} - \left(\frac{K_1}{4Q_1}\right)a_+(\rho_{o2})\right]\right)\right. \\
\left. - \frac{\ell_+^-(\tilde{\rho})}{\pi} \log\left(1 + \frac{\pi}{\ell_+^-(\tilde{\rho})} \sqrt{\frac{a_+(\rho_{o2})}{\pi}} - \left(\frac{K_1}{4Q_1}\right)a_+(\rho_{o2})\right)\right], \quad \text{if } K_1 > 0,
\]

where we set

\[
\beta_{K_1}^1 = \beta_{K_1}^1(M_*) = \left\{ \begin{array}{ll}
1 + \left(\frac{K_1}{4Q_1}\right)\frac{a(M)}{\pi - \left(\frac{K_1}{4Q_1}\right)a(M)} & , \quad \text{if } K_1 = 0 \\
1, & \quad \text{if } K_1 \geq 0.
\end{array} \right.
\]

Applying analogous operations to \(\Gamma_i^-(\rho_{o_i} - \epsilon_o), \ 2 \leq i \leq k,\) and denoting

\[
Q_i = \max\{1, \text{the number of convex angles of } \Gamma_i^-(\rho_{o_i} - \epsilon_o)\}
\]

and setting

\[
\hat{\ell}_i = \frac{\beta_{K_1}^1}{\pi} \frac{\ell_+^-(\rho_{o_i})}{\pi} \log\left(1 + \frac{\pi}{\beta_{K_1}^1 \ell_+^-(\rho_{o_i})} \sqrt{\frac{a(M)}{\pi}} - \left(\frac{K_1}{4Q_1}\right)a(M)\right),
\]
we obtain analogously

\[
\rho_{0i} - \rho_{0(i+1)} \geq \begin{cases} 
\frac{1}{Q_i} (r_1(\rho_{0(i+1)}) - r_1(\rho_{0i})) - \frac{\hat{\ell}_i}{Q_i}, & \text{if } K_1 = 0, \\
\frac{2}{\sqrt{Q_i K_1}} \left[ \arctan \left( \frac{\sqrt{|K_1|}}{2\sqrt{Q_i}} r_1(\rho_{0(i+1)}) \right) - \arctan \left( \frac{\sqrt{|K_1|}}{2\sqrt{Q_i}} r_1(\rho_{0i}) \right) \right] - \frac{\hat{\ell}_i}{Q_i}, & \text{if } K_1 < 0, \\
\frac{2}{\sqrt{Q_i K_1}} \left[ \arctan \left( \frac{\sqrt{|K_1|}}{2\sqrt{Q_i}} r_1(\rho_{0(i+1)}) \right) - \arctan \left( \frac{\sqrt{|K_1|}}{2\sqrt{Q_i}} r_1(\rho_{0i}) \right) \right] - \frac{\hat{\ell}_i}{Q_i}, & \text{if } K_1 > 0,
\end{cases}
\]

where

\[
\hat{\beta}_K^i = \hat{\beta}_K^i(M_*) = \begin{cases} \frac{\pi}{\pi - (\frac{K_1}{4Q}) a(M)}, & \text{if } K_1 = 0 \\
1, & \text{if } K_1 \geq 0.
\end{cases}
\]

Thus, in view of (21), it is easy to see that, setting

\[
Q = Q(M_*) = \max_i Q_i,
\]

and

\[
L_* = L_*(M_*) = \begin{cases} 0, & \text{if } \rho_{0i} \text{ in (21) does not exist}, \\
\ell_i^-(\hat{\rho}), & \text{if } k = 1 \text{ in (21)}, \\
\frac{1}{2} \text{(perimeter of } M_*), & \text{if } k > 1 \text{ in (21)}.
\end{cases}
\]

the following theorem holds true.

**Theorem 2.1** There holds

\[
\bar{\rho}(M) \geq \begin{cases} \frac{1}{Q_i} \sqrt{\frac{a(M)}{\pi}} - \frac{\hat{\ell}}{Q}, & \text{if } K_1 = 0, \\
\frac{1}{\sqrt{|Q| K_1}} \arctan \left( \frac{\sqrt{|K_1|}}{2\sqrt{Q}} \sqrt{\frac{a(M)}{\pi - (\frac{K_1}{4Q}) a(M)}} \right) - \frac{\hat{\ell}}{Q}, & \text{if } K_1 < 0, \\
\frac{1}{\sqrt{|Q| K_1}} \arctan \left( \frac{\sqrt{|K_1|}}{2\sqrt{Q}} \sqrt{\frac{a(M)}{\pi - (\frac{K_1}{4Q}) a(M)}} \right) - \frac{\hat{\ell}}{Q}, & \text{if } K_1 > 0,
\end{cases}
\]

where

\[
\hat{\ell} = \sum_{i=1}^{k} \hat{\ell}_i \leq k \frac{L_* \hat{\beta}^i_{K_1}}{\pi} \log \left( 1 + \frac{\pi}{L_* \hat{\beta}^i_{K_1}} \sqrt{\frac{a(M)}{\pi - (\frac{K_1}{4Q}) a(M)}} \right),
\]

and

\[
\hat{\beta}^i_{K_1} = \hat{\beta}^i_{K_1}(M_*) = \begin{cases} \frac{\pi}{\pi - (\frac{K_1}{4Q}) a(M)}, & \text{if } K_1 = 0, \\
1, & \text{if } K_1 \geq 0.
\end{cases}
\]

Furthermore, setting

\[
\beta_{K_1} = \beta_{K_1}(M_*) = \begin{cases} 1, & \text{if } K_1 \leq 0, \\
1 + \frac{(K_1)}{4Q} \frac{a(M)}{\pi - (\frac{K_1}{4Q}) a(M)} = \frac{\pi}{\pi - (\frac{K_1}{4Q}) a(M)}, & \text{if } K_1 < 0,
\end{cases}
\]
we have, analogously to (24),
\[-\frac{dr_*(\rho)}{d\rho}(\rho) \leq \frac{Q\beta_{K_1}}{\pi r_*(\rho)}\sqrt{(\pi r_*(\rho))^2 + (L_\ast \beta_{K_1})^2}\]
and hence, analogously to (25),
\[-\frac{dr_*(\rho)}{d\rho}(\rho) \leq \frac{Q\beta_{K_1}}{\pi r_*(\rho)}[\pi r_*(\rho) + L_\ast \beta_{K_1}].\]

The last two inequalities yield, respectively, Theorem 1.4 and Theorem 1.5 below.

**Theorem 2.2**

\[
\bar{\rho}(M) \geq \begin{cases} 
\frac{1}{2Q\pi\beta_{K_1}} \sqrt{\pi^2 \left( \frac{a(M)}{\pi - \left(\frac{K_1}{4Q}\right)a(M)} \right) + (L_\ast \beta_{K_1})^2} - L_\ast \beta_{K_1}, & \text{if } L_\ast > 0, \\
\frac{1}{2Q\beta_{K_1}} \sqrt{\frac{a(M)}{\pi - \left(\frac{K_1}{4Q}\right)a(M)}}, & \text{if } L_\ast = 0.
\end{cases}
\]

**Theorem 2.3** Suppose that $L_\ast > 0$. Setting
\[
h_1 = h_1(M) = \left[\frac{a(M)}{\pi - \left(\frac{K_1}{4Q}\right)a(M)}\right] / L_\ast \beta_{K_1},
\]
we have
\[
\bar{\rho}(M) \geq \frac{1}{Q\beta_{K_1}} \sqrt{\frac{a(M)}{\pi - \left(\frac{K_1}{4Q}\right)a(M)} \left(1 - \frac{1}{\pi h_1(M)} \log(1 + \pi h_1(M))\right)}.
\]

In this connection, we notice that, by suitably choosing $(m - 1)$ crosscuts and using the argument of page 44 of [4], there holds
\[
L_\ast \leq (m - 1)\sqrt{a(M) + (\text{the perimeter of } M) = L_{**}(M)}, \text{ say.}
\]
Thus, we have

**Corollary 2.4** Setting
\[
h_{**}(M) = \left[\frac{a(M)}{\pi - \left(\frac{K_1}{4Q}\right)a(M)}\right] / L_{**}\beta_{K_1},
\]
we have
\[
\bar{\rho}(M) \geq \frac{1}{Q\beta_{K_1}} \sqrt{\frac{a(M)}{\pi - \left(\frac{K_1}{4Q}\right)a(M)} \left(1 - \frac{1}{\pi h_{**}(M)} \log(1 + \pi h_{**}(M))\right)}.
\]
3 The Torsion Problem

We begin our consideration of the torsion problem with a useful lemma:

**Lemma 3.1** Let \( E \) be a region in a two-dimensional Riemannian manifold \( M \) endowed with a Riemannian metric
\[
d\sigma^2 = e^{\nu(\xi)}ds^2,
\]
\( \xi = (\xi_1, \xi_2) \in D, D \) being a domain in the \((\xi_1, \xi_2)\)-plane.
Let \( u : E \to \mathbb{R} \) be of class \( C^2 \). Let us set
\[
E(\mu) = \{ x : x \in E, u(x) \geq \mu \},
\]
\( \Gamma(\mu) = \partial E(\mu) \), and
\[
a(\nu) = \int_{E(\nu)} e^{\nu(\xi)}d\xi_1d\xi_2 = \text{the area of } E(\mu).
\]
Suppose, for almost all \( \mu \), \( \inf \mu < \mu < \sup \mu \), there hold
(i) \( \Gamma(\mu) \) consists of finitely many disjoint simple closed curves,
(ii) \( \Gamma(\mu) \) is of class \( C^2 \),
(iii) \( \frac{Du}{dn} < 0 \) on \( \Gamma(\mu) \), \( n \) being the outward unit normal on \( E(\mu) \).
If the Gauss curvature of \( M \) is bounded above by the number \( K_0 \), then, for almost all \( \mu \), \( \inf \mu < \mu < \sup \mu \), there holds
\[
- \left( \frac{da}{d\mu} \right) / a(\mu) \geq \frac{4\pi - K_0 a(\mu)}{\int |\nabla u|d\sigma}.
\]

A proof of this result for a *simply-connected* region \( E \) is a straightforward modification of that on page 81 of [2], using coarea formula, Schwartz’s inequality and Bol-Fiala-Huber’s isoperimetric inequality. Theorem 1 enables us to generalize this result to regions \( E \) of finite connectivity.

2.1. Consider a region \( E \subset M, E \) : measurable, and consider a subregion \( E^\delta \subset E \) with \( H_2(E^\delta) = (1 - \delta)H_2(E) \), where \( H_2 \) denotes the Haussdorff two-dimensional measure and \( M \) is a two-dimensional Riemannian manifold endowed with a Riemannian metric
\[
d\sigma^2 = e^{\nu(\xi)}ds^2,
\]
\( \xi = (\xi_1, \xi_2), D \) being a domain in the \((\xi_1, \xi_2)\)-plane.
Let \( \eta(x) : E \subset E^\delta \to \mathbb{R} \) be the function satisfying
\[ \begin{cases} \Delta \eta = -2 & \text{in } E \setminus E^\delta, \\ \eta = 0 & \text{on } \partial E, \\ \eta = c & \text{on } \partial E^\delta, \end{cases} \]  

in which the constant \( c \) is not known apriori but is determined from the condition

\[ \oint_{\partial E^\delta} \frac{\partial \eta}{\partial n} d\sigma = 2H_2(E^\delta), \]  

where \( n \) is the outward pointing normal of \( \partial E^\delta \) with respect to \( E \setminus E^\delta \).

We note first that the inequality

\[ \eta > 0 \]  

in \( E \setminus E^\delta \) has been established on page 63 of [2].

Let us set

\[
E_1(\mu) = \{ x : x \in M, \mu(x) \geq \mu \}, \\
\Gamma_1(\mu) = \{ x : x \in M, \mu(x) = \mu \}, \\
a_1(\mu) = H_2(E_1(\mu));
\]

moreover, \( \mu(a_1) \) is defined to be the inverse of \( a_1(\mu) \). The following result holds.

**Lemma 3.2** Suppose the gradient of \( \eta \) vanishes at points on \( \Gamma(\mu_0) \), where \( \mu_0 \) is not a relative maximum value of \( \eta \) and \( \mu_0 \neq c \). Then these points must be isolated (with respect to the relative topology of \( \Gamma_1(\mu) \)).

To prove this lemma, we shall examine closely the topology of \( \Gamma_1(\mu) \), which will become important in our future discussion.

Indeed, the condition that

\[ \oint_{\partial E^\delta} \frac{\partial \eta}{\partial n} d\sigma \geq 0 \]

and

\[ \eta \equiv c \quad \text{on } \partial E^\delta \]

yield a tubular neighborhood of \( \partial E^\delta \) in \( E \setminus E^\delta \) consisting of level sets \( \tilde{\Gamma}_1(c - \epsilon) \) on which \( \eta(x) = c - \epsilon \) for positive and sufficiently small \( \epsilon \). Each of such level sets \( \tilde{\Gamma}_1(\mu - \epsilon) \) is a simple closed curve, provided no critical point of \( \eta \) lies on it. Denote \( \epsilon_1 \) as the least upper bound of such \( \epsilon \) that \( \tilde{\Gamma}_1(c - \epsilon) \) is a simple closed curve. The inequality (29) yields that \( \epsilon_1 > 0 \). Let

\[ E_{\epsilon_1} = \bigcup_{0 < \epsilon < \epsilon_1} \tilde{\Gamma}_1(c - \epsilon). \]
We may observe that $\partial E_{\epsilon_1} \setminus \partial E^\delta$ cannot consist entirely of critical points of $\eta$ for otherwise we could obtain the following contradiction

\[
0 \geq -2H_2(E_{\epsilon_1}) = \int_{E_{\epsilon_1}} \Delta \eta H_2(dx) = \int_{\partial E^\delta} \frac{\partial \eta}{\partial n} d\sigma + \int_{\partial E_{\epsilon_1} \setminus \partial E^\delta} \frac{\partial \eta}{\partial n} H_1(dx) = 2H_2(E^\delta) + 0,
\]

where we use (28) to obtain the last equality. Thus, since $\eta \equiv 0$ on $\partial E$, each critical point $P$ of $\eta$ on $\partial E_{\epsilon_1} \setminus \partial E^\delta$ must lie on a simple closed curve $\Gamma_P$ on which $\eta = c - \epsilon_1$ and which encloses a compact, simply-connected region $E_P$ disjoint from $\partial E \cup \partial E^\delta$ and in the closure of $E \setminus E_{\epsilon_1}$.

Since

\[
\Delta \eta = -2 < 0 \quad \text{in} \quad E \setminus E^\delta
\]

and $E_P$ is compact, we know that $\eta$ takes at least one relative maximum value in the interior of $E_P$ and $\eta \geq c - \epsilon_1$ throughout $E_P$.

Were some critical points on $\partial E_{\epsilon_1}$ not isolated (with respect to the relative topology of $\partial E_{\epsilon_1}$), these critical points on $\partial E_{\epsilon_1}$ would take interior relative minimum value of $\eta$ in $E \setminus E^\delta$, contradicting the superharmonicity of $\eta$. Hence the critical points of $\eta$ on $\partial E_{\epsilon_1}$ must be isolated (with respect to the relative topology of $\partial E_{\epsilon_1}$). Likewise, we know that the critical points of $\eta$ on $\Gamma_P$ must be isolated (with respect to the relative topology of $\Gamma_P$). Hence

\[
\Gamma_P \cap \partial E_{\epsilon_1} = \{P\}
\]

and $\eta$ keeps decreasing in the direction of the outer normal of $\partial E_{\epsilon_1}$ (with respect to $E_{\epsilon_1}$) and $\Gamma_P$ (with respect to $E_P$) at non-critical points of $\eta$ on $\Gamma_P \cap \partial E_{\epsilon_1}$. The previous argument for the region $E_{\epsilon_1} \cup E_P$ can then be modified and applied to the region $E \setminus (E_{\epsilon_1} \cup E_P)$ to conclude Lemma 2.2.

Incidentally, the following result is established by reviewing our discussion closely and naming the relative maximum value of $\eta$ in $E \setminus E^\delta$ in the order

\[
\eta_1 \leq \eta_2 \leq \cdots \leq \eta_n = \eta_{\text{max}} = \sup_{x \in E \setminus E^\delta} \eta(x).
\]

**Lemma 3.3** Each level set $\Gamma(\eta_i)$ $i = 1, \cdots, n$, consists of isolated points and possibly some Jordan arcs (with two distinct endpoints). Each component of $\Gamma(\eta_i)$, $i = 1, \cdots, n$, lies in a compact, simply-connected region $E_P$, for some critical point $P$ of $\eta$ which is neither a relative maximum nor a relative minimum point of $\eta$; moreover, $\eta$ takes constant value $\eta_P$ on the boundary $\Gamma_P$ of $E \setminus P$,

\[
\eta(P) < c \quad \text{and} \quad \eta(P) \leq \eta\big|_{E_P},
\]

where $\eta\big|_{E_P}$ denotes the restriction of $\eta$ to $E_P$. 

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2.2. Thus, we may apply Lemma 2.1 to obtain
\[ -\frac{1}{d\mu/d\alpha_1} \geq \frac{(4\pi - K_0a_1(\mu))a_1(\mu)}{f_{\Gamma_1(\mu)} |\text{grad}\eta|d\sigma}. \]  

(30)

On the other hand, by Green’s identity, we have
\[ \oint_{\Gamma_1(\mu)} |\text{grad}\eta|d\sigma = 2 \int_{E \setminus E^\delta \cap E_1(\mu)} 1 \, d\tau + 2H_2(E^\delta) = 2a_1(\mu), \]

using (27) and (28). From (30) and (31), it follows that

\[ -\frac{1}{d\mu/d\alpha_1} \geq 2\pi - \frac{K_0}{2}a_1(\mu), \]

for almost all $\mu$, $\mu \neq c, \mu_1, \cdots, \mu_n (= \eta_{\text{max}})$. Thus, we obtain that, for $0 < \mu < c$,

\[
\mu(a_1) \leq \begin{cases} 
\frac{1}{2\pi}(H_2(E) - \mu_1(a_1)), & \text{if } K_0 = 0, \\
\frac{1}{\sqrt{\pi|K_0|}} \arctan\left(\sqrt{|K_0|}H_2(E)\right) - \frac{1}{\sqrt{\pi|K_0|}} \arctan\left(\sqrt{|K_0|a_1(\mu)}\right), & \text{if } K_0 < 0, \\
\frac{1}{\sqrt{\pi|K_0|}} \arctanh\left(\sqrt{K_0}H_2(E)\right) - \frac{1}{\sqrt{\pi|K_0|}} \arctanh\left(\sqrt{K_0a_1(\mu)}\right), & \text{if } K_0 > 0 \text{ and (1) holds}, 
\end{cases}
\]

while, for $c < \mu < \eta_{\text{max}}$,

\[
\mu(a_1) \leq \begin{cases} 
\frac{1}{2\pi}(H_2(E) - \mu_1(a_1)) - \frac{a_1(\mu)}{2\pi}, & \text{if } K_0 = 0, \\
\frac{1}{\sqrt{\pi|K_0|}} \left[ \arctan\left(\sqrt{|K_0|}H_2(E)\right) - \arctan\left(\sqrt{|K_0|a_1(\mu)}\right) \right] - \frac{A_0}{\sqrt{\pi|K_0|}}, & \text{if } K_0 < 0, \\
\frac{1}{\sqrt{\pi|K_0|}} \left[ \arctanh\left(\sqrt{K_0}H_2(E)\right) - \arctanh\left(\sqrt{K_0a_1(\mu)}\right) \right] - \frac{A_1}{\sqrt{\pi|K_0|}}, & \text{if } K_0 > 0 \text{ and (1) holds}, 
\end{cases}
\]

in which

\[ A_0 = \arctan\left(\frac{\sqrt{|K_0|}H_2(E_1(c))}{2\sqrt{\pi}}\right) - \arctan\left(\frac{\sqrt{|K_0|}[H_2(E_1(c)) - H_2(E^\delta)]}{2\sqrt{\pi}}\right), \]

and

\[ A_1 = \arctanh\left(\frac{\sqrt{K_0}H_2(E_1(c))}{2\sqrt{\pi}}\right) - \arctanh\left(\frac{\sqrt{K_0}[H_2(E_1(c)) - H_2(E^\delta)]}{2\sqrt{\pi}}\right), \]

\[ \geq \arctanh\left(\frac{\sqrt{K_0}}{H_2(E^\delta)2\sqrt{\pi}}\right). \]


From this readily follows that

\[
\eta_{\text{max}} \leq \begin{cases}
\frac{1}{2\pi} (H_2(E) - \mu(a_1)), & \text{if } K_0 = 0, \\
\frac{1}{\sqrt{\pi|K_0|}} \arctan \left( \frac{\sqrt{|K_0|} |H_2(E) - H_2(E^\delta)|}{2\sqrt{\pi}} \right), & \text{if } K_0 < 0, \\
\frac{1}{\sqrt{\pi|K_0|}} \arctanh \left( \frac{\sqrt{K_0} H_2(E)}{2\sqrt{\pi}} \right) - \frac{1}{\sqrt{\pi K_0}} \arctanh \left( \frac{\sqrt{K_0} H_2(E^\delta)}{2\sqrt{\pi}} \right), & \text{if } K_0 > 0 \text{ and } (1) \text{ holds},
\end{cases}
\]

(32)

and, if \( K_0 = 0 \),

\[
a_1(\mu) \leq \begin{cases}
H_2(E) - 2\pi \mu, & \text{for } 0 < \mu < c, \\
H_2(E) - H_2(E^\delta) - 2\pi \mu, & \text{for } c < \mu < \eta_{\text{max}},
\end{cases}
\]

(33)

if \( K_0 < 0 \),

\[
a_1(\mu) \leq \begin{cases}
H_2(E) - \frac{2\sqrt{\pi}}{\sqrt{-K_0}} \tan \sqrt{\pi|K_0|}\mu(a_1), & \text{for } 0 < \mu < c, \\
H_2(E) - H_2(E^\delta) - \frac{2\sqrt{\pi}}{\sqrt{-K_0}} \tan \sqrt{\pi|K_0|}\mu(a_1), & \text{for } c < \mu < \eta_{\text{max}},
\end{cases}
\]

(34)

while if \( K_0 > 0 \),

\[
a_1(\mu) \leq \begin{cases}
\frac{2\sqrt{\pi}}{K_0} \tan \left[ \arctanh \left( \frac{\sqrt{K_0} H_2(E^\delta)}{2\sqrt{\pi}} \right) - \sqrt{\pi K_0} \mu(a_1) \right], & \text{for } 0 < \mu < c, \\
\frac{2\sqrt{\pi}}{K_0} \tan \left[ \arctanh \left( \frac{\sqrt{K_0} H_2(E^\delta)}{2\sqrt{\pi}} \right) - \arctanh \left( \frac{\sqrt{K_0} H_2(E)}{2\sqrt{\pi}} \right) - \sqrt{\pi K_0} \mu(a_1) \right], & \text{for } c < \mu < \eta_{\text{max}},
\end{cases}
\]

(35)

2.3. We now proceed to obtain a lower bound of \( \eta_{\text{max}} \) and \( c \) in terms of \( \delta \) and \( H_2(E) \). For this, we note that, as indicated previous to and in Lemma 2, the set \( E \setminus E^\delta \) consists of one or more simply-connected, compact components. By (33), (34) and (35), we have

\[
H_2(E \setminus E^\delta) \leq \begin{cases}
H_2(E) - H_2(E^\delta) - 2\pi c, & \text{if } K_1 = 0, \\
H_2(E) - H_2(E^\delta) - \frac{2\sqrt{\pi}}{\sqrt{-K_0}} \left[ \tan \sqrt{\pi|K_0|} c \right], & \text{if } K_1 < 0, \\
\frac{2\sqrt{\pi}}{\sqrt{-K_0}} \tan \left[ \arctanh \left( \frac{\sqrt{K_0} H_2(E^\delta)}{2\sqrt{\pi}} \right) - \arctanh \left( \frac{\sqrt{K_0} H_2(E)}{2\sqrt{\pi}} \right) \right] - 2\pi c, & \text{if } K_1 > 0.
\end{cases}
\]

(36)

Denote the domain \( E \setminus E_1(c) \) as \( E' \). Although \( E' \) is multiply-connected, bounded by piecewise smooth simple closed curves \( \partial E, \partial E^\delta, \Gamma_1, \ldots, \Gamma_{q-1}, \) say, we can transform \( E' \) into a simply-connected domain \( E'' \) by introducing \( q - 1 \) suitably chosen crosscuts \( C_1, \ldots, C_{q-1} \). We consider in the resultant simply-connected domain \( E'' \), analogously to (27), the problem

\[
\begin{align*}
\Delta \nu &= -2 & \text{in } E'' \\
\nu &= 0 & \text{on } (\partial E \cup \partial E^\delta \bar{\Gamma}_1 \cup \cdots \cup \bar{\Gamma}_{q-2}) \cup (C_1 \cup \cdots \cup C_{q-1}).
\end{align*}
\]

(37)
This, (29) and an application of the maximum principle for the harmonic function to \( \eta - \nu \) yield

\[
\eta \geq \nu, \tag{38}
\]

in \( E'' \); in particular

\[
c \geq \nu_{\text{max}} = \sup_{x \in E''} \nu(x). \tag{39}
\]

On the other hand, the function \( \nu \) defined by (37) has the representation

\[
\nu(y) = 2 \int_{E''} G_{E''}(x, y) d\tau,
\]

where \( G_{E''}(x, y) \) is the Green’s function for the Laplacian with singular point \( y \) and subject to the condition \( G_{E''} = 0 \) on \( \partial E'' \). We may set, analogously to page 60 of [2],

\[
E''(\tilde{g}) = \{ x : x \in E'', G_{E''}(x, y) \geq \tilde{g} \},
\]

\[
\Gamma''(\tilde{g}) = \{ x : x \in E'', G_{E''}(x, y) = \tilde{g} \},
\]

\[
a''(\tilde{g}) = H_2(E''(\tilde{g})).
\]

Setting \( \tilde{g}(a'') \) to be the inverse \( \tilde{g}(a'') \) of \( a''(\tilde{g}) \) and using Lemma 2.1, we have

\[
- \frac{1}{d\tilde{g}/da''} \geq \frac{(4\pi - K_0 a''(\tilde{g}))a''(\tilde{g})}{\int_{\Gamma''(\tilde{g})} |\text{grad} G_{E''}| d\sigma}.
\]

But, since \( G_{E''} \) is the Green’s function of Laplacian, we have, for all \( \tilde{g} \geq 0 \),

\[
\int_{\Gamma''(\tilde{g})} \left| \frac{\partial G_{E''}}{\partial n} \right| d\sigma = 1,
\]

whence

\[
- \frac{d\tilde{g}}{da''} \leq \frac{1}{(4\pi - K_0 a''(\tilde{g}))a''(\tilde{g})} = \frac{1}{K_0} \frac{1}{4\pi - K_0 a''(\tilde{g})} + \frac{1}{4\pi a''(\tilde{g})}. \tag{40}
\]

An integration of (40) from \( a'' \) to \( a(E'') \) yields

\[
\tilde{g}(a'') \geq \frac{1}{4\pi} \log \left( \frac{1}{a''(\tilde{g})} \right) + \frac{1}{4\pi} \log \left( 1 - \frac{K_0}{4\pi} a''(\tilde{g}) \right) + \lim_{\tilde{g} \to \infty} \left( \frac{1}{4\pi} \log a''(\tilde{g}) + \tilde{g} \right). \tag{41}
\]

As we take isothermal coordinates on \( E'' \), we may set

\[
G_{E''}(x, y) = \frac{1}{2\pi} \log \left( \frac{R_y(E'')}{\text{dist}(x, y)} \right) + H(x, y),
\]
where \( \text{dist}(x,y) \) denotes the distance between \( x \) and \( y \) on \( M \), \( H(x,y) \) is a harmonic function with \( H(y,y) = 0 \) and \( R_y(E'') \) is the so-called conformal radius of \( E'' \) with respect \( y \).

In view of Theorem 0.2, the geodesic circle on \( E'' \) with center \( y \) and of radius \( r \) has area greater than \( \pi r^2 \) if \( K_0 \leq 0 \) and approaching to \( \pi r^2 \) as \( r \) approaches to zero if \( K_0 > 0 \). Hence, no matter \( K_0 = 0, > 0, \) or \( < 0 \), we have

\[
\lim_{\tilde{g} \to \infty} \left( \frac{1}{\pi} \log \tilde{\rho} + \tilde{\rho}(M) \right) \geq \frac{1}{4\pi} (\log \pi (R_y(E''))^2)
\]

This together with (41), gives us

\[
a''(\tilde{g}) \geq \frac{1}{4\pi} \log \frac{\pi (R_y(E''))^2}{a''(\tilde{g})} + \frac{1}{4\pi} \log \left(1 - \frac{K_0}{4\pi} a''(\tilde{g})\right).
\]

At this point, we give two definitions.

**Definition 3.4** For \( B \subseteq M, \ B \) measurable, let \( B^* \) be the disk

\[
\{(x_1, x_2) : \sqrt{(x_1)^2 + (x_2)^2} < R\}
\]

in the \((x_1, x_2)\)-plane such that \( H_2(B) = \pi R^2 \).

**Definition 3.5** The function

\[
G_{E''}(x_1, x_2) : (E'')^* \setminus \{(0, 0)\} \rightarrow R
\]

is obtained from \( G_{E''}(x, y) : E'' \setminus \{y\} \rightarrow R \) as follows:

\[
G_{E''}(x_1, x_2) = \sup\{\tilde{g} : (x_1, x_2) \in (E''(\tilde{g}))^*\}.
\]

Thus, (42) means that, in the \((x_1, x_2)\)-plane

\[
G_{E''}(x_1, x_2) \geq \frac{1}{2\pi} \log \frac{R_y(E'')}{\sqrt{(x_1)^2 + (x_2)^2}} + \frac{1}{4\pi} \log \left(1 - \frac{K_0}{4} ((x_1)^2 + (x_2)^2)\right).
\]

Hence, if \( K_0 \neq 0 \), by Definition 2.5,

\[
\nu(y) = 2 \int_{E''} G_{E''}(x, y) d\tau = 2 \int_{(E'')^*} G^*_y(x_1, x_2) dx_1 dx_2
\]

\[
\geq 2 \int_{(E'')^*} \frac{1}{2\pi} \log \frac{R_y(E'')}{\sqrt{(x_1)^2 + (x_2)^2}} dx_1 dx_2
\]

\[
+ \frac{1}{2} \int_0^{\sqrt{a(M)/\pi}} \log \left(1 - \frac{K_0}{4} r^2\right) r dr
\]

\[
= \frac{a(E'')}{4\pi} \log \left(1 - \frac{K_0 a(E'')}{4\pi}\right) - \frac{a(E'')}{4\pi} - \frac{1}{K_0} \log \left(1 - \frac{K_0 a(E'')}{4\pi}\right)
\]

\[
+ \int_{(E'')^*} \frac{1}{2\pi} \log \frac{R_y(E'')}{\sqrt{(x_1)^2 + (x_2)^2}} dx_1 dx_2.
\]
We may also note that, setting in the \((x_1, x_2)\)-plane

\[
E_y = \{(x_1, x_2) : \sqrt{(x_1)^2 + (x_2)^2} < R_y(E'')\},
\]
the Green’s function of \(E_y\) for the Laplacian, with singularity at \((0, 0)\) and subject to the condition that \(y = 0\) on \(\partial E_y\) is

\[
G_{E_y}((x_1, x_2), (0, 0)) = \frac{1}{2\pi} \log \frac{R_y(E'')}{\sqrt{(x_1)^2 + (x_2)^2}}.
\]

And the solution to the problem

\[
\begin{align*}
\Delta w &= -2 & \text{in } E_y \\
w &= 0 & \text{on } \partial E_y
\end{align*}
\]

is

\[
w(x_1, x_2) = \frac{(R_y(E''))^2}{2} - \frac{(x_1)^2 + (x_2)^2}{2}.
\]

Hence

\[
2 \int_{(E'')} \frac{1}{2\pi} \log \frac{R_y(E'')}{\sqrt{(x_1)^2 + (x_2)^2}} \, dx_1 \, dx_2 = 2 \int_{E_y} G_{E_y}((x_1, x_2), (0, 0)) \\
= w(0, 0) = \frac{(R_y(E''))^2}{2}.
\]

Inserting this into (43) and setting

\[
\delta''_1 = \delta_1(E'') = \left(1 - \frac{4\pi}{K_0 a(E'')}\right) \log \left(1 - \frac{K_0 a(E'')}{4\pi}\right) - 1,
\]

we obtain

\[
\nu(y) \geq \frac{(R_y(E''))^2}{2} + \frac{a(E'')}{4\pi} \delta''_1,
\]

which, together with (39), yields

\[
c \geq \max_{y \in E''} \frac{(R_y(E''))^2}{2} + \frac{a(E'')}{4\pi} \delta''_1.
\]

To estimate a lower bound of \(R_y(E'')\), we set, for \(y \in E''\), \(G_{B_{r_y}(y)}(x, y)\) to be the Green’s function for the Laplacian of largest inscribed geodesic ball \(B_{r_y}(y)\) in \(E''\) with center \(y\) and radius \(r_y\). Thus,

\[
G_{B_{r_y}(y)}(x, y) = \frac{1}{2\pi} \log \frac{r_y}{\text{dist}(x, y)}.
\]
An application of the maximum principle for Laplacian to $G_{E''} - G_{B_\nu(y)}$ yields

$$G_{E''} \geq G_{B_\nu(y)}$$

in $B_\nu(y) \setminus \{y\}$; hence,

$$R_y(E'') \geq r_y.$$ 

Inserting this into (45), we obtain

$$c \geq \tilde{\rho}(E'') + \frac{\delta''}{4\pi}a(E''),$$

where $\tilde{\rho}(E'')$ is the largest radius of inscribed circle in $E''$, recalling the notation introduced in Part 1.

We observe that by considering the region $E \setminus E^\delta$ instead of $E''$ in the previous discussion, we obtain analogous results for $\eta_{\text{max}}$. To sum up, we formulate

**Lemma 3.6** Suppose $\eta$ satisfies (27) and (28). Set $\eta_{\text{max}} = \sup_{x \in E \setminus E^\delta} \eta(x)$, there holds

$$c \geq \max_{y \in E'} \left( \frac{R_y(E'')^2}{2} + \frac{a(E')}{4\pi} \delta'' \right),$$

$$\eta_{\text{max}} \geq \max_{y \in E \setminus E^\delta} \left( \frac{R_y((E \setminus E^\delta)')^2}{2} + \frac{a(E \setminus E^\delta)}{4\pi} \delta_1 \right),$$

where $E''$ and $(E \setminus E^\delta)'$ are simply-connected regions obtained respectively from $E' = (E \setminus E^\delta) \setminus E_1(c)$ and $E \setminus E^\delta$ by introducing suitable crosscuts, $\delta''$ is defined in (44) and

$$\delta_1 = \delta_1(E \setminus E^\delta) + \left( \log \left( 1 - \frac{K_0 a(E \setminus E^\delta)}{4\pi} \right) \right) \left( 1 - \frac{4\pi}{K_0 a(E \setminus E^\delta)} \right) - 1.$$ 

Hence, there also holds

$$c \geq \left( \frac{\tilde{\rho}(E')}{2} + \frac{\delta''}{4\pi}a(E'') \right),$$

$$\eta_{\text{max}} \geq \left( \frac{\tilde{\rho}(E \setminus E^\delta)}{2} + \frac{\delta_1}{4\pi}a(E \setminus E^\delta) \right).$$

In view of Theorem 1.5 and Theorem 1.6, we thus obtain
**Theorem 3.7** Let us set \( Q(E'') \) and \( Q(E \setminus E^\delta) \) as in (25). Also, set \( L_*(E'') \) and \( L_*(E \setminus E^\delta) \) as in (26) (\( E'' \) being defined in the beginning of this section). We have

\[
c \geq \left( \frac{h(E')}{2(\beta K_1)^2} + \frac{\delta''}{4\pi} \right) a(E'),
\]
and

\[
\eta_{\text{max}} \geq \left( \frac{h(E \setminus E^\delta)}{2(\beta K_1)^2} + \frac{\delta_1}{4\pi} \right) a(E \setminus E^\delta),
\]

where

\[
h(E') = \frac{1}{(Q(E''))^2(\pi - (\frac{K_1}{4Q(E'')})a(E'))} \left( 1 - \frac{\log(1 + \pi h_1(E''))}{\pi h_1(E'')} \right),
\]
and

\[
h(E \setminus E^\delta) = \frac{1}{(Q(E \setminus E^\delta))^2(\pi - (\frac{K_1}{4Q(E \setminus E^\delta)})a(E \setminus E^\delta))} \left( 1 - \frac{\log(1 + \pi h_1((E \setminus E^\delta)''))}{\pi h_1((E \setminus E^\delta)'')} \right),
\]
in which \( h_1(E'') \) and \( h_1((E \setminus E^\delta)'') \) is defined in Theorem 1.6 and Corollary 1.7, and \( K_1 \) is the lower bound of the Gauss curvature of \( M \).

2.4. We thus obtain in the case where

\[
\frac{h(E \setminus E^\delta)}{(\beta K_1)^2} + \frac{\delta_1}{4\pi} \geq 0
\]

a lower bound of \( \eta_{\text{max}} \) in terms of \( E, \delta, K_0, Q(E \setminus E^\delta) \) and the perimeter of \( E \setminus E^\delta \), recalling Corollary 1.7 and recalling that \( a(E \setminus E^\delta) = (1 - \delta)a(E) \). We are interested in finding the situations where we are able to obtain a lower bound of \( c \) in terms of the same set of values. For this, we proceed to find out situations where we can estimate \( R_y(E') \) in terms of this set of values.

For example, if \( \ell^- (\tilde{\rho}(E_1(c))) = 0 \), then we obviously have

\[
\max_{y \in E'} R_y(E') = \max_{y \in E \setminus E^\delta} R_y(E \setminus E^\delta).
\]

Let us set

\[
\tilde{\rho}^c = \tilde{\rho}(E_1(c)) \quad \text{and} \quad \tilde{\rho}^\delta = \tilde{\rho}(E \setminus E^\delta).
\]

From the discussion previous to Lemma 2.2 and Lemma 2.3, it is easy to see that if \( c \neq \eta_{\text{max}} \), then the set \( \Gamma^- (\tilde{\rho}^c) \) consists of isolated points and open Jordan arcs. Suppose, for \( E \setminus E^\delta \), the set \( \Gamma^- (\tilde{\rho}^\delta) \) consists of one or more arcs with \( p \)
as the total number of endpoints, (cf. Figure 4). If \( \ell^-(\tilde{\rho}^c) \leq \ell^*_-(\tilde{\rho}^\delta) + p\tilde{\rho}^\delta \), then (47) remains true. If, on the other hand, \( \ell^-(\tilde{\rho}^c) \geq \ell^*_-(\tilde{\rho}^\delta) + p\tilde{\rho}^\delta \), we have

\[
R_y(E') \geq \frac{1}{2}(\tilde{\rho}^\delta - \tilde{\rho}^c) \geq \frac{1}{2} \sqrt{a(E \setminus E^\delta)} - \sqrt{2} \sqrt{\frac{h_c}{\beta_{K_1}}} \sqrt{a(E \setminus E^\delta)} - 2\pi c,
\]

by (36), Theorem 1.3 and Theorem 1.6, where \( \hat{\beta}_{K_0} = \hat{\beta}_{K_0}(E_1(c)) \) and

\[
h_c = \frac{1}{\sqrt{1 - (\frac{K_0}{4})(a(E \setminus E^\delta) - 2\pi c)}} \left(1 - \frac{\log(1 + \pi h_o(E_1(c))))}{\pi h_o(E_1(c))}\right),
\]

with \( h_o(E_1(c)) \) being defined in Theorem 1.3.

The inequality (48) suggests us to find situations where

\[
\frac{\sqrt{h(E \setminus E^\delta)}}{\beta_{K_1}} \geq \sqrt{2} \sqrt{\frac{h_c}{\beta_{K_0}}}.
\]

If \( \ell^-(\tilde{\rho}^c) \leq \ell^*_-(\tilde{\rho}^\delta) + p\tilde{\rho}^\delta \), then

\[
\frac{\sqrt{a(E \setminus E^\delta)} - 2\pi c}{\ell^-(\tilde{\rho}^c)} \leq \frac{\sqrt{a(E \setminus E^\delta)} - 2\pi c}{\ell^*_-(\tilde{\rho}^\delta) + p\tilde{\rho}^\delta} \leq \frac{\ell^*_-(\tilde{\rho}^\delta)}{\ell^*_-(\tilde{\rho}^\delta) + p\tilde{\rho}^\delta} \frac{\sqrt{a(E \setminus E^\delta)}}{\ell^*_-(\tilde{\rho}^\delta)},
\]

and then we have, in virtue of Theorem 1.6,

\[
\frac{\sqrt{a(E \setminus E^\delta)} - 2\pi c}{\ell^-(\tilde{\rho}^c)} \leq \left[1 + \frac{ph_1^\delta}{Q^\delta} \left(1 - \frac{\log(1 + \pi h_1^\delta)}{\pi h_1^\delta}\right)\right]^{-1} \frac{\sqrt{a(E \setminus E^\delta)}}{\ell^*_-(\tilde{\rho}^\delta)},
\]

where \( h_1^\delta = h_1((E \setminus E^\delta).Q^\delta = Q(E \setminus E^\delta) \). Thus, if \( k = 1 \) in (21), then we have

\[
h_o(E_1(c)) \leq \beta^c(K_0, K_1) \left[1 + \frac{ph_1^\delta}{Q^\delta} \left(1 - \frac{\log(1 + \pi h_1^\delta)}{\pi h_1^\delta}\right)\right]^{-1} h_1^\delta,
\]

where we set

\[
\beta^c(K_0, K_1) = \frac{\beta_{K_1}}{\beta_{K_0}} \sqrt{\frac{\pi - (\frac{K_0}{4})a(E \setminus E^\delta)}{\pi - (\frac{K_0}{4})(a(E \setminus E^\delta) - 2\pi c)}}.
\]

As is easy to see that, for all \( a > 0 \),

\[
\frac{a^2}{2(1 + a)} \leq a - \log(1 + a) \leq \frac{a^2}{2}
\]
and also
\[ \frac{a^2}{2(1 + a)} \leq a - \log(1 + a) \frac{a^2}{1 + a}, \]
whence, for all \( a > 0 \)
\[ \frac{a}{2(1 + a)} \leq 1 - \frac{1}{a} \log(1 + a) \leq \frac{a}{2}, \tag{51} \]
and also
\[ \frac{a}{2(1 + a)} \leq 1 - \frac{1}{a} \log(1 + a) \leq \frac{a}{1 + a}. \tag{52} \]
Hence, since
\[ \frac{\sqrt{h(E \setminus E^\delta)}}{\beta_{K_1}} \left/ \frac{\sqrt{2}\sqrt{h_c}}{\beta_{K_0}} \right. \geq \frac{1}{\beta_{K_0}} \frac{1 - \frac{\log(1 + \pi h_1^\delta)}{\pi h_1^\delta}}{1 - \frac{\log(1 + \pi h_0(E_1(c)))}{\pi h_0(E_1(c)))}}, \tag{53} \]
we have, by (51) and (49),
\[ \frac{\sqrt{h(E \setminus E^\delta)}}{\beta_{K_1}} \left/ \frac{\sqrt{2}\sqrt{h_c}}{\beta_{K_0}} \right. \geq 1 + \frac{p \pi (h_1^\delta)^2}{2Q^\delta} \frac{1}{1 + \pi h_1^\delta} \frac{1}{\sqrt{2}(1 + \pi h_1^\delta)}, \]
while, by (52) and (49),
\[ \frac{\sqrt{h(E \setminus E^\delta)}}{\beta_{K_1}} \left/ \frac{\sqrt{2}\sqrt{h_c}}{\beta_{K_0}} \right. \geq \left( 1 + \frac{p \pi (h_1^\delta)^2}{2Q^\delta} \frac{1}{1 + \pi h_1^\delta} \right) \left( 1 + \pi h_0(E_1(c)) \right) \left( \frac{1}{2\sqrt{2}(1 + \pi h_1^\delta)} \right) \]
If we have
\[ p > 2\sqrt{2}Q^\delta \pi, \]
then \( \frac{p}{2\sqrt{2}Q^\delta} > \pi \), and then there exists a positive number \( b \) such that
\[ \frac{1 + \frac{p \pi (h_1^\delta)^2}{2Q^\delta} \frac{1}{1 + \pi h_1^\delta}}{\sqrt{2}(1 + \pi h_1^\delta)} > \sqrt{\frac{b + 2}{b}}, \tag{54} \]
for sufficiently large \( h_1^\delta \). Hence
\[ \frac{\sqrt{h(E \setminus E^\delta)}}{\beta_{K_1}} > \sqrt{\frac{b + 2}{b}} \frac{\sqrt{2}\sqrt{h_c}}{\beta_{K_0}}. \tag{55} \]
We observe that, for all \( s, t \in R \),
\[ (s - t)^2 = \left( \frac{s^2}{b+1} - \frac{t^2}{b} \right) + \left( \frac{bs^2}{b+1} - 2st + \frac{b+1}{b} t^2 \right) \]
\[ = \frac{1}{b+1} \left( s^2 - \frac{b+1}{b} t^2 \right) + \left( \frac{b}{b+1} s - \frac{b+1}{b} t \right)^2, \]
Setting \( s = \frac{1}{2} \sqrt{\frac{h(E \setminus E^\delta)}{\beta_{K_1}}} \sqrt{a(E \setminus E^\delta)} \) and \( t = \sqrt{2} \frac{h(E \setminus E^\delta)}{\beta_{K_0}} \sqrt{a(E \setminus E^\delta)} - 2\pi c \) in this identity and inserting what results in into the inequality (55), we obtain, with the aid of Lemma 2.6 and (48) that
\[
c \geq \left[ 1 - \frac{1}{8b(b+1)(\beta_{K_1})^2} \right] \frac{h(E)}{8b(b+1)(\beta_{K_1})^2} a(E \setminus E^\delta) + \frac{2}{8b} \frac{h(E \setminus E^\delta)}{(\beta_{K_1})^2} (2\pi c) + \frac{\delta''}{4\pi} (2\pi c).
\] (56)

If there holds
\[
\frac{\pi}{2b} \frac{h(E \setminus E^\delta)}{(\beta_{K_1})^2} + \frac{\delta''}{2} > 1,
\] (57)
then (56) is a contradiction, which concludes that
\[c = \eta_{\text{max}},\]
and hence yields as estimate of \( c \) of the desired type in this case.

If, on the other hand, (57) fails to hold, then we obtain from (56) that
\[
\left[ 1 - \frac{\pi}{2b} \frac{h(E \setminus E^\delta)}{(\beta_{K_1})^2} - \frac{\delta''}{2} \right] c \geq \frac{h(E \setminus E^\delta)}{8b(b+1)(\beta_{K_1})^2} a(E \setminus E^\delta),
\]
and hence
\[
c \geq \left( 1 - \frac{\delta''}{2} \right)^{-1} \frac{h(E \setminus E^\delta)}{8b(b+1)(\beta_{K_1})^2} a(E \setminus E^\delta),
\] (58)

\[
\geq \begin{cases} 
(1 - \frac{\delta'}{2})^{-1} \frac{h(E \setminus E^\delta)}{8b(b+1)(\beta_{K_1})^2} a(E \setminus E^\delta), & \text{if } K_0 = 0, \\
\frac{h(E \setminus E^\delta)}{8b(b+1)(\beta_{K_1})^2} a(E \setminus E^\delta), & \text{if } K_0 \geq 0 \text{ and (1) holds.}
\end{cases}
\]

Suppose that
\[2 \leq p \leq 2\sqrt{2}Q^\delta \pi.\]
If \( k = 1 \) in (21) and if, instead of (49), there holds
\[
h_o(E_1(c)) \leq \beta^c(K_0, K_1) \left[ 2\sqrt{2}\pi h_1^\delta \left( 1 - \frac{\log\left( 1 + \frac{\pi h_1^\delta}{\pi h_1^\delta} \right)}{\pi h_1^\delta} \right) \right]^{-1} h_1^\delta,
\] (59)
the estimates made in the previous paragraph for \( c \) is still valid. If, on the other hand, (59) fails to hold, then, by (52), we have
\[
h_o(E_1(c)) \leq \beta^c(K_0, K_1) \frac{h_1^\delta \left( 1 + \pi h_1^\delta \right)}{1 + \pi h_1^\delta + 2\sqrt{2}(\pi h_1^\delta)^2}.
\]
If there holds additionally
\[
\beta^c(K_0, K_1) \geq 2\sqrt{2}Q^\delta,
\] (60)
then we have

$$\beta^c(K_0, K_1) \frac{h_1^\delta(1 + \pi h_1^\delta)}{1 + \pi h_1^\delta + 2\sqrt{2}(\pi h_1^\delta)^2} \geq 2\sqrt{2}Q^\delta \sqrt{\frac{b_1 + 2}{b_1}}, \tag{61}$$

for some positive number $b_1$ depending on the lower bound of $h_1^\delta$. Therefore, there holds the inequality

$$\frac{1 + \pi h_0(E_1(c))}{2\sqrt{2}} > \pi Q^\delta \sqrt{\frac{b_1 + 2}{b_1}}.$$

Hence, in virtue of (53), we know that either $c = \eta_{\text{max}}$ or (58) holds true for $b = b_1$. We thus proceed to find situations where (60) is satisfied.

If $K_0 \geq K_1 > 0$, then, in view of (50), we have

$$\beta^c(K_0, K_1) = \frac{\pi}{\pi - (\frac{K_1}{4Q})a(E \setminus E^6)} \frac{\sqrt{\pi - (\frac{K_1}{4Q})a(E \setminus E^6)}}{\sqrt{\pi - (\frac{K_1}{4Q})[a(E \setminus E^6) - 2\pi c]}}$$

which and (1) prevent the occurrence of (60). If $K_0 \geq 0 \geq K_1$, then

$$\beta^c(K_0, K_1) = \frac{\sqrt{\pi - (\frac{K_1}{4Q})a(E \setminus E^6)}}{\sqrt{\pi - (\frac{K_1}{4Q})[a(E \setminus E^6) - 2\pi c]}} \geq \frac{\sqrt{\pi - (\frac{K_1}{4Q})a(E \setminus E^6)}}{\sqrt{\pi}},$$

and hence (60) holds true under the additional condition

$$|K_1|a(E \setminus E^6) \geq 4Q^\delta[8\pi^3(Q^\delta)^2 + \pi]. \tag{62}$$

Finally, if $0 \geq K_0 \geq K_1$, then

$$\beta^c(K_0, K_1) = \frac{\sqrt{\pi - (\frac{K_1}{4Q})a(E \setminus E^6)}}{\sqrt{\pi - (\frac{K_1}{4Q})[a(E \setminus E^6) - 2\pi c]}} \geq \frac{\sqrt{\pi - (\frac{K_1}{4Q})a(E \setminus E^6)}}{\sqrt{\pi}},$$

and hence again (60) holds true if (62) does.

To sum up, we formulate...
Theorem 3.8 Suppose \( k = 1 \) for \( E \setminus E^\delta \) in (21) and \( \Gamma_* (\tilde{\rho}(E \setminus E^\delta)) \) consists of one or more arcs with \( p \) as the total number of endpoints. If we have

\[
p > 2\sqrt{2}(Q^\delta(E \setminus E^\delta))\pi,
\]
then the inequality (54) holds true for sufficiently large \( h(E \setminus E^\delta) \) and for some positive number \( b \), (where \( h(E \setminus E^\delta) \) is defined in Theorem 3.3) and then one of the following possibilities occurs:

(1) \( c = \eta_{\text{max}} \),

(2) The inequalities in (58) holds true.

If we have, instead,

\[
2 \leq p < 2\sqrt{2}(Q^\delta(E \setminus E^\delta))\pi,
\]
then either the inequality (54) holds true for sufficiently large \( h(E \setminus E^\delta) \) and for some positive number \( b \), whence one of the above-mentioned possibilities occurs, or else after choosing \( K_1 < 0 \) and \( |K_1| \) so large that (62) holds, the inequality (61) holds true for some positive number \( b_1 \) and one of the above-mentioned possibilities occurs with \( b = b_1 \).

4 The First Eigenvalue for Laplacian of a Multiply Connected Surface with Boundary

Consider the solution of the problem

\[
\begin{cases}
\Delta \eta = -2 & \text{on } M, \\
\eta = 0 & \text{on } \partial M,
\end{cases}
\]

where \( M \) is a multiply-connected, compact, two-dimensional Riemannian manifold with boundary. Setting \( \eta_{\text{max}} = \sup_{x \in M} \eta(x) \), we obtain from (32) and the consideration made in Part 2

\[
\eta_{\text{max}} \leq \begin{cases}
\frac{1}{2\pi} H_2(M), & \text{if } K_0 = 0, \\
\frac{1}{\sqrt{|K_0|}} \arctan \left( \frac{\sqrt{-K_0} H_2(M)}{2\sqrt{\pi}} \right), & \text{if } K_0 < 0 \\
\frac{1}{\sqrt{|K_0|}} \arctanh \left( \frac{\sqrt{K_0} H_2(M)}{2\sqrt{\pi}} \right), & \text{if } K_0 > 0 \text{ and (1) holds},
\end{cases}
\]

(63)

where \( K_0 \) is an upper bound of the Gauss curvature of \( M \).
We may use \( \eta \) as the trial function in the Rayleigh quotient for the first eigenvalue \( \lambda_1 \) of the fixed membrane problem on \( M \). Thus,
\[
\lambda_1 \leq \frac{\int_M |\text{grad} \, \eta|^2 H_2(\text{d}x)}{\int_M \eta^2 H_2(\text{d}x)}.
\]
To estimate the right hand side of this inequality, we may set as before for \( 0 < \mu < c \)
\[
E_1(\mu) = \{ x : x \in M, \eta(x) \geq \mu \},
\]
\[
\Gamma_1(\mu) = \partial E_1(\mu)
\]
\[
a_1(\mu) = H_2(E_1(\mu)),
\]
and
\[
S_1(\mu) = \int_{E_1(\mu)} |\text{grad} \, \eta|^2 H_2(\text{d}x).
\]
The coarea formula and Green’s identity yield
\[
\frac{dS_1}{d\mu}(\mu) = \oint_{\Gamma_1(\mu)} \left| \text{grad} \, \eta \right|^2 H_1(\text{d}x) = 2a_1(\mu).
\]
Hence, by (33), (34) and (35),
\[
-\frac{dS_1}{d\mu}(\mu) \leq \begin{cases} 
2(H_2(M) - 2\pi \mu), & \text{if } K_0 = 0, \\
2 \left[ H_2(M) - \frac{2\sqrt{\pi}}{\sqrt{-K_0}} \arctan(\sqrt{\pi|K_0|}\eta(a_1)) \right], & \text{if } K_0 < 0, \\
2 \left[ \frac{2\sqrt{\pi}}{\sqrt{-K_0}} \arctanh \left( \frac{\eta(a_1) H_2(M)}{2\sqrt{\pi}} \right) - 2\pi \mu \eta(a_1) \right], & \text{if } K_0 > 0 \text{ and (1) holds}. 
\end{cases}
\]
Hence, if \( K_0 = 0 \),
\[
\int_{E_1(\mu)} |\text{grad} \, \eta|^2 H_2(\text{d}x) \leq 2\eta_{\max} H_2(M) - 2\pi(\eta_{\max})^2, \tag{64}
\]
if \( K_0 < 0 \),
\[
\eta_{\max} \leq 2\eta_{\max} H_2(M) - \frac{2\sqrt{\pi}}{\sqrt{-K_0}} \eta_{\max} \arctan(\sqrt{\pi|K_0|}\eta_{\max}) \tag{65}
\]
\[- \frac{1}{K_0} \log \left| 1 - \pi K_0(\eta_{\max})^2 \right|, \]
while if \( K_0 > 0 \),
\[
\eta_{\max} \leq 2\eta_{\max} \left[ \frac{2\sqrt{\pi}}{\sqrt{K_0}} \arctanh \left( \frac{\sqrt{K_0} H_2(M)}{2\sqrt{\pi}} \right) \right] - 2\pi(\eta_{\max})^2. \tag{66}
\]
Moreover, setting

$$S_2(\mu) = \int_{E_1(\mu)} \eta^2 H_2(dx),$$

we obtain from an application of the coarea formula

$$-\frac{dS_2}{d\mu}(\mu) = \mu^2 \int \frac{H_2(dx)}{|\text{grad} \eta|} = \mu^2 (-\frac{da_1}{d\mu}),$$

which, together with Lemma 2.1 and (31) yields

$$-\frac{dS_2}{d\mu}(\mu) = \mu^2 \frac{(4\pi - K_0 a_1(\mu))a_1(\mu)}{\frac{H_2}{1(\mu)} |\text{grad} \eta|^2 H_1(dx)} \geq \left(2\pi - \frac{K_0}{2} a_1(\mu)\right)\mu^2.$$

Hence, from (33), (34) and (35), we have

$$-\frac{dS_2}{d\mu}(\mu) = \begin{cases} 
2\pi \mu^2, & \text{if } K_0 = 0 \\
2\pi \mu^2 - \frac{K_0}{2} H_2(M)\mu^2 - 2\frac{\sqrt{\tan(\sqrt{K_0} |\mu|)} |\mu|}{\sqrt{K_0} |\mu|}, & \text{if } K_0 < 0 \\
2\pi \mu^2 - \frac{\sqrt{\pi}}{\sqrt{K_0}} \mu^2 \arctanh\left(\frac{\sqrt{K_0} H_2(M)}{2\sqrt{\pi}}\right) + K_0 \mu^3, & \text{if } K_0 > 0 
\end{cases}$$

Hence, if $K_0 = 0$,

$$\int_M \eta^2 H_2(dx) \geq \frac{2\pi}{3} (\eta_{\text{max}})^3.$$

If $K_0 < 0$, since

$$\arctan(\sqrt{\pi} |K_0| |\mu|) < \sqrt{\pi} |K_0| |\mu|,$$

we have

$$-\frac{dS_2}{d\mu}(\mu) = 2\pi \mu^2 - \frac{K_0}{2} H_2(M)\mu^2 + 2\pi K_0 \mu^3,$$

which yields

$$\int_M \eta^2 H_2(dx) \geq \frac{2\pi}{3} (\eta_{\text{max}})^3 - \frac{K_0}{3} H_2(M)(\eta_{\text{max}})^3 + \frac{\pi K_0}{2} (\eta_{\text{max}})^4. \quad (67)$$

If $K_0 > 0$,

$$\int_M \eta^2 H_2(dx) \geq \frac{2\pi}{3} (\eta_{\text{max}})^3 - \frac{\sqrt{\pi}}{3\sqrt{K_0}} \arctan\left(\frac{\sqrt{K_0} H_2(M)}{2\sqrt{\pi}}\right)(\eta_{\text{max}})^3 + \frac{K_0}{4} (\eta_{\text{max}})^4. \quad (68)$$

Denoting as before that $K_1$ is a lower bound of the Gauss curvature of $M$ and

$$G = G(M) = \frac{h(M)}{2(\beta_{K_1}(M))^2} + \frac{\delta_1(M)}{4\pi},$$

isoperimetric inequalities
we have from Theorem 2.7

$$\eta_{\text{max}} \geq G\eta_2(M).$$

Moreover, setting

$$A = A(M, K_0) = \begin{cases} \arctan\left(\frac{\sqrt{-K_0}H_2(M)}{2\sqrt{\pi}}\right), & \text{if } K_0 = 0, \\ \arctanh\left(\frac{\sqrt{K_0}H_2(M)}{2\sqrt{\pi}}\right), & \text{if } K_0 < 0, \end{cases}$$

we have, by (63),

$$\eta_{\text{max}} \leq \begin{cases} \frac{1}{2\pi} H_2(M), & \text{if } K_0 = 0 \\ \frac{1}{\sqrt{\pi|K_0|}} A = \frac{A}{\sqrt{\pi|K_0|}}, & \text{if } K_0 \neq 0. \end{cases}$$

Hence, if $K_0 = 0$, by (64) and (67),

$$\lambda_1 \leq \frac{1}{2\pi} H_2(M) - 2\pi G^2(H_2(M))^2$$

$$= \frac{3}{2\pi^2 G^3(H_2(M))^2} - \frac{3}{G} (H_2(M))^{-1},$$

and if $K_0 < 0$, by (65) and (68),

$$\lambda_1 \leq \frac{1}{\sqrt{\pi|K_0|}} A - \frac{1}{K_0} \log |1 + A^2|$$

$$= \frac{1}{\sqrt{\pi|K_0|}} A - \frac{1}{K_0} \log |1 + A^2|$$

$$= \frac{(2\pi^3 - K_0H_2(M))G^3(H_2(M))^3}{(2\pi^3 - K_0H_2(M))G^3(H_2(M))^3} + \frac{1}{2\pi K_0} A^4$$

$$= \frac{2\sqrt{\pi|K_0|}G\eta_2(M)[\arctan(\sqrt{\pi|K_0|}GH_2(M))]}{(2\pi^3 - K_0H_2(M))G^3(H_2(M))^3} + \frac{1}{2\pi K_0} A^4$$

$$= \frac{2\pi K_0(2\pi - K_0H_2(M))G^3(H_2(M))^3 - A^4}{(2\pi^3 - K_0H_2(M))G^3(H_2(M))^3}$$

$$= \frac{2\pi(2\pi K_0(2\pi - K_0H_2(M))G^3(H_2(M))^3 - A^4}{(2\pi^3 - K_0H_2(M))G^3(H_2(M))^3}$$

while if $K_0 > 0$, by (66) and (69),

$$\lambda_1 \leq \frac{4}{K_0} A^2 - 2\pi G^2(H_2(M))^2$$

$$= \frac{2\pi K_0H_2(M)[\arctan(\sqrt{\pi|K_0|}GH_2(M))]}{2\pi^3 - K_0H_2(M))G^3(H_2(M))^3} + \frac{1}{2\pi K_0} A^4$$

$$= \frac{12\pi K_0A^2 - 6\pi^2(K_0)^2 G^2(H_2(M))^2}{2\pi^2(K_0)^2 G^3(H_2(M))^3} + \frac{3\pi(K_0)^3 G^4(H_2(M))^4}{2} - A^4.$$


References


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