An Interesting Orbit-Induced Partition of the Class of all Groups

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Abstract

Given that the set of endomorphisms of a group is contained in the set of distributive elements of its endomorphism near-ring, which, in turn, is contained in the endomorphism near-ring, we show that the class of all groups is partitioned into four nonempty subclasses when all combinations of these inclusions, proper or non-proper, are considered. Furthermore, a characterization of each subclass is given in terms of the orbits of the underlying group.

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1 Introduction

The endomorphism ring of an abelian group has been studied extensively and over a long period of time. By studying the relationship between the endomorphism ring $End(G)$ and its underlying group G , ring theory has been used to study problems in group theory with great success. In contrast, when the group is nonabelian, $End(G)$ is never additively closed, although it is a semigroup with respect to function composition. However, if we let $(M_0(G), +, \circ)$ denote the near-ring of zero preserving mappings on G , then the subgroup of $(M_0(G), +)$ generated by $End(G)$ is a near-ring, called the endomorphism near-ring of G , and denoted by $\mathcal{E}(G)$. Here, all near-rings are right and zero-symmetric, compatible with the composition of function convention: $(f \circ g)(x) = (fg)(x) = f(g(x))$. Recall that a right near-ring is a triple $(N, +, \cdot)$ such that $(N, +)$ is a not necessarily abelian group, $(N, \cdot)$ is a semigroup, and $(n + m) \cdot r = n \cdot r + m \cdot r$, for all $n, m, r \in N$.

We now identify an important substructure of $\mathcal{E}(G)$: the set of distributive elements of $\mathcal{E}(G)$. Let A and B be sets of functions with ranges in a groupoid $(T, +)$, and let f be a function with a range also in T . We say f is *left distributive over* (A, B) if $f(\alpha + \beta) = f\alpha + f\beta$, for each $\alpha \in A$ and $\beta \in B$. The set of all functions that are left distributive over (A, B) will be denoted by ${}_d(A, B)$. If $A = B$, we write ${}_dA$ instead of ${}_d(A, A)$. In a similar fashion, we define their right-sided analogs: $(A, B)_d$ and A_d . Let $\mathcal{D}_l(A) = {}_dA \cap A$, the set of left distributive elements of A . Similarly, let $\mathcal{D}_r(A) = A_d \cap A$, the set of right distributive elements of A . Let $\mathcal{D}(A) = {}_dA \cap A_d \cap A$, the set of distributive elements of A . A near-ring is *distributive* if each element is distributive. A near-ring is *distributively generated* (abbreviated d.g.) if it is additively generated by a set of distributive elements. Thus, $\mathcal{E}(G)$ is a d.g. near-ring since each element of $\text{End}(G)$ is distributive.

If $\mathcal{E}(G)$ is our set of functions, then $\mathcal{D}(\mathcal{E}(G)) = \mathcal{D}_l(\mathcal{E}(G))$, since we automatically have $\mathcal{D}_r(\mathcal{E}(G)) = \mathcal{E}(G)$, as our near-rings are right near-rings. Note that the following containments always hold for any group G :

$$\text{End}(G) \subseteq \mathcal{D}(\mathcal{E}(G)) \subseteq \mathcal{E}(G).$$

Ostensibly, the class of all groups is partitioned into four subclasses corresponding to how $\mathcal{D}(\mathcal{E}(G))$ “sits” between $\text{End}(G)$ and $\mathcal{E}(G)$. In particular, $\mathcal{D}(\mathcal{E}(G))$ may coincide with both $\text{End}(G)$ and $\mathcal{E}(G)$, with $\mathcal{E}(G)$ only, with $\text{End}(G)$ only, or with neither $\text{End}(G)$ nor $\mathcal{E}(G)$. The obvious question is: are these four subclasses nonempty? The result “ $\mathcal{E}(G) = \text{End}(G)$ if and only if G is abelian,” a well-known result with an obscure pedigree, shows us that one of the subclasses is nonempty. In the middle of the 20th Century, Fröhlich [9] proved that “ $\mathcal{E}(G) = \mathcal{D}(\mathcal{E}(G))$ if and only if G is an E -group.” (An E -group was originally defined as a group G in which each element commutes with any of its endomorphic images. Now the condition “ $\mathcal{E}(G)$ is a ring” serves as the definition.) Of course, all abelian groups are E -groups. But it was not until 1971 that nonabelian examples were found by Faudree [8], establishing that a second subclass of the alleged partition is nonempty. The open question of whether the remaining two subclasses of groups are nonempty was the motivation for this paper.

2 The Four Subclasses Via Orbits

Let E denote the set of “generalized Endomorphisms” or elements of $\mathcal{E}(G)$. Let D denote the “Distributive elements of $\mathcal{E}(G)$ ” or $\mathcal{D}(\mathcal{E}(G))$. Let H denote the set of “Homomorphisms of $\mathcal{E}(G)$ ” or $\text{End}(G)$. These assignments provide a heuristic for remembering the following definitions.

Definition 2.1. A group G has the $E \rightarrow D$ property if $\mathcal{E}(G) = \mathcal{D}(\mathcal{E}(G))$.

Definition 2.2. A group G has the $D \rightarrow H$ property if $\mathcal{D}(\mathcal{E}(G)) = \text{End}(G)$.

With the aid of these two properties, we now name the four partitioning subclasses as follows:

Definition 2.3. A group G is an E - H group if G has both the $E \rightarrow D$ and $D \rightarrow H$ properties (i.e., $\mathcal{E}(G) = \text{End}(G)$ or $E = H$).

Definition 2.4. A group G is an E - D group if G has the $E \rightarrow D$ property but not the $D \rightarrow H$ property.

Definition 2.5. A group G is a D - H group if G has the $D \rightarrow H$ property but not the $E \rightarrow D$ property.

Definition 2.6. A group G is a D - D group if G has neither the $E \rightarrow D$ nor the $D \rightarrow H$ properties.

For convenience, we present this information in the following alternate form:

$$\begin{aligned} G \text{ is an } E\text{-}H \text{ group} &\iff \text{End}(G) = \mathcal{D}(\mathcal{E}(G)) = \mathcal{E}(G); \\ G \text{ is an } E\text{-}D \text{ group} &\iff \text{End}(G) \subsetneq \mathcal{D}(\mathcal{E}(G)) = \mathcal{E}(G); \\ G \text{ is a } D\text{-}H \text{ group} &\iff \text{End}(G) = \mathcal{D}(\mathcal{E}(G)) \subsetneq \mathcal{E}(G); \\ G \text{ is a } D\text{-}D \text{ group} &\iff \text{End}(G) \subsetneq \mathcal{D}(\mathcal{E}(G)) \subsetneq \mathcal{E}(G). \end{aligned}$$

Recently, in the dissertation of Diener [7], a similar problem was considered. He studied distributive elements in centralizer near-rings. In particular, let S be a semigroup of endomorphisms of G and let

$$M_S(G) = \{f : G \rightarrow G \mid f(0) = 0, f\sigma = \sigma f, \forall \sigma \in S\}.$$

Let $\text{End}_S(G) = \text{End}(G) \cap M_S(G)$. Then $\text{End}_S(G) \subseteq \mathcal{D}(M_S(G)) \subseteq M_S(G)$. Diener considered when the containments were proper or nonproper, for certain classes of S and G . This extends previous work done in this area by Maxson and Meldrum [17].

Now we characterize E - H , E - D , D - H and D - D groups in terms of the orbits of G . In this paper, the orbits of G are the fully invariant subgroups of G of the form $\mathcal{E}(G)(a) := \{f(a) : f \in \mathcal{E}(G)\}$, where $a \in G$. The following result of Fröhlich [9, Theorem 4.4.3] will be useful.

Proposition 2.7 (Fröhlich). *Let $(N, +, \cdot)$ be a distributively generated near-ring with a left or right unity, generated by $(S_d, \cdot)$, a semigroup of distributive elements. The following are equivalent:*

- (i) N is a ring;
- (ii) N is distributive;

- (iii) For all $a, b \in S_d$, $a + b = b + a$;
- (iv) $(N^2, +)$ is abelian.

Theorem 2.8. *Let $\mathcal{R}$ be the property that all orbits are abelian. Let $\mathcal{S}$ be the property that if $f \in \mathcal{E}(G)$ is a homomorphism when restricted to each orbit, then $f \in \text{End}(G)$. Then*

- (i) *E - H groups are exactly the groups satisfying $\mathcal{R}$ and $\mathcal{S}$;*
- (ii) *E - D groups are exactly the groups satisfying $\mathcal{R}$ and not $\mathcal{S}$;*
- (iii) *D - H groups are exactly the groups satisfying not $\mathcal{R}$ and $\mathcal{S}$;*
- (iv) *D - D groups are exactly the groups satisfying not $\mathcal{R}$ and not $\mathcal{S}$.*

Proof. To prove the theorem we will show that, for a group G , property $\mathcal{R}$ is equivalent to the $E \rightarrow D$ property, and property $\mathcal{S}$ is equivalent to the $D \rightarrow H$ property. By Proposition 2.7, $\mathcal{E}(G)$ is a ring if and only if $(\mathcal{E}(G), +)$ is abelian if and only if $\mathcal{E}(G) = \mathcal{D}(\mathcal{E}(G))$. Thus, G has the $E \rightarrow D$ property if and only if for each $a \in G$, $(f + h)(a) = (h + f)(a)$ for each $f, g \in \mathcal{E}(G)$ if and only if $\mathcal{E}(G)(a)$ is abelian for each $a \in G$ if and only if G has property $\mathcal{R}$.

To show G has property $\mathcal{S}$ if and only if G has the $D \rightarrow H$ property, it suffices to show that if $f \in \mathcal{E}(G)$, then $f \in \mathcal{D}(\mathcal{E}(G))$ if and only if $f|_{\mathcal{E}(G)(a)}$ is a homomorphism for each $a \in G$. Let $f \in \mathcal{D}(\mathcal{E}(G))$, $c \in G$, and $x, y \in \mathcal{E}(G)(c)$. Then for some $g, h \in \mathcal{E}(G)$,

$$f(x + y) = f(g(c) + h(c)) = f(g + h)(c) = (fg + fh)(c) = f(x) + f(y);$$

hence, $f|_{\mathcal{E}(G)(a)}$ is a homomorphism for each $a \in G$. Now let $f, g, h \in \mathcal{E}(G)$, $c \in G$, and assume $f|_{\mathcal{E}(G)(a)}$ is a homomorphism for each $a \in G$. We have that

$$\begin{aligned} f(g + h)(c) &= f|_{\mathcal{E}(G)(c)}(g(c) + h(c)) = f|_{\mathcal{E}(G)(c)}(g(c)) + f|_{\mathcal{E}(G)(c)}(h(c)) \\ &= (fg + fh)(c), \end{aligned}$$

implying that $f \in \mathcal{D}(\mathcal{E}(G))$. □

For E - H and E - D groups, we present better characterizations, which will lead us to identify an enormous variety of examples of these two classes of groups. We then find some sufficient conditions for a group to be a D - H group, and use these to find many examples belonging to this subclass. Finally, we use properties of E - D and D - H groups to construct examples of D - D groups.

3 *E-H* Groups

Let $(T, +)$ be a groupoid and let A and B be sets. We say $\{A, B\}$ is a $(T, +)$ -commutative pair if $A, B \subseteq T$ and $a + b = b + a$, for each $a \in A$ and $b \in B$. If $A = B$, we say A is a $(T, +)$ -commutative set.

Lemma 3.1. *Let $\alpha, \beta \in \text{End}(G)$. The following are equivalent:*

- (i) $\alpha + \beta \in \text{End}(G)$;
- (ii) $\{\alpha(G), \beta(G)\}$ is a G -commutative pair.

Proof. This follows from a routine calculation. □

Proposition 3.2. *The following are equivalent:*

- (i) G is an E - H group (G has both the $E \rightarrow D$ and $D \rightarrow H$ properties);
- (ii) $\mathcal{E}(G) \subseteq \text{End}(G)$;
- (iii) $\{f(G), h(G)\}$ is a G -commutative pair for each $f, h \in \mathcal{E}(G)$;
- (iv) $\{\alpha(G), \beta(G)\}$ is a G -commutative pair for each $\alpha, \beta \in \text{End}(G)$;
- (v) G is abelian;
- (vi) $1_G + 1_G \in \text{End}(G)$;
- (vii) $\gamma + \xi \in \text{End}(G)$, for some $\gamma, \xi \in \text{Aut}(G)$.

Proof. G is an E - H group if and only if G has both the $E \rightarrow D$ and $D \rightarrow H$ properties, if and only if $\mathcal{E}(G) \subseteq \text{End}(G)$. This establishes that (i) $\Leftrightarrow$ (ii). It is clear that (iii) $\Rightarrow$ (iv). Assume (iv) and note that since $1_G \in \text{End}(G)$, we have that $1_G(G) = G$ is a G -commutative set. In other words, G is abelian. The implication (v) $\Rightarrow$ (iii) is immediate. Thus, we have established the equivalence of (iii), (iv) and (v). Since (v) $\Rightarrow$ (ii) follows by a routine calculation, we only need to show (ii) $\Rightarrow$ (iv) to establish the equivalence of (i)–(v). Assume (ii) and let $\alpha, \beta \in \text{End}(G)$. Then $\alpha + \beta \in \mathcal{E}(G)$ implies that $\alpha + \beta \in \text{End}(G)$. By Lemma 3.1, we have that $\{\alpha(G), \beta(G)\}$ is a G -commutative pair. Now we show that (v), (vi) and (vii) are equivalent. Clearly, we have that (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii). Assume (vii). By Lemma 3.1, we have that $\{\gamma(G), \xi(G)\}$ is a G -commutative pair. But $\gamma, \xi \in \text{Aut}(G)$, so we have that G is abelian. Thus (vii) $\Rightarrow$ (v); so we are done. □

Remark. Much of Proposition 3.2 is already known. It is provided here for completeness and to exhibit some not so well-known characterizations using new techniques.

By Proposition 3.2, a group has both the $E \rightarrow D$ and $D \rightarrow H$ properties if and only if it is an abelian group. This allows us to establish the following three corollaries:

Corollary 3.3. *A group G is an E - D group if and only if G is a nonabelian group with the $E \rightarrow D$ property.*

Corollary 3.4. *A group G is a D - H group if and only if G is a nonabelian group with the $D \rightarrow H$ property.*

Corollary 3.5. *A group G is a D - D group if and only if G is a nonabelian group that is not an E - D or D - H group.*

4 E - D Groups

Just as E - H groups have already been characterized, so too have E - D groups. In 1958, Fröhlich (see Proposition 2.7) gave several characterizations of E -groups. So if we include the hypothesis that G is nonabelian, we get several characterizations of E - D groups from his results. Nonetheless, it was conjectured that nonabelian E -groups did not exist. It was Faudree [8] who proved the conjecture false in 1971, by constructing the first example of a nonabelian E -group—the first E - D group. (See Malone [15] for a lively account of the events leading to this discovery.)

In this section, we give new characterizations of E -groups and different proofs of previously known characterizations (as mentioned, see Fröhlich and also Malone [13]). We begin by determining exactly when the sum of two distributive elements is distributive.

Lemma 4.1. *Let $f, h \in \mathcal{D}(\mathcal{E}(G))$. The following are equivalent:*

- (i) $f + h \in \mathcal{D}(\mathcal{E}(G))$;
- (ii) $\{f(\mathcal{E}(G)(a)), h(\mathcal{E}(G)(a))\}$ is a G -commutative pair, for each $a \in G$.

Proof. This follows from a routine calculation. □

We use Proposition 4.1 to establish parts of the following theorem:

Theorem 4.2. *The following are equivalent:*

- (i) G is either an E - H or E - D group;
- (ii) $\mathcal{E}(G) \subseteq \mathcal{D}(\mathcal{E}(G))$ (i.e., G has the $E \rightarrow D$ property);
- (iii) $\{f(\mathcal{E}(G)(a)), h(\mathcal{E}(G)(a))\}$ is a G -commutative pair, for each $a \in G$ and $f, h \in \mathcal{E}(G)$;
- (iv) $\{\alpha(\mathcal{E}(G)(a)), \beta(\mathcal{E}(G)(a))\}$ is a G -commutative pair, for each $a \in G$ and $\alpha, \beta \in \text{End}(G)$;

- (v) every orbit of G is abelian;
- (vi) $\text{End}(G)(a)$ is a G -commutative set for each $a \in G$;
- (vii) $(\mathcal{E}(G), +)$ is abelian;
- (viii) G is an E -group (i.e., $\mathcal{E}(G)$ is a ring);
- (ix) $\{1_G, \text{End}(G)\}$ is an $(\mathcal{E}(G), +)$ -commutative pair;
- (x) $1_G + \alpha \in {}_d(1_G, \text{End}(G))$, for each $\alpha \in \text{End}(G)$;
- (xi) $\text{End}(G)$ is an $(\mathcal{E}(G), +)$ -commutative set;
- (xii) $1_G + 1_G \in \mathcal{D}(\mathcal{E}(G))$;
- (xiii) $\gamma + \xi \in \mathcal{D}(\mathcal{E}(G))$, for some $\gamma, \xi \in \text{Aut}(G)$;
- (xiv) $(\text{End}(G))^2$ is an $(\mathcal{E}(G), +)$ -commutative set;
- (xv) $f|_{\mathcal{E}(G)(a)}$ is a homomorphism for each $a \in G$ and $f \in \mathcal{E}(G)$.

Proof. The equivalence of (ii), (vii), (viii), (xi) and (xiv) are established in Proposition 2.7. By definition, G has the $E \rightarrow D$ property if and only if G is either an E - H or E - D group. Thus (i) $\Leftrightarrow$ (ii). Assume (iii) and note that (iii) $\Rightarrow$ (iv) is obvious. Assume (iv). Since $1_G \in \text{End}(G)$ and $1_G(\mathcal{E}(G)(a)) = \mathcal{E}(G)(a)$ is a G -commutative set, for each $a \in G$, we have that every orbit of G is abelian; hence, (iv) $\Rightarrow$ (v). The implication (v) $\Rightarrow$ (vi) is clear. Since $\mathcal{E}(G) = \langle \text{End}(G) \rangle_{(M_0(G), +)}$, we obtain that (vi) $\Rightarrow$ (vii). Assume (vii) and let $f, h, s, t \in \mathcal{E}(G)$ and $a \in G$. Then we have that $f(s(a)) + h(t(a)) = (fs + ht)(a) = (ht + fs)(a) = h(t(a)) + f(s(a))$. Thus, (vii) $\Rightarrow$ (iii) and we have established the equivalence of (i)–(viii). Assume (ix). Let $\alpha, \beta \in \text{End}(G)$. Then we have that

$$\begin{aligned}
 (1_G + \alpha)(1_G + \beta) &= (\alpha + 1_G)(1_G + \beta) = \alpha + \alpha\beta + 1_G + \beta = 1_G + \alpha + \alpha\beta + \beta \\
 &= 1_G + \alpha + (\alpha + 1_G)\beta = 1_G + \alpha + (1_G + \alpha)\beta \\
 &= 1_G + \alpha + \beta + \alpha\beta = (1_G + \alpha)1_G + (1_G + \alpha)\beta.
 \end{aligned}$$

Thus (ix) $\Rightarrow$ (x). Assume (x) and let $\alpha, \beta \in \text{End}(G)$. We have that

$$(1_G + \alpha)(1_G + \beta) = (1_G + \alpha)1_G + (1_G + \alpha)\beta = 1_G + \alpha + \beta + \alpha\beta,$$

and

$$(1_G + \alpha)(1_G + \beta) = 1_G(1_G + \beta) + \alpha(1_G + \beta) = 1_G + \beta + \alpha + \alpha\beta.$$

Therefore, $\alpha + \beta = \beta + \alpha$, establishing that (x) $\Rightarrow$ (xi). It is clear that (xi) $\Rightarrow$ (ix), which establishes the equivalence of (i)–(xi). It is clear that (xii) $\Rightarrow$ (xiii). Assume (xiii). By Lemma 4.1, $\gamma + \xi \in \mathcal{D}(\mathcal{E}(G))$ implies that, for each $a \in G$, $\{\gamma(\mathcal{E}(G)(a)), \xi(\mathcal{E}(G)(a))\}$ is a G -commutative pair. But $\gamma(\mathcal{E}(G)(a)) = \mathcal{E}(G)(a)$, for each $\gamma \in \text{Aut}(G)$, since $\gamma(\gamma^{-1}f)(a) = f(a)$, for each $f \in \mathcal{E}(G)$. Thus, $\mathcal{E}(G)(a)$ is abelian for each $a \in G$. Hence, (xiii) $\Rightarrow$ (v). Assume (v). Then $1_G(\mathcal{E}(G)(a))$ is a G -commutative set, which implies that $1_G + 1_G \in \mathcal{D}(\mathcal{E}(G))$ by Lemma 4.1. This establishes (v) $\Rightarrow$ (xii) and the equivalence of (i)–(xiv). We complete the logical circuit by showing that (ii) $\Leftrightarrow$ (xv). Assume (ii) and let $f \in \mathcal{E}(G)$ and $a \in G$. Let $g(a), h(a) \in \mathcal{E}(G)(a)$. Then $f(g(a) + h(a)) = f(g + h)(a) = (fg + fh)(a) = f(g(a)) + g(h(a))$, which implies that $f|_{\mathcal{E}(G)(a)}$ is a homomorphism. Assume (xv) and let $f \in \mathcal{E}(G)$ and $a \in G$. For $g, h \in \mathcal{E}(G)$, we have that

$$\begin{aligned} f(g + h)(a) &= f(g(a) + h(a)) = f|_{\mathcal{E}(G)(a)}(g(a) + h(a)) \\ &= f|_{\mathcal{E}(G)(a)}(g(a)) + f|_{\mathcal{E}(G)(a)}(h(a)) = (fg + fh)(a); \end{aligned}$$

hence $f \in \mathcal{D}(\mathcal{E}(G))$. □

Corollary 4.3. *G is an E - D group if and only if G is a nonabelian E -group.*

Proof. This follows directly from Theorem 4.2 and Corollary 3.3. □

Remark. We have established that G is an E - D group if and only if G is nonabelian and satisfies any one of the properties in Theorem 4.2. Furthermore, we have that

$$\begin{aligned} \{E\text{-groups}\} &= \{E\text{-}H \text{ groups}\} \cup \{E\text{-}D \text{ groups}\} \\ &= \{\text{groups with the } E \rightarrow D \text{ property}\}, \end{aligned}$$

where the braces denote classes. Also note that Malone [13, Theorem 6] has established the equivalence of (ix) and (xi) (from Theorem 4.2), using a different proof.

Example 4.4. If p is an odd prime, the Faudree groups F_{p^8} , introduced in 1971 [8], are E - D groups. This is an infinite family of finite p -groups. It was Malone [14] who pointed out that p must be odd for Faudree's construction to work.

Example 4.5. Many of the groups constructed by Jonah and Konvisser [12] are E - D groups. All of their groups have the property that $(\text{Aut}(G), \circ)$ is abelian. Malone [14] designates the ones that are E - D groups as JK groups, and notes that, like the F_{p^8} groups, the JK groups have order p^8 . Since p can be any prime natural number, this construction yielded the first published example of an E - D group that is a 2-group.

Example 4.6. Caranti [4] constructs several new examples of finite E - D groups.

Example 4.7. Using homological algebra techniques, Caranti, Franciosi and de Giovanni [5] constructed the first published examples of infinite E - D groups. These include both torsion and torsion-free examples.

5 D - H Groups

In 1971, the same year that Faudree exhibited the first E - D group, Maxson considered the problem of determining what additional conditions are needed on a group G to insure that G is abelian when $\mathcal{E}(G)$ is a ring. (It had just been determined by Faudree [8] that the conjecture “ $\mathcal{E}(G)$ a ring implies G abelian” was false.) In particular, he gives a complete answer [16, Theorem 2.4] in the restricted setting of finitely generated groups. This theorem will be proved at the end of this section as a corollary of results in this paper.

Theorem 5.1 (Maxson). *Let G be finitely generated and let $\mathcal{E}(G)$ be a ring. The following are equivalent:*

- (i) G is abelian;
- (ii) G is a cyclic $\mathcal{E}(G)$ -module.

Following Maxson, a group G is a *cyclic $\mathcal{E}(G)$ -module* if there exists an element c in G such that $\mathcal{E}(G)(c) = G$. The element $c \in G$ is called the $\mathcal{E}(G)$ -generator of G . (The Quaternion group is an example of a nonabelian cyclic $\mathcal{E}(G)$ -module.) Note that Johnson [11] has shown that a group G is a cyclic $\mathcal{E}(G)$ -module if and only if G is invariantly simple. (Recall that a group is invariantly simple if it has no proper nontrivial fully invariant subgroups.)

Using results established here, we easily obtain the following generalization of Maxson’s result without any restriction on G .

Proposition 5.2. *Let $\mathcal{E}(G)$ be a ring. The following are equivalent:*

- (i) G is abelian;
- (ii) G has the $D \rightarrow H$ property.

Proof. By Proposition 3.2, we have that (i) $\Rightarrow$ (ii). Assume (ii). Since $\mathcal{E}(G)$ a ring implies that G has the $E \rightarrow D$ property by Theorem 4.2, G has both the $E \rightarrow D$ and $D \rightarrow H$ properties. Thus G is an E - H group; hence it is abelian by Proposition 3.2. $\square$

In this section we identify and study the nonabelian members of the class of groups with the $D \rightarrow H$ property, i.e., the D - H groups. See Diener [7, Lemma II.2] and Maxson and Meldrum [17, Theorem 1.1] for similar results in the area of centralizer near-rings.

Proposition 5.3.

- (i) *Let G be a group such that for each $a, b \in G$, $\{a, b\} \subseteq \mathcal{E}(G)(c)$, for some $c \in G$. Then G has the $D \rightarrow H$ property.*
- (ii) *Cyclic $\mathcal{E}(G)$ -modules have the $D \rightarrow H$ property.*
- (iii) *nonabelian groups satisfying the hypothesis of (i) or (ii) are D - H groups.*

Proof. To prove (i), let $f \in \mathcal{D}(\mathcal{E}(G))$ and $a, b \in G$. Then for some $g, h \in \mathcal{E}(G)$,

$$\begin{aligned} f(a + b) &= f(g(c) + h(c)) = f(g + h)(c) = (fg + fh)(c) \\ &= f(g(c)) + f(h(c)) = f(a) + f(b). \end{aligned}$$

This establishes that G has the $D \rightarrow H$ property. It is clear that a cyclic $\mathcal{E}(G)$ -module satisfies the hypothesis of (i); so (ii) holds. Part (iii) follows from Corollary 3.4. $\square$

Corollary 5.4. *Nonabelian invariantly simple groups are D - H groups.*

Proposition 5.5. *Let X be a set. Let $G = F\langle X \rangle$ denote the free group on X .*

- (i) *G is a cyclic $\mathcal{E}(G)$ -module;*
- (ii) *if $|X| > 1$, G is D - H group;*
- (iii) *every nontrivial group is the quotient of a D - H group.*

Proof. To prove (i), we define for each $a \in G$, a mapping $f_a : X \rightarrow G$ via $f_a(x) = a$, for each $x \in X$. By the Universal Mapping Theorem, there exists $\overline{f_a} \in \text{End}(G)$, for each $a \in G$, such that $\overline{f_a}|_X = f_a$. So for each $a \in G$ and any $x \in X$, $\overline{f_a}(x) = a$. Thus, $\text{End}(G)(x) = G$, for each $x \in X$. Therefore, G is a cyclic $\mathcal{E}(G)$ -module. To prove (ii), note that if $|X| > 1$, then $G = F\langle X \rangle$ is nonabelian. By (i) and Proposition 5.3 (iii), G is a D - H group. To prove (iii), note that every group is the quotient of a free group, and use (ii). $\square$

Proposition 5.6. *Any nonabelian orbit of any group is a D - H group.*

Proof. We first establish that any orbit S of a group G is a cyclic $\mathcal{E}(S)$ -module. Let $a \in G$, and let $S = \mathcal{E}(G)(a)$. If $\mathcal{E}(G)(a) = G$ then $S = G$ is a cyclic $\mathcal{E}(S)$ -module. Suppose $S \subsetneq G$. Since S is a fully invariant subgroup of G , $f|_S \in \mathcal{E}(S)$, for each $f \in \mathcal{E}(G)$. Since $a \in \mathcal{E}(G)(a) = S$, we have that $S = \mathcal{E}(G)(a) = \mathcal{E}(G)|_S(a) \subseteq \mathcal{E}(S)(a)$. Thus $\mathcal{E}(S)(a) = S$ and $a \in S$. Therefore S is a cyclic $\mathcal{E}(S)$ -module. It follows by Proposition 5.3 (iii), that a nonabelian orbit is a D - H group. $\square$

Corollary 5.7.

- (i) *Every D - H group G is either a cyclic $\mathcal{E}(G)$ -module or contains a proper nontrivial D - H subgroup.*
- (ii) *Every D - D group contains a proper nontrivial D - H subgroup.*

Proof. By Theorem 4.2, D - H and D - D groups must contain a nonabelian orbit S , which is a cyclic $\mathcal{E}(G)$ -module by Proposition 5.6. Furthermore, a D - D group G cannot be a cyclic $\mathcal{E}(G)$ -module by Proposition 5.3 (ii); so S is proper. $\square$

Proposition 5.8. *Let G be a nonabelian group such that each proper orbit is abelian. Then G is either a D - H or E - D group.*

Proof. If $\mathcal{E}(G)(a) \subsetneq G$, for each $a \in G$, then G is an E - D group by Theorem 4.2 and Corollary 3.3. Otherwise, $\mathcal{E}(G)(a) = G$, for some $a \in G$. So G is a D - H group by Proposition 5.3 (iii). $\square$

Proposition 5.8 provides an excellent “sieve” technique for locating D - H groups via the computer. While it theoretically will collect all E - D groups, it identifies many D - H groups as well, especially in groups of small order.

Example 5.9. All nonabelian groups of order at most 30 are D - H groups. This was established using the SONATA [19] package, which is built on top of the GAP [10] group analysis program.

Corollary 5.10. *A group which is not an E -group and has all proper orbits abelian is a D - H group.*

Proof. This follows since a group which is not an E -group is nonabelian and not an E - D group. See Theorem 4.2 (i) and (viii). $\square$

Example 5.11. Let p be an odd prime. Then D_p , the dihedral group of order $2p$, is a D - H group. Since Chandy [6] has shown that all E -groups are nilpotent of class at most 3, we have that D_p is not an E - H or E - D group. Since all subgroups of D_p are abelian, D_p is a D - H group by Corollary 5.10.

Example 5.12. The Tarski groups (infinite groups all of whose proper non-trivial subgroups have order a fixed prime) are D - H groups. These are some of the most pathological examples of nonsolvable p -groups. (See Ol' šanskii [18] for examples of Tarski groups.) The fact that these are D - H groups follows from the Corollary 5.10 (since every proper subgroup is abelian) or from Corollary 5.4 (since these groups are also simple).

Up to now, all examples of D - H groups have been nonabelian cyclic $\mathcal{E}(G)$ -modules. But before we can give an example of a D - H group that is not a cyclic $\mathcal{E}(G)$ -module, we must establish some properties of cyclic $\mathcal{E}(G)$ -modules. We begin by introducing some notation and definitions.

The following definitions are, essentially, those of Weinstein [20], but altered slightly to reflect our use of additive notation for groups. Let $\{A_i\}_{i \in I}$ be an indexed set of groups with indexing set I . Let 0_{A_i} denote the identity element of A_i . Let $\mathcal{O}_{A_i, A_j}$ denote the homomorphism that sends each element of A_i to 0_{A_j} . If $i = j$, we write $\mathcal{O}_{A_i}$ instead of $\mathcal{O}_{A_i, A_i}$, for the identity element of $(\mathcal{E}(A_i), +)$. Define

$$\sum \{A_i\}_{i \in I} := \{g | g : I \rightarrow \cup \{A_i\}_{i \in I}, g(i) \in A_i \forall i \in I\}.$$

We call $\sum \{A_i\}_{i \in I}$ the *unrestricted direct sum* of $\{A_i\}_{i \in I}$. The group operation on $\sum \{A_i\}_{i \in I}$ is defined componentwise by the equation $(g + h)(i) = g(i) + h(i)$, where $g, h \in \sum \{A_i\}_{i \in I}$ and $i \in I$. If $g \in \sum \{A_i\}_{i \in I}$, let $S(g) = \{i \in I | g(i) \neq 0_{A_i}\}$. Define

$$\sum \bigoplus \{A_i\}_{i \in I} := \left\{ g | g \in \sum \{A_i\}_{i \in I}, |S(g)| < \infty \right\}.$$

We call $\sum \bigoplus \{A_i\}_{i \in I}$ the *restricted direct sum* of $\{A_i\}_{i \in I}$.

Let $G = \sum \{A_i\}_{i \in I}$ and let $\omega_i \in \mathcal{E}(A_i)$, for each $i \in I$. Let $\omega : G \rightarrow G$ be defined via $\omega(g)(i) = \omega_i(g(i))$, for each $i \in I$. We write $\omega = (\omega_i)_{i \in I}$. It is clear that $\omega \in \mathcal{E}(G)$. In some settings, all elements of $\mathcal{E}(G)$ are of this form.

Lemma 5.13. *Let $\{A_i\}_{i \in I}$ be an indexed set of groups such that $\text{Hom}(A_i, A_j) = \{\mathcal{O}_{A_i, A_j}\}$, for each $i, j \in I$, $i \neq j$. Let $G = \sum \bigoplus \{A_i\}_{i \in I}$. Then for each $\omega \in \text{End}(G)$, $\omega = (\omega_i)_{i \in I}$, where $\omega_i \in \text{End}(A_i)$.*

Proof. Let $\{\Pi_i\}_{i \in I}$ be the coordinate projections and let $\{\Gamma_i\}_{i \in I}$ be the coordinate injections. First we show that $\omega\Gamma_i(a) \in \Gamma_i(A_i)$, for each $i \in I$, $a \in A_i$, and $\omega \in \text{End}(G)$. Suppose $\omega\Gamma_i(a) \notin \Gamma_i(A_i)$, for some $i \in I$, $a \in A_i$, $\omega \in \text{End}(G)$. Then there exists A_j , $j \neq i$, such that $0_{A_j} \neq \Pi_j\omega\Gamma_i(a) \in A_j$. But by hypothesis, $\Pi_j\omega\Gamma_i \in \text{Hom}(A_i, A_j) = \{\mathcal{O}_{A_i, A_j}\}$, which is a contradiction.

Now let $\omega \in \text{End}(G)$ and $g \in G$. Since $|S(g)| < \infty$, we have that $g = \sum_{i \in S(g)} \Gamma_i g(i)$. Thus $\omega(g) = \sum_{i \in S(g)} \omega\Gamma_i g(i)$. Since we have shown that

$\omega\Gamma_i(A_i) \subseteq \Gamma_i(A_i)$, for each $i \in I$, we obtain that

$$\begin{aligned}\omega(g)(i) &= \Pi_i \omega(g) = \Pi_i \sum_{i \in S(g)} \omega\Gamma_i g(i) = \sum_{i \in S(g)} \Pi_i \omega\Gamma_i g(i) \\ &= (\Pi_i \omega\Gamma_i)(g(i)).\end{aligned}$$

Therefore, $\omega = (\Pi_i \omega\Gamma_i)_{i \in I}$. Since $\Pi_i \omega\Gamma_i \in \text{End}(A_i)$, we are done. $\square$

Proposition 5.14. *Let $\{A_i\}_{i \in I}$ be an indexed set of groups such that A_i is a cyclic $\mathcal{E}(A_i)$ -module, for each $i \in I$. Then $G = \sum \{A_i\}_{i \in I}$ is a cyclic $\mathcal{E}(G)$ -module.*

Proof. Let $c \in G$ be defined via $c(i) = \text{any } \mathcal{E}(A_i)\text{-generator of } A_i$, for each $i \in I$. Then if $g \in G$, there exists $w_i \in \mathcal{E}(A_i)$, such that $g(i) = \omega_i(c(i))$, for each $i \in I$. Thus $g = (\omega_i)_{i \in I}(c)$. Since $(\omega_i)_{i \in I} \in \mathcal{E}(G)$, G is a cyclic $\mathcal{E}(G)$ -module. $\square$

Theorem 5.15. *Let $\{A_i\}_{i \in I}$ be an indexed set of groups such that A_i is a cyclic $\mathcal{E}(A_i)$ -module for each $i \in I$. Then*

- (i) $G = \sum \bigoplus \{A_i\}_{i \in I}$ has the $D \rightarrow H$ property;
- (ii) Let $\text{Hom}(A_i, A_j) = \{\mathcal{O}_{A_i, A_j}\}$ for each $i, j \in I, i \neq j$. Then G is a cyclic $\mathcal{E}(G)$ -module if and only if at most finitely many A_i are nontrivial.

Proof. To prove (i), let $g, h \in G$. We show that there exists $c \in G$ such that $\{g, h\} \subseteq \mathcal{E}(G)(c)$. Then we are done by Propositions 5.2 and 5.3 (i). Let $J = S(g) \cup S(h)$ and note that $|J| < \infty$. Let $c \in G$ be defined by

$$c(i) = \begin{cases} \text{any } \mathcal{E}(A_i)\text{-generator of } A_i, & \text{if } i \in J; \\ 0_{A_i}, & \text{if } i \notin J. \end{cases}$$

Let $\omega = (\omega_i)_{i \in I}$, where

$$\omega_i = \begin{cases} f \in \mathcal{E}(A_i) \text{ such that } g(i) = f(c(i)), & \text{if } i \in S(g); \\ \mathcal{O}_{A_i}, & \text{if } i \notin S(g). \end{cases}$$

Then $\omega \in \mathcal{E}(G)$. Similarly, define $\nu = (\nu_i)_{i \in I} \in \mathcal{E}(G)$ by

$$\nu_i = \begin{cases} f \in \mathcal{E}(A_i) \text{ such that } h(i) = f(c(i)), & \text{if } i \in S(h); \\ \mathcal{O}_{A_i}, & \text{if } i \notin S(h). \end{cases}$$

Thus, $g = \omega(c)$ and $h = \nu(c)$, which implies that $\{g, h\} \subseteq \mathcal{E}(G)(c)$. Therefore, G has the $D \rightarrow H$ property.

To prove (ii), assume that infinitely many A_i are nontrivial. We show that an $\mathcal{E}(G)$ -generator of G cannot exist. Suppose c is such a generator. Let

$j \in I \setminus S(c)$ (which is nonempty since infinitely many A_i are nontrivial) and let $\omega \in \mathcal{E}(G)$. Then by Proposition 5.13, $\omega = (\omega_i)_{i \in I}$, where $\omega_i \in \mathcal{E}(A_i)$. Pick $g \in G$ such that $g(j) \neq 0_{A_j}$ (there is such a g since infinitely many A_i are nontrivial). Then we have that $\omega_j(c(j)) = \omega_j(0_{A_j}) = 0_{A_j}$; hence $g \notin \mathcal{E}(G)(c)$, a contradiction. The converse follows from Proposition 5.14. $\square$

Before we give our example of D - H group that is not a cyclic $\mathcal{E}(G)$ -module, we prove the following:

Lemma 5.16. *A cyclic group G is a cyclic $\mathcal{E}(G)$ -module.*

Proof. Let G be a cyclic group with generator a . Let $S = \{n1_G : n \in \mathbb{Z}\}$. Then $S \subseteq \text{End}(G)$ and $S(a) = G$. Thus, $\mathcal{E}(G)(a) = G$. $\square$

Example 5.17. Let q be an odd prime and let $\mathcal{P}$ be any infinite set of prime natural numbers not including q . Let C_p denote a cyclic group of order p . Then $G = D_q \oplus \sum \bigoplus \{C_p\}_{p \in \mathcal{P}}$ is a D - H group but not a cyclic $\mathcal{E}(G)$ -module. To ascertain this, let $I = \mathcal{P} \cup \{q\}$ and

$$A_i = \begin{cases} D_i, & \text{if } i = q; \\ C_i, & \text{if } i \neq q. \end{cases}$$

Then, we have that $G = \sum \bigoplus \{A_i\}_{i \in I}$. Note that $G = C_p$ is a cyclic $\mathcal{E}(G)$ -module by the lemma above. That $G = D_q$ is a cyclic $\mathcal{E}(G)$ -module is established in the proof of Proposition 5.8. By Theorem 5.15 (i), G has the $D \rightarrow H$ property. Since G is nonabelian, G is a D - H group. However, by Theorem 5.15 (ii), G is *not* a cyclic $\mathcal{E}(G)$ -module.

Recall that Maxson proved that if G is finitely generated and $\mathcal{E}(G)$ is a ring, then G is abelian if and only if G is a cyclic $\mathcal{E}(G)$ -module. This fact is easy to prove using results established in this paper: First assume that G is abelian. Since G is also finitely generated, it is the finite direct sum of cyclic groups. By Lemma 5.16 and Proposition 5.14, G is a cyclic $\mathcal{E}(G)$ -module. Now assume that G is a cyclic $\mathcal{E}(G)$ -module. Then it has the $D \rightarrow H$ property by Proposition 5.3 (ii). Since G is an E -group, it also has the $E \rightarrow D$ property. Thus, by Proposition 3.2, G must be abelian.

6 D - D Groups

It is possible to construct a D - D group as a direct sum of a D - H group and an E - D group using the following two theorems. The first result [13, Theorem 4] is due to Malone.

Theorem 6.1 (Malone). *Let G be a group with the $E \rightarrow D$ property and let $\gamma \in \text{End}(G)$. Then $\gamma(G)$ has the $E \rightarrow D$ property.*

We will prove an analogous result with respect to the $D \rightarrow H$ property. But first we introduce another substructure of $\mathcal{E}(G)$: Let $G = (G, +)$ be a group, not necessarily abelian, with identity element 0, and let K be a nonzero subgroup of G . Let $\text{Hom}(G, K)$ denote the set of homomorphisms from G into K and let $\mathcal{H}(G, K)$ denote the subgroup of $(\mathcal{E}(G), +)$, generated by $\text{Hom}(G, K)$. Since $\text{Hom}(G, K)$ is a semigroup under function composition, then $\mathcal{H} := \mathcal{H}(G, K)$ is a near-ring under pointwise addition and function composition. Note that an element $f \in \mathcal{H}$ can be written as $f = \sum_{j \in J} \varepsilon_j \sigma_j$ where $\sigma_j \in \text{Hom}(G, K)$, $\varepsilon_j = \pm 1$, and J is some linearly ordered, finite indexing set. We will let $\mathcal{O}_G$ denote the zero element of the near-ring $(\mathcal{H}(G, K), +, \circ)$ and we will assume $\mathcal{H}$ is non-trivial and let $\mathcal{H}^\#$ denote the set $\mathcal{H} \setminus \{\mathcal{O}_G\}$. Note that Birkenmeier, Heatherly and Pilz [1], [2] initiated the study of the $\mathcal{H}(G, K)$ substructure.

The following result [3, Proposition 2.7] identifies a useful connection between the structure of a group G and $\mathcal{H}(G, K)$ having a one-sided unity. It is useful to assign $\text{Ker}\mathcal{H} := \bigcap \{h^{-1}(0) : h \in \mathcal{H}\}$. The justification for the use of the symbol $\text{Ker}\mathcal{H}$ follows from the observation that $\bigcap \{h^{-1}(0) : h \in \mathcal{H}\} = \bigcap \{\ker h : h \in \text{Hom}(G, K)\}$. Thus $\text{Ker}\mathcal{H}$ is a normal subgroup of G .

Proposition 6.2. *The following are equivalent:*

- (i) $\mathcal{H}(G, K)$ contains a right unity r ;
- (ii) there exists an idempotent $r \in \mathcal{H}^\#$ such that $G = \text{Ker}\mathcal{H} + r(G)$;
- (iii) there exists an idempotent $r \in \mathcal{H}^\#$ such that $\rho_r : \mathcal{H} \rightarrow \mathcal{E}(\overline{K})|_{r(G)}$, defined via $\rho_r(f) = f|_{r(G)}$, for each $f \in \mathcal{H}$, is an injective group homomorphism.

This structural result will help us establish the first of two lemmas that will assist us in proving the main theorem of this section. Recall that ${}_dA$ is the set of functions that left distribute over A .

Lemma 6.3. *Let $K \leq G$. If $f \in {}_d\mathcal{E}(K)$ and $\mathcal{H}(G, K)$ has a right unity, then $f \in {}_d\mathcal{H}(G, K)$.*

Proof. Let $f \in {}_d\mathcal{E}(K)$, $a \in G$ and $h, g \in \mathcal{H}(G, K)$. Let r be a right unity of $\mathcal{H}(G, K)$. By Proposition 6.2, $G = \text{Ker}\mathcal{H} + r(G)$; hence $a = n + r(b)$, for some $n \in \text{Ker}\mathcal{H}$ and $b \in G$. Thus, we have that

$$\begin{aligned} f(h+g)(a) &= f(h(n+r(b)) + g(n+r(b))) = f(hr(b) + gr(b)) \\ &= f(h|_K + g|_K)(r(b)) = (f(h|_K) + f(g|_K))(r(b)) = (fh + fg)(a). \end{aligned}$$

□

Lemma 6.4. *Let $K \leq G$. If $f \in \mathcal{D}(\mathcal{H}(G, K))$ and $\mathcal{H}(G, K)$ has a right unity in $\text{Hom}(G, K)$, then $f \in \mathcal{D}(\mathcal{E}(G))$.*

Proof. Let $f \in \mathcal{D}(\mathcal{H}(G, K))$, $a \in G$ and $h, g \in \mathcal{E}(G)$. Let r be a right unity of $\mathcal{H}(G, K)$ in $\text{Hom}(G, K)$. Then

$$\begin{aligned} f(h+g)(a) &= fr(h(a)+g(a)) = f(rh(a)+rg(a)) = f(rh+rg)(a) \\ &= (frh+frg)(a) = (fh+fg)(a). \end{aligned}$$

□

As promised, we obtain the following analog to Theorem 6.1

Theorem 6.5. *Let G be a group with the $D \rightarrow H$ property and let $\gamma \in \text{End}(G)$. If γ is a right unity of $\mathcal{H}(G, \gamma(G))$, then $\gamma(G)$ has the $D \rightarrow H$ property.*

Proof. Let $f \in \mathcal{D}(\mathcal{E}(\gamma(G)))$. By Lemma 6.3, $f \in {}_d\mathcal{H}(G, \gamma(G))$. Since $\gamma \in \mathcal{D}(\mathcal{H}(G, \gamma(G)))$ and $f\gamma \in \mathcal{H}(G, \gamma(G))$, we have that $f\gamma \in \mathcal{D}(\mathcal{H}(G, \gamma(G)))$. By Lemma 6.4, $f\gamma \in \mathcal{D}(\mathcal{E}(G))$, which implies that $f\gamma \in \text{End}(G)$, since G has the $D \rightarrow H$ property. If we let $\gamma(a), \gamma(b) \in \gamma(G)$, we have that $f(\gamma(a) + \gamma(b)) = f\gamma(a+b) = f\gamma(a) + f\gamma(b)$. Therefore, $f \in \text{End}(\gamma(G))$. □

We next establish two additional properties of $\mathcal{H}(G, K)$.

Lemma 6.6. *If $G = A \oplus B$ and $\text{Hom}(A, B) = \{\mathcal{O}_{A,B}\}$, then π_B , the projection of G onto B , is a right unity of $\mathcal{H}(G, B)$.*

Proof. Let $f \in \mathcal{H}(G, B)$ and let $x \in G = A \oplus B$. Then $f\pi_B(x) = f\pi_B(a+b) = f(b)$. Since $\text{Hom}(G, B)|_A \subseteq \text{Hom}(A, B) = \{\mathcal{O}_{A,B}\}$, we have that

$$\begin{aligned} f(x) &= f(a+b) = \sum_{j \in J} \varepsilon_j \sigma_j(a+b) = \sum_{j \in J} \varepsilon_j (\sigma_j(a) + \sigma_j(b)) \\ &= \sum_{j \in J} \varepsilon_j \sigma_j(b) = f(b). \end{aligned}$$

Thus π_B is a right unity of $\mathcal{H}(G, B)$. □

Lemma 6.7. *If $G = A \oplus B$, π_B is a right unity of $\mathcal{H}(G, B)$, and B is an E -group, then $\mathcal{H}(G, B)$ is a ring.*

Proof. We only need to show that $f+g = g+f$ for all $f, g \in \text{Hom}(G, B)$. Then the result follows from Proposition 2.7, since $\mathcal{H}(G, B)$ is distributively generated by $\text{Hom}(G, B)$. Let $f, g \in \text{Hom}(G, B)$. Noting that $f\pi_B, g\pi_B \in \mathcal{E}(B)$, we obtain that $f+g = f\pi_B + g\pi_B = g\pi_B + f\pi_B = g+f$, since $\mathcal{E}(B)$ is a ring by the definition of an E -group. □

Armed with Theorems 6.1 and 6.5 and the previous two lemmas, we are finally ready to build our first D - D group.

Proposition 6.8. *Let $G = A \oplus B$, where A is a D - H group, B is an E - D group, and $\text{Hom}(A, B) = \{\mathcal{O}_{A,B}\}$. Then G is a D - D group.*

We present three methods of proof: The first one is indirect, the second direct, and the third constructive. The purpose is to illustrate various aspects of the theory developed.

Proof. By Lemma 6.6, π_B is a right unity of $\mathcal{H}(G, B)$.

Method I: If G is an E - D group, then $\pi_A(G) = A$ is an E - D group by Theorem 6.1, a contradiction. If G is a D - H group, then $\pi_B(G) = B$ is a D - H group by Theorem 6.5, a contradiction. Since G is not abelian, it is not an E - H group; hence G is a D - D group.

Method II: Note that $\text{Hom}(G, B) \subsetneq \mathcal{H}(G, B)$, since $\overline{B} = B$ is nonabelian. By Lemma 6.7, $\mathcal{H}(G, B)$ is a ring, so $\mathcal{H}(G, B) = \mathcal{D}(\mathcal{H}(G, B))$. By Lemma 6.4, $\mathcal{D}(\mathcal{H}(G, B)) \subseteq \mathcal{D}(\mathcal{E}(G))$. By Theorem 6.1, G is not an E -group. Thus, $\mathcal{D}(\mathcal{E}(G)) \subsetneq \mathcal{E}(G)$, by Theorem 4.2. It follows that $\text{End}(G) \subsetneq \mathcal{D}(\mathcal{E}(G)) \subsetneq \mathcal{E}(G)$. Therefore, G is a D - D group.

Method III: We show that $1_G + 1_G$ is an element in $\mathcal{E}(G)$ not in $\mathcal{D}(\mathcal{E}(G))$, and that $\pi_B + \pi_B$ is an element in $\mathcal{D}(\mathcal{E}(G))$ not in $\text{End}(G)$. Then G is a D - D group by definition. Clearly, $1_G + 1_G \in \mathcal{E}(G)$. But $1_G + 1_G \notin \mathcal{D}(\mathcal{E}(G))$ by Theorem 4.2 (xii), since G is not an E -group by Theorem 6.1. By Proposition 3.1, $\pi_B + \pi_B \in \text{End}(G)$ if and only if $\pi_B(G) = B$ is abelian. Since B is nonabelian, $\pi_B + \pi_B \notin \text{End}(G)$. By Proposition 4.1, $\pi_B + \pi_B \in \mathcal{D}(\mathcal{E}(G))$ if and only if $\pi_B(\mathcal{E}(G)(a))$ is abelian for each $a \in G$. Since $\mathcal{H}(G, B)$ is a ring by Lemma 6.7, and $\pi_B \alpha \in \mathcal{H}(G, B)$, for any $\alpha \in \mathcal{E}(G)$, we are done. $\square$

Example 6.9. Let p, q be distinct odd primes. Let $G = D_q \oplus B$, where B is an E - D group of order p^8 from Example 4.4 or 4.5. It is clear that all hypotheses of Proposition 6.8 are satisfied.

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