B-Binomial Convolution and Associated Arithmetical Functions and Their Properties

Selin INAG and Hasan SENAY

Secondary Mathematics Teaching Program
Graduate School of Natural and Applied Sciences
Education Faculty, Selcuk University, Konya, Turkey
inag_s@hotmail.com, hsenay@selcuk.edu.tr

Abstract. In this paper we considered Apostol’s article [2] and we defined for B-Binomial convolution associated arithmetical function S. We show that this function and B-Binomial convolution associated with Ramanujan Sums are connected.

Keywords: Arithmetic functions, Convolution, Binomial convolution, Ramanujan Sum

1. INTRODUCTION

Let \( n \) and \( d \)’s standart forms are \( n = \prod p_k^{n_k} \) and \( d = \prod p_k^{d_k} \) respectively. We know that \( d \mid n \) if and only if \( d_k \leq n_k \) [6]. Let \( \binom{n_k}{d_k} \) be classical Binomial coefficient. For positive integer \( n \) and \( d \) is a divisor of \( n \), the function \( B(n, d) \) is defined by,

\[
B(n, d) = \prod_k \binom{n_k}{d_k} = \begin{cases} 1, & \text{if } d_k = n_k \lor d_k = 0 \\ 0, & \text{if } d_k > n_k \end{cases}
\]

For each positive integer \( n_k \), we choose a nonempty subset of the set of divisors of \( n_k \), that is \( B(n_k) = \{ d_k : d_k \leq n_k \land B(n, d) = 1 \} \). Let

\[
B(n, d) = \prod_{d_k \leq n_k} \binom{n_k}{d_k} = \begin{cases} 1, & \text{if } d_k \in B(n_k) \\ 0, & \text{if } d_k \notin B(n_k) \end{cases}
\]

With this for arithmetical functions \( f \) and \( g \), we define B-Binomial convolution as;
\[ b_{(f*B^g)}(n_k) = \sum_{d_k \leq n_k} f(d_k)g\left(\frac{n_k}{d_k}\right) B(n,d). \]

\( b_{(f*B^g)}(n_k) \) generalize the Dirichlet convolution. In fact, providing \( d_k \in B(n_k) \), from equation (1) we directly obtain

\[ (f*B^g)(n_k) = \sum_{d_k \in B(n_k)} f(d_k)g\left(\frac{n_k}{d_k}\right). \]

**Theorem 1.** For fixed \( n_k \), Equation (1) expresses \( b_{(f*B^g)}(n_k) \) as a Dirichlet convolution,

\[ b_{(f*B^g)}(n_k) = (a_{B'} \ast \mu^{-1})(n_k), \]

where

\[ a_{B'}(d_k) = f(d_k)g\left(\frac{n_k}{d_k}\right) B(n,d). \]

**Proof.**

\[ (a_{B'} \ast \mu^{-1})(n_k) = \sum_{d_k \leq n_k} a_{B'}(d_k)\mu^{-1}\left(\frac{n_k}{d_k}\right) \]

\[ = \sum_{d_k \leq n_k} f(d_k)g\left(\frac{n_k}{d_k}\right) B(n,d)\mu^{-1}\left(\frac{n_k}{d_k}\right) \]

\[ = \sum_{d_k \leq n_k} f(d_k)g\left(\frac{n_k}{d_k}\right) B(n,d) \]

\[ = b_{(f*B^g)}(n_k) \]

which proves Theorem 1.

**Theorem 2.** For fixed \( n_k \), Equation (1) expresses \( b_{(f*B^g)}(n_k) \) as a Dirichlet convolution

\[ b_{(f*B^g)}(n_k) = (a_{B'} \ast g)(n_k), \]

where

\[ a_{B'}(d_k) = f(d_k)B(n,d). \]
Proof.

\[(a_B \ast g)(n_k) = \sum_{d_k \leq n_k} a_{B^*}(d_k) g\left(\frac{n_k}{d_k}\right)\]

\[= \sum_{d_k \leq n_k} f(d_k) B(n, d) g\left(\frac{n_k}{d_k}\right)\]

\[= \sum_{d_k \leq n_k} f(d_k) B\left(\frac{n_k}{d_k}\right) B(n, d)\]

\[= b_{(f \ast B g)}(n_k)\]

which proves Theorem 2.

Definition 1. If \( k \) is a nonnegative integer, the function \( \zeta_k \) is defined by \( \zeta_k(n) = n^k \). The function \( \zeta = \zeta_0 \) is called the zeta function and for all \( n \), \( \zeta(n) = 1 \) \[4\]

Definition 2. For \( n = \prod p_k^{a_k} \), Liouville function is defined by \( \lambda(1) = 1 \) and \( \lambda(n) = (-1)^{a_1 + a_2 + \ldots + a_k} \) \[1\].

Theorem 3. \( \zeta^{-1} = \lambda \) where \( \zeta(n) = 1 \) \[2\].

The following inversion theorem for B-Binomial convolution is given by Haukkanen and \( \lambda \) Liouville function is served as \( \mu \) Mobius function in classical inversion theorem \[3\].

Theorem 4. (Inversion Theorem) For all \( n \);

\[f(n) = \sum_{d \mid n} B(n, d) g(d)\]

if and only if

\[g(n) = \sum_{d \mid n} B(n, d) f(d) \lambda\left(\frac{n}{d}\right)\].

\[4\].

2. RAMANUJAN SUMS FOR B-BINOMIAL CONVOLUTION

Definition 3. For any integers \( a_k \) and \( b_k \), let \( e(a_k, b_k) = e^{\frac{2\pi i a_k}{b_k}} \) and let \( n_k \) be an integer positive, negative or zero and \( r_k \) be a positive integer. Then Ramanujan Sum for B-Binomial convolution is defined by,
\[ C_B(n_k, r_k) = \sum_{(n_k, r_k) = 1} e(n_k x_k, r_k) \]

It should be noted that the sum is taken over all \( x_k \) such that \( 1 \leq x_k \leq r_k \) and \((x_k, r_k) = 1\), but it could be over any reduced residue system \((\text{Mod } r_k)\). This is because, if \( x_k \equiv x'_k \pmod{r_k} \) then \( e(n_k x_k, r_k) = e(n_k x'_k, r_k) \).

For fixed \( r_k \), and with \( n_k \) restricted to the positive integers, we obtain an arithmetical function \( C_B(., r_k) \).

On the other hand, for fixed \( n_k \), we obtain an arithmetical function \( C_B(n_k, .) \).

Classical Ramanujan Sum has following property,

**Proposition 5.** For all \( n \) and \( r \),

\[ C(n, r) = \sum_{d|(n, r)} d \mu \left( \frac{r}{d} \right) \]

[5].

We introduce and prove an analogue property of the classical Ramanujan Sum, called Ramanujan Sum for B-Binomial Convolution in our M.s. Thesis.

**Theorem 6.** For all \( n_k \) and \( r_k \),

\[ C_B(n_k, r_k) = \sum_{d_k \leq (n_k, r_k)} d_k \lambda_B \left( \frac{n_k}{d_k} \right) \]

[4].

In Theorem 6 when \( r_k = 0 \), we obtain Euler’s function related to Ramanujan Sum. Indeed

\[ C_B(n_k, r_k) = \sum_{d_k \leq (n_k, r_k)} d_k \lambda_B \left( \frac{n_k}{d_k} \right) \]

\[ C_B(n_k, r_k) = \sum_{d_k \leq (n_k, 0)} d_k \lambda_B \left( \frac{n_k}{d_k} \right) \]

\[ C_B(n_k, r_k) = \sum_{d_k \leq n_k} d_k \lambda_B \left( \frac{n_k}{d_k} \right) \]

\[ C_B(n_k, r_k) = \sum_{d_k \leq n_k} d_k \lambda_B \left( \frac{n_k}{d_k} \right) \]
$C_B(n_k, 0) = \varphi_B(n_k)$.

In the following theorem we prove some basic properties of Ramanujan Sum for B-Binomial convolution.

**Definition 4.** Let $h$ and $g$ be multiplicative arithmetical functions and consider the sum.

$$s_{(f*Bg)}(n, r) = \sum_{d_k \leq (n, r), d_k \in B(n) \atop d_k \in B(n)} B(n, d) h(d) g\left(\frac{n_k}{d_k}\right) \lambda_B\left(\frac{n_k}{d_k}\right)$$

where $n_k$ is an integer and $r_k$ is a positive integer. For B-Binomial convolution associated arithmetical function $S$ is defined by

$$S(n_k) = s_{(f*Bg)}(n_k, 0)$$

for all $n_k$.

**Theorem 7.** Let $h = \zeta_1$ and $g = \zeta_0$. Then

$$s_{(f*Bg)}(n, r) = C_B(n, r), \quad S(n_k) = \varphi_B(n_k).$$

**Proof.** For $h = \zeta_1 = d_k$ and $g = \zeta_0 = 1$

$$s_{(f*Bg)}(n_k, r_k) = \sum_{d_k \leq (n_k, r_k), d_k \in B(n)} B(n, d) h(d) g\left(\frac{n_k}{d_k}\right) \lambda_B\left(\frac{n_k}{d_k}\right)$$

$$= \sum_{d_k \leq (n_k, r_k), d_k \in B(n)} 1. d_k \lambda_B\left(\frac{n_k}{d_k}\right)$$

$$= C_B(n_k, r_k).$$

Now again for $h = \zeta_1 = d_k$ and $g = \zeta_0 = 1$ and $r_k = 0$; we clearly obtain $(n_k, r_k) = (n_k, 0) = n_k$. 
\[ S(n_k) = s_{(f \ast B g)}(n_k, r_k) = \sum_{d_k \leq (n_k, r_k) \atop d_k \in B(n_k)} B(n, d) h(d_k) g \left( \frac{n_k}{d_k} \right) \lambda_B \left( \frac{n_k}{d_k} \right) \]

\[ = \sum_{d_k \leq (n_k, 0) \atop d_k \in B(n_k)} B(n, d) h(d_k) g \left( \frac{n_k}{d_k} \right) \lambda_B \left( \frac{n_k}{d_k} \right) \]

\[ = \sum_{d_k \leq n_k \atop d_k \in B(n_k)} d_k \lambda_B \left( \frac{n_k}{d_k} \right) \]

\[ S(n_k) = \varphi_B(n_k) \]

which proves theorem 6. Here, we obtain at the same time as a bonus
\[ S(n_k) = s_{(f \ast B g)}(n_k, 0) = C_B(n_k, 0) = \varphi_B(n_k). \]

REFERENCES


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