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B-Binomial Convolution and Associated Arithmetical Functions and Their Properties

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Abstract. In this paper we considered Apostol's article [2] and we defined for B-Binomial convolution associated arithmetical function S. We show that this function and B-Binomial convolution associated with Ramanujan Sums are connected.

Keywords: Arithmetic functions, Convolution, Binomial convolution, Ramanujan Sum

1. Introduction

Let n and d's standart forms are $n=\prod\limits_{k} p_k^{n_k}$ and $d=\prod\limits_{k} p_k^{d_k}$ respectively. We know that $d|n$ if and only if $d_k \le n_k$ [6]. Let $\binom{n_k}{d_k}$ be classical Binomial coefficent. For positive integer n and d is a divisor of n, the function $B(n,d)$ is defined by,

$$B(n,d)=\prod_{k}\binom{n_k}{d_k}=\begin{cases} 1, & if \quad d_k=n_k \ \vee \ d_k=0 \\ 0, & if \quad d_k>n_k \end{cases}$$

For each positive integer n_k, we choose a nonempty subset of the set of divisors of n_k, that is $B(n_k)=\{d_k : \ d_k \le n_k \ \wedge B(n,d)=1\}$.Let

$$B(n,d)=\prod_{d_k\le n_k}\binom{n_k}{d_k}=\begin{cases} 1, & if \qquad d_k\in B(n_k) \\ 0, & if \qquad d_k\notin B(n_k) \end{cases}$$

With this for arithmetical functions f and g, we define B-Binomial convolution as;

$$b_{(f*_B g)}(n_k) = \sum_{d_k \leq n_k} f(d_k) g\left(\frac{n_k}{d_k}\right) B(n,d). \qquad ($$

$b_{(f*_B g)}(n_k)$ generalize the Dirichlet convolution. In fact, providing $d_k \in B(n_k)$, from equation (1) we directly obtain

$$(f *_B g)(n_k) = \sum_{d_k \in B(n_k)} f(d_k) g\left(\frac{n_k}{d_k}\right).$$

Theorem 1. *For fixed* n_k *, Equation* (1) *expresses* $b_{(f*_B g)}(n_k)$ *as a dirichlet convolution,*

$$b_{(f*_B g)}(n_k) = (a_{B'} * \mu^{-1})(n_k),$$

where

$$a_{B'}(d_k) = f(d_k) g\left(\frac{n_k}{d_k}\right) B(n,d).$$

Proof.

$$\begin{aligned}
(a_{B'} * \mu^{-1})(n_k) &= \sum_{d_k \leq n_k} a_{B'}(d_k) \mu^{-1}\left(\frac{n_k}{d_k}\right) \\
&= \sum_{d_k \leq n_k} f(d_k) g\left(\frac{n_k}{d_k}\right) B(n,d) \mu^{-1}\left(\frac{n_k}{d_k}\right) \\
&= \sum_{d_k \leq n_k} f(d_k) g\left(\frac{n_k}{d_k}\right) B(n,d) \\
&= b_{(f*_B g)}(n_k)
\end{aligned}$$

which proves Theorem 1.

Theorem 2. *For fixed* n_k, *Equation* (1) *expresses* $b_{(f*_B g)}(n_k)$ *as a Drichlet convolution*

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$$b_{(f*_B g)}(n_k) = (a_{B''} * g)(n_k),$$

where

$$a_{B''}(d_k) = f(d_k) B(n,d).$$

Proof.

$$\begin{aligned}(a_{B^{"}} * g)(n_k) &= \sum_{d_k \leq n_k} a_{B^{"}}(d_k) g\left(\frac{n_k}{d_k}\right) \\ &= \sum_{d_k \leq n_k} f(d_k) B(n,d) g\left(\frac{n_k}{d_k}\right) \\ &= \sum_{d_k \leq n_k} f(d_k) g\left(\frac{n_k}{d_k}\right) B(n,d) \\ &= b_{(f *_B g)}(n_k)\end{aligned}$$

which proves Theorem 2.

Definition 1. *If k is a nonnegative integer, the function ζ_k is defined by $\zeta_k(n) = n^k$.The function $\zeta = \zeta_0$ is called the zeta function and for all n, $\zeta(n) = 1$ [4]*

Definition 2. *For $n = \prod_k p_k^{a_k}$, Liouville function is defined by $\lambda(1) = 1$ and $\lambda(n) = (-1)^{a_1+a_2+\ldots+a_k}$ [1].*

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Theorem 3. $\zeta^{-1} = \lambda$ *where $\zeta(n) = 1$ [2].*

The following inversion theorem for B-Binomial convolution is given by Haukkanen and λ Liouville function is served as μ Mobius function in classical inversion theorem *[3].*

Theorem 4. *(Inversion Theorem) For all n;*

$$f(n) = \sum_{d|n} B(n,d) g(d)$$

if and only if

$$g(n) = \sum_{d|n} B(n,d) f(d) \lambda\left(\frac{n}{d}\right).$$

[4].

2. RAMANUJAN SUMS FOR B-BINOMIAL CONVOLUTION

Definition 3. *For any integers a_k and b_k , let $e(a_k\ , b_k\ ) = e^{\frac{2\pi i a_k}{b_k}}$ and let n_k be an integer positive, negative or zero and r_k be a positive integer. Then Ramanujan Sum for B-Binomial convolution is defined by,*

$$C_B(n_k, r_k) = \sum_{(n_k, r_k)_b = 1} e(n_k x_k, r_k)$$

It should be noted that the sum is taken over all x_k such that $1 \leq x_k \leq r_k$ and $(x_k, r_k) = 1$, but it could be over any reduced residue system $(Mod\ r_k)$. This is becouse, if $x_k \equiv x_k^{'}$ $(Mod\ r_k)$ then $e(n_k x_k, r_k) = e(n_k x_k^{'}, r_k)$.

For fixed r_k, and with n_k restricted to the positive integers, we obtain an arithmetical function $C_B(., r_k)$.

On the other hand, for fixed n_k ,we obtain an arithmetical function $C_B(n_k, .)$.

Classical Ramanujan Sum has following property,

Proposition 5. *For all n and r,*

$$C(n, r) = \sum_{d|(n,r)} d\mu\left(\frac{r}{d}\right)$$

[5].

We introduce and prove an analogue property of the classical Ramanujan Sum, called Ramanujan Sum for B-Binomial Convolution in our M.s. Thesis.

Theorem 6. *For all n_k and r_k,*

$$C_B(n_k, r_k) = \sum_{\substack{d_k \leq (n_k, r_k) \\ d_k \in B(n_k)}} d_k \lambda_B \left(\frac{n_k}{d_k}\right)$$

[4].

In Theorem 6 when $r_k = 0$, we obtain Euler's function related to Ramanujan Sum. Indeed

$$C_B(n_k, r_k) = \sum_{\substack{d_k \leq (n_k, r_k) \\ d_k \in B(n_k)}} d_k \lambda_B \left(\frac{n_k}{d_k}\right)$$

$$C_B(n_k, r_k) = \sum_{\substack{d_k \leq (n_k, 0) \\ d_k \in B(n_k)}} d_k \lambda_B \left(\frac{n_k}{d_k}\right)$$

$$C_B(n_k, r_k) = \sum_{\substack{d_k \leq n_k \\ d_k \in B(n_k)}} d_k \lambda_B \left(\frac{n_k}{d_k}\right)$$

$$C_B(n_k, 0) = \varphi_B(n_k).$$

In the following theorem we prove some basic properties of Ramanujan Sum for B-Binomial convolution.

Definition 4. *Let h and g be multiplicative arithmetical functions and consider the sum.*

$$s_{(f*_Bg)}(n_k, r_k) = \sum_{\substack{d_k \leq (n_k, r_k) \\ d_k \in B(n_k)}} B(n, d)h(d_k)g\left(\frac{n_k}{d_k}\right)\lambda_B\left(\frac{n_k}{d_k}\right)$$

where n_k is an integer and r_k is a positive integer. For B-Binomial convolution associated arithmetical function S is defined by

$$S(n_k) = s_{(f*_Bg)}(n_k, 0) \qquad \text{for all } n_k.$$

Theorem 7. *Let $h = \zeta_1$ and $g = \zeta_0$ .Then*

$$s_{(f*_Bg)}(n_k, r_k) = C_B(n_k, r_k), \qquad S(n_k) = \varphi_B(n_k).$$

Proof. For $h = \zeta_1 = d_k$ and $g = \zeta_0 = 1$

$$s_{(f*_Bg)}(n_k, r_k) = \sum_{\substack{d_k \leq (n_k, r_k) \\ d_k \in B(n_k)}} B(n, d)h(d_k)g\left(\frac{n_k}{d_k}\right)\lambda_B\left(\frac{n_k}{d_k}\right)$$

$$= \sum_{\substack{d_k \leq (n_k, r_k) \\ d_k \in B(n_k)}} 1.d_{k.}1.\lambda_B\left(\frac{n_k}{d_k}\right)$$

$$= C_B(n_k, r_k).$$

Now again for $h = \zeta_1 = d_k$ and $g = \zeta_0 = 1$ and $r_k = 0$; we clearly obtain $(n_k, r_k) = (n_k, 0) = n_{k.}$

$$S(n_k) = s_{(f*_Bg)}(n_k, r_k) = \sum_{\substack{d_k \leq (n_k, r_k) \\ d_k \in B(n_k)}} B(n,d)h(d_k)g\left(\frac{n_k}{d_k}\right)\lambda_B \left(\frac{n_k}{d_k}\right)$$

$$= \sum_{\substack{d_k \leq (n_k, 0) \\ d_k \in B(n_k)}} B(n,d)h(d_k)g\left(\frac{n_k}{d_k}\right)\lambda_B \left(\frac{n_k}{d_k}\right)$$

$$= \sum_{\substack{d_k \leq n_k \\ d_k \in B(n_k)}} d_k\lambda_B \left(\frac{n_k}{d_k}\right)$$

$$S(n_k) = \varphi_B(n_k)$$

which proves theorem 6. Here, we obtain at the same time as a bonus $S(n_k) = s_{(f*_Bg)}(n_k, 0) = C_B(n_k, 0) = \varphi_B(n_k)$. ∎

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