Tauberian Conditions in Terms of General Control Modulo of Oscillatory Behavior of Integer Order of Sequences

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Abstract

Let (u_n) be a sequence of real numbers and let L be any $(C, 1)$ regular and additive limitable method. In this paper we prove that if the classical control modulo of the oscillatory behavior of (u_n) belonging to some space of sequences is a Tauberian condition for L , then convergence or $(C, 1)$ convergence of (u_n) out of L -limitability of (u_n) is recovered depending on the conditions on the general control modulo of the oscillatory behavior of integer order m of (u_n) .

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1 Introduction

Tauber's first theorem [5] asserted that an Abel limitable sequence $u = (u_n)$ is convergent if

$$\omega_n^{(0)}(u) = n\Delta u_n. \quad (1)$$

To describe this, we say that the condition (1) is a Tauberian condition for the Abel limitable method. Tauber [5] proved further that the weaker condition

$$\sigma_n^{(1)}(\omega^{(0)}(u)) = \frac{1}{n+1} \sum_{k=0}^n k\Delta u_k \quad (2)$$

is also a Tauberian condition for the Abel limitable method. Meyer-König and Tietz [3] showed that if (1) is a Tauberian condition for any regular and additive method L , then (2) is a Tauberian condition for L . Recently, Çanak,

Dik and Dik [1] have used the concept of the general control modulo defined by Stanojević [4] and obtained several results which state that for specially choosen spaces $\mathcal{A}$ and $\mathcal{B}$ of sequences, if $(\omega_n^{(0)}(u)) \in \mathcal{A}$ is Tauberian condition for any $(C, 1)$ regular and additive method L , then $(\omega_n^{(1)}(u)) \in \mathcal{B}$ is a Tauberian condition for L . Çanak and Totur [2] extended the results in [1] and proved that if $(\omega_n^{(0)}(u)) \in \mathcal{A}$ is Tauberian condition for any $(C, 1)$ regular and additive method L , then for any integer $m \geq 1$, $(\omega_n^{(m)}(u)) \in \mathcal{B}$ is a Tauberian condition for L .

In this paper we prove that if $(\omega_n^{(0)}(u))$ belonging to some space of sequences is a Tauberian condition for L , then convergence or $(C, 1)$ convergence of (u_n) out of L -limitability of (u_n) is recovered depending on the conditions on $(\omega_n^{(m)}(u))$ for any integer $m \geq 1$.

2 Definitions and Notations

For a sequence $u = (u_n)$ and for each integer $m \geq 1$ we define $(n\Delta)_m u_n = n\Delta((n\Delta)_{m-1} u_n) = (n\Delta)_{m-1}(n\Delta u_n)$ where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n := n(u_n - u_{n-1})$. The symbol $u_n = o(1)$ means that $u_n \rightarrow 0$ as $n \rightarrow \infty$. The classical control modulo of the oscillatory behavior of (u_n) is denoted by $\omega_n^{(0)}(u) = n\Delta u_n$. The general control modulo of the oscillatory behavior of order m of (u_n) is defined inductively by

$$\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega_n^{(m-1)}(u))$$

where $\sigma_n^{(1)}(u) = \frac{1}{n+1} \sum_{k=0}^n u_k$. The Kronecker identity $u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u)$ where $V_n^{(0)}(\Delta u) = \frac{1}{n+1} \sum_{k=0}^n k\Delta u_k$ is well known and used in various steps of proofs of theorems. For each integer $m \geq 1$ and for all nonnegative integers n , we inductively define sequences related to (u_n) such as $V_n^{(m)}(\Delta u) = \sigma_n^{(1)}(V_n^{(m-1)}(\Delta u))$ and $\sigma_n^{(m)}(u) = \sigma_n^{(1)}(\sigma_n^{(m-1)}(u))$ where $\sigma_n^{(0)}(u_n) = u_n$. A sequence (u_n) is Abel limitable to s if $\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n = s$ and $(C, 1)$ limitable to s if $\lim_n \sigma_n^{(1)}(u) = s$. A sequence (u_n) is slowly oscillating if $\lim_{\lambda \rightarrow 1^+} \overline{\lim}_n \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{j=n+1}^k \Delta u_j \right| = 0$ where $[\lambda n]$ denotes the integer part of λn . A sequence (u_n) is moderately oscillatory if for $\lambda > 1$, $\overline{\lim}_n \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{j=n+1}^k \Delta u_j \right| < +\infty$. Let L be any limitable method. If $u = (u_n)$ is L limitable to s , we write $L - \lim_n u_n = s$. A limitation method L is called additive if $L - \lim_n u_n = s$ and $L - \lim_n v_n = t$ imply $L - \lim_n (u_n + v_n) = s + t$. The limitable method L is said to be regular if $\lim_n u_n = s$ implies $L - \lim_n u_n = s$. The limitable method L is said to be $(C, 1)$ regular if $L - \lim_n u_n = s$ implies $L - \lim_n \sigma_n^{(1)}(u) = s$. That every regular limitable method is $(C, 1)$ regular is clear. A condition on (u_n) is $(C, 1)$ Tauberian condition for L if we recover only convergence of $(C, 1)$ means of (u_n) out of L -limitability of (u_n) and the condition on (u_n) .

3 Main Results

Throughout this paper we require L to be $(C, 1)$ regular and additive. Let $\mathcal{B}$, $\mathcal{N}$, $\mathcal{S}$ and $\mathcal{M}$ denote the space of sequences bounded, converging to 0, slowly oscillating and moderately oscillating, respectively. Denote by $\mathcal{A}$ the space of sequences such that $\mathcal{N} \subseteq \mathcal{A}$.

We prove the following theorems.

Theorem 3.1 *If $(\omega_n^{(0)}(u)) \in \mathcal{A}$ is a Tauberian condition for L , then $(\sigma_n^{(1)}(\omega^{(0)}(u))) \in \mathcal{A}$ is a Tauberian condition for L .*

Theorem 3.2 *If $(\omega_n^{(0)}(u)) \in \mathcal{A}$ is a Tauberian condition for L , then for any integer $m \geq 1$, $(\omega_n^{(m)}(u)) \in \mathcal{A}$ is also a Tauberian condition for L .*

Theorem 3.3 *Let $\mathcal{U}$ and $\mathcal{V}$ be spaces of sequences such that $(u_n - \sigma_n^{(1)}(u)) \in \mathcal{U}$ for every $(u_n) \in \mathcal{V}$ and $\mathcal{N} \subseteq \mathcal{U}$. If $(\omega_n^{(0)}(u)) \in \mathcal{U}$ is a Tauberian condition for L , then for any integer $m \geq 1$, $(\omega_n^{(m)}(u)) \in \mathcal{V}$ is a Tauberian condition for L .*

Theorem 3.4 *Let $\mathcal{U}$ and $\mathcal{V}$ be spaces of sequences such that $(\sigma_n^{(1)}(u)) \in \mathcal{U}$ for every $(u_n) \in \mathcal{V}$ and $\mathcal{N} \subseteq \mathcal{U}$. If $(\omega_n^{(0)}(u)) \in \mathcal{U}$ is a $(C, 1)$ Tauberian condition for L , then for any integer $m \geq 1$, $(\omega_n^{(m)}(u)) \in \mathcal{V}$ is a Tauberian condition for L .*

4 Proof of Theorems

To prove this theorems, we need the following observation.

Observation 4.1 *For each integer $m \geq 1$, $\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u)$.*

Proof We do the proof by induction. By definition, for $m = 1$ we have $\omega_n^{(1)}(u) = \omega_n^{(0)}(u) - \sigma_n^{(1)}(\omega^{(0)}(u)) = n\Delta u_n - V_n^{(0)}(\Delta u) = n\Delta V_n^{(0)}(\Delta u)$. Assume the observation is true for $m = k$. That is, assume that

$$\omega_n^{(k)}(u) = (n\Delta)_k V_n^{(k-1)}(\Delta u). \quad (3)$$

We must show that the observation is true for $m = k + 1$. That is, we must show that $\omega_n^{(k+1)}(u) = (n\Delta)_{k+1} V_n^{(k)}(\Delta u)$. Again by definition, $\omega_n^{(k+1)}(u) = \omega_n^{(k)}(u) - \sigma_n^{(1)}(\omega^{(k)}(u))$. By (3),

$$\begin{aligned} \omega_n^{(k+1)}(u) &= (n\Delta)_k V_n^{(k-1)}(\Delta u) - (n\Delta)_k V_n^{(k)}(\Delta u) &= (n\Delta)_k (V_n^{(k-1)}(\Delta u) - V_n^{(k)}(\Delta u)) \\ & &= (n\Delta)_k (n\Delta V_n^{(k)}(\Delta u)) \\ & &= (n\Delta)_{k+1} V_n^{(k)}(\Delta u). \end{aligned}$$

Thus, we conclude that observation is true for each integer $m \geq 1$.

Proof of Theorem 3.1 Assume that $(\omega_n^{(0)}(u)) \in \mathcal{A}$ is a Tauberian condition for L . Let $(\sigma_n^{(1)}(\omega^{(0)}(u))) \in \mathcal{A}$ and $L - \lim_n u_n = s$. Since L is $(C, 1)$ regular, we have $L - \lim_n V_n^{(0)}(\Delta u) = s$. From the identity $\sigma_n^{(1)}(\omega^{(0)}(u)) = n\Delta V_n^{(0)}(\Delta u)$, we deduce that $V_n^{(0)}(\Delta u) = 0$ by the assumption. Since $\mathcal{N} \subseteq \mathcal{A}$, we conclude that $(V_n^{(0)}(\Delta u)) = (n\Delta\sigma_n^{(1)}(u)) \in \mathcal{A}$. It then follows that $\lim_n \sigma_n^{(1)}(u) = s$ by $(C, 1)$ regularity of L . By Kronecker identity, we obtain $\lim_n u_n = s$.

In the case $\mathcal{A} = \mathcal{N}$, we have Meyer-Konig and Tietz's theorem [3]. In Theorem 3.1, $\mathcal{A}$ can be also taken as $\mathcal{B}$, $\mathcal{S}$ or $\mathcal{M}$.

Proof of Theorem 3.2 Assume that $(\omega_n^{(0)}(u)) \in \mathcal{A}$ is a Tauberian condition for L . Let $(\omega_n^{(m)}(u)) \in \mathcal{A}$. Since $L - \lim_n (n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) = 0$,

$$(n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) = o(1) \quad (4)$$

by assumption. Since $L - \lim_n (n\Delta)_{m-2} V_n^{(m-1)}(\Delta u) = 0$, by (4) we obtain

$$(n\Delta)_{m-2} V_n^{(m-1)}(\Delta u) = o(1). \quad (5)$$

From the identity $(n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) = (n\Delta)_{m-2} V_n^{(m-2)}(\Delta u) - (n\Delta)_{m-2} V_n^{(m-1)}(\Delta u)$, (4) and (5), we conclude that

$$(n\Delta)_{m-2} V_n^{(m-2)}(\Delta u) = o(1). \quad (6)$$

Similarly, we obtain

$$(n\Delta)_{m-3} V_n^{(m-3)}(\Delta u) = o(1). \quad (7)$$

Continuing in this vein, we deduce that

$$V_n^{(0)}(\Delta u) = o(1). \quad (8)$$

Since $V_n^{(0)}(\Delta u) = n\Delta\sigma_n^{(1)}(u) = o(1)$ and $L - \lim_n \sigma_n^{(1)}(u) = L - \lim_n u_n$, we have

$$\lim_n \sigma_n^{(1)}(u) = L - \lim_n u_n \quad (9)$$

by assumption. This completes the proof.

Proof of Theorem 3.3 Assume that $(\omega_n^{(0)}(u)) \in \mathcal{U}$ is a Tauberian condition for L . Let $(\omega_n^{(m)}(u)) \in \mathcal{V}$. Then

$$(n\Delta((n\Delta)_m V_n^{(m)}(\Delta u))) \in \mathcal{U}. \quad (10)$$

Since $L - \lim_n (n\Delta)_m V_n^{(m)}(\Delta u) = 0$,

$$(n\Delta)_m V_n^{(m)}(\Delta u) = o(1) \quad (11)$$

by assumption. Since $L - \lim_n (n\Delta)_{m-1} V_n^{(m)}(\Delta u) = 0$, by (11) we obtain

$$(n\Delta)_{m-1} V_n^{(m)}(\Delta u) = o(1). \quad (12)$$

From the identity $(n\Delta)_m V_n^{(m)}(\Delta u) = (n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) - (n\Delta)_{m-1} V_n^{(m)}(\Delta u)$, (11) and (12), we have

$$(n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) = o(1). \quad (13)$$

Continuing in this vein, we deduce that

$$V_n^{(0)}(\Delta u) = o(1). \quad (14)$$

Since $V_n^{(0)}(\Delta u) = n\Delta\sigma_n^{(1)}(u) = o(1)$ and $L - \lim_n \sigma_n^{(1)}(u) = L - \lim_n u_n$, we conclude

$$\lim_n \sigma_n^{(1)}(u) = L - \lim_n u_n \quad (15)$$

by assumption. This completes the proof.

As special cases to sequence spaces in Theorem 3.3, spaces $\mathcal{U}$ and $\mathcal{V}$ can be taken as follows.

- i) $\mathcal{U} = \mathcal{B}, \mathcal{V} = \mathcal{S}$
- ii) $\mathcal{U} = \mathcal{B}, \mathcal{V} = \mathcal{M}$
- iii) $\mathcal{U} = \mathcal{V} = \mathcal{B}$
- iv) $\mathcal{U} = \mathcal{V} = \mathcal{N}$.

Proof of Theorem 3.4 Assume that $(\omega_n^{(0)}(u)) \in \mathcal{U}$ is a Tauberian condition for L . Let $(\omega_n^{(m)}(u)) \in \mathcal{V}$. Then

$$(\sigma_n^{(1)}(\omega_n^{(m)}(u))) = ((n\Delta)_m V_n^{(m)}(\Delta u)) \in \mathcal{U}. \quad (16)$$

Additivity and (C,1) regularity of L and $L - \lim_n u_n = s$ imply $L - \lim_n (n\Delta)_{m-1} V_n^{(m)}(\Delta u) = 0$. Since $(n\Delta)((n\Delta)_{m-1} V_n^{(m)}(\Delta u)) \in \mathcal{U}$, we obtain

$$(n\Delta)_{m-1} V_n^{(m)}(\Delta u) = o(1) \quad (17)$$

by assumption. From the identity $(n\Delta)_m V_n^{(m)}(\Delta u) = (n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) - (n\Delta)_{m-1} V_n^{(m)}(\Delta u)$, (16) and (17), it follows that $((n\Delta)_{m-1} V_n^{(m-1)}(\Delta u)) \in \mathcal{U}$. Additivity and (C,1) regularity of L and $L - \lim_n u_n = s$ imply $L - \lim_n (n\Delta)_{m-2} V_n^{(m-1)}(\Delta u) = 0$. Since $(n\Delta)((n\Delta)_{m-2} V_n^{(m-1)}(\Delta u)) \in \mathcal{U}$, we conclude by assumption that

$$(n\Delta)_{m-2} V_n^{(m-1)}(\Delta u) = o(1). \quad (18)$$

Additivity and (C,1) regularity of L and $L - \lim_n u_n = s$ imply $L - \lim_n (n\Delta)_{m-3} V_n^{(m-1)}(\Delta u) = 0$. Since $(n\Delta)((n\Delta)_{m-3} V_n^{(m-1)}(\Delta u)) \in \mathcal{U}$, we deduce by assumption that

$$(n\Delta)_{m-3} V_n^{(m-1)}(\Delta u) = o(1). \quad (19)$$

From the identity $(n\Delta)_{m-2}V_n^{(m-1)}(\Delta u) = (n\Delta)_{m-3}V_n^{(m-2)}(\Delta u) - (n\Delta)_{m-3}V_n^{(m-1)}(\Delta u)$, (18) and (19), it follows that

$$(n\Delta)_{m-3}V_n^{(m-2)}(\Delta u) = o(1). \quad (20)$$

Continuing in this vein, we have

$$n\Delta V_n^{(2)}(\Delta u) = o(1). \quad (21)$$

From $L - \lim_n V_n^{(2)}(\Delta u) = 0$, we have by assumption

$$V_n^{(2)}(\Delta u) = o(1). \quad (22)$$

From the identity $n\Delta V_n^{(2)}(\Delta u) = V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u)$, (21) and (22), we have

$$V_n^{(1)}(\Delta u) = o(1). \quad (23)$$

Since $V_n^{(1)}(\Delta u) = n\Delta\sigma_n^{(2)}(u) = o(1)$ and $L - \lim_n \sigma_n^{(1)}(u) = L - \lim_n \sigma_n^{(2)}(u)$, we have

$$\lim_n \sigma_n^{(2)}(u) = L - \lim_n \sigma_n^{(1)}(u). \quad (24)$$

From the identity $\sigma_n^{(1)}(u) - \sigma_n^{(2)}(u) = V_n^{(1)}(\Delta u)$, (23) and (24), we have

$$\lim_n \sigma_n^{(1)}(u) = L - \lim_n \sigma_n^{(1)}(u). \quad (25)$$

As special cases to sequence spaces in Theorem 3.4, spaces $\mathcal{U}$ and $\mathcal{V}$ can be taken as follows.

- i) $\mathcal{U} = \mathcal{S}, \mathcal{V} = \mathcal{S}$
- ii) $\mathcal{U} = \mathcal{S}, \mathcal{V} = \mathcal{M}$
- iii) $\mathcal{U} = \mathcal{V} = \mathcal{B}$
- iv) $\mathcal{U} = \mathcal{V} = \mathcal{N}$.

References

- [1] İ. Çanak, M. Dik and F. Dik, *On a Theorem of W. Meyer-König and H. Tietz*, Int. J. Math. Math. Sci. **15** (2005), 2491 - 2496.
- [2] İ. Çanak and Ü. Totur, *Tauberian conditions for a general limitable method*, Submitted (2006).
- [3] W. Meyer-König and H. Tietz, *On Tauberian conditions of type o*, Bull. Amer. Math. Soc. **73** (1967), 926 - 927.
- [4] Č. V. Stanojević, *Analysis of Divergence: Applications to the Tauberian Theory*, Graduate Research Seminar, University of Missouri - Rolla (1999).
- [5] A. Tauber, *Ein Satz aus der Theorie der unendlichen Reihen*, Monatsh. f. Math. **8** (1897), 273 - 277.

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