An Interesting Orbit-Induced Partition of the Class of all Groups

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Abstract

Given that the set of endomorphisms of a group is contained in the set of distributive elements of its endomorphism near-ring, which, in turn, is contained in the endomorphism near-ring, we show that the class of all groups is partitioned into four nonempty subclasses when all combinations of these inclusions, proper or non-proper, are considered. Furthermore, a characterization of each subclass is given in terms of the orbits of the underlying group.

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1 Introduction

The endomorphism ring of an abelian group has been studied extensively and over a long period of time. By studying the relationship between the endomorphism ring $\text{End}(G)$ and its underlying group $G$, ring theory has been used to study problems in group theory with great success. In contrast, when the group is nonabelian, $\text{End}(G)$ is never additively closed, although it is a semigroup with respect to function composition. However, if we let $(M_0(G), +, \circ)$ denote the near-ring of zero preserving mappings on $G$, then the subgroup of $(M_0(G), +)$ generated by $\text{End}(G)$ is a near-ring, called the endomorphism near-ring of $G$, and denoted by $\mathcal{E}(G)$. Here, all near-rings are right and zero-symmetric, compatible with the composition of function convention: $(f \circ g)(x) = (fg)(x) = f(g(x))$. Recall that a right near-ring is a triple $(N, +, \cdot)$ such that $(N, +)$ is a not necessarily abelian group, $(N, \cdot)$ is a semigroup, and $(n + m) \cdot r = n \cdot r + m \cdot r$, for all $n, m, r \in N$. 
We now identify an important substructure of $\mathcal{E}(G)$: the set of distributive elements of $\mathcal{E}(G)$. Let $A$ and $B$ be sets of functions with ranges in a groupoid $(T, +)$, and let $f$ be a function with a range also in $T$. We say $f$ is left distributive over $(A, B)$ if $f(\alpha + \beta) = f\alpha + f\beta$, for each $\alpha \in A$ and $\beta \in B$.

The set of all functions that are left distributive over $(A, B)$ will be denoted by $\mathcal{D}(A, B)$. If $A = B$, we write $\mathcal{D}(A)$ instead of $\mathcal{D}(A, A)$. In a similar fashion, we define their right-sided analogs: $(A, B)\mathcal{D}$ and $A\mathcal{D}$. Let $\mathcal{D}(A) = A\mathcal{D} \cap \mathcal{D}(A)$, the set of distributive elements of $A$. A near-ring is distributive if each element is distributive. A near-ring is distributively generated (abbreviated d.g.) if it is additively generated by a set of distributive elements. Thus, $\mathcal{E}(G)$ is a d.g. near-ring since each element of $\text{End}(G)$ is distributive.

If $\mathcal{E}(G)$ is our set of functions, then $\mathcal{D}(\mathcal{E}(G)) = \mathcal{D}(\mathcal{E}(G)) = \mathcal{E}(G)$, as our near-rings are right near-rings.

Ostensibly, the class of all groups is partitioned into four subclasses corresponding to how $\mathcal{D}(\mathcal{E}(G))$ “sits” between $\text{End}(G)$ and $\mathcal{E}(G)$. In particular, $\mathcal{D}(\mathcal{E}(G))$ may coincide with both $\text{End}(G)$ and $\mathcal{E}(G)$, with $\mathcal{E}(G)$ only, with $\text{End}(G)$ only, or with neither $\text{End}(G)$ nor $\mathcal{E}(G)$. The obvious question is: are these four subclasses nonempty? The result “$\mathcal{E}(G) = \text{End}(G)$ if and only if $G$ is abelian,” a well-known result with an obscure pedigree, shows us that one of the subclasses is nonempty. In the middle of the 20th Century, Fröhlich [9] proved that “$\mathcal{E}(G) = \mathcal{D}(\mathcal{E}(G))$ if and only if $G$ is an $E$-group.” (An $E$-group was originally defined as a group $G$ in which each element commutes with any of its endomorphic images. Now the condition “$\mathcal{E}(G)$ is a ring” serves as the definition.) Of course, all abelian groups are $E$-groups. But it was not until 1971 that nonabelian examples were found by Faudree [8], establishing that a second subclass of the alleged partition is nonempty. The open question of whether the remaining two subclasses of groups are nonempty was the motivation for this paper.

2 The Four Subclasses Via Orbits

Let $E$ denote the set of “generalized Endomorphisms” or elements of $\mathcal{E}(G)$. Let $D$ denote the “Distributive elements of $\mathcal{E}(G)$” or $\mathcal{D}(\mathcal{E}(G))$. Let $H$ denote the set of “Homomorphisms of $\mathcal{E}(G)$” or $\text{End}(G)$. These assignments provide a heuristic for remembering the following definitions.

**Definition 2.1.** A group $G$ has the $E \rightarrow D$ property if $\mathcal{E}(G) = \mathcal{D}(\mathcal{E}(G))$. 
Definition 2.2. A group $G$ has the $D \to H$ property if $D(E(G)) = \text{End}(G)$.

With the aid of these two properties, we now name the four partitioning subclasses as follows:

Definition 2.3. A group $G$ is an $E-H$ group if $G$ has both the $E \to D$ and $D \to H$ properties (i.e., $E(G) = \text{End}(G)$ or $E = H$).

Definition 2.4. A group $G$ is an $E-D$ group if $G$ has the $E \to D$ property but not the $D \to H$ property.

Definition 2.5. A group $G$ is a $D-H$ group if $G$ has the $D \to H$ property but not the $E \to D$ property.

Definition 2.6. A group $G$ is a $D-D$ group if $G$ has neither the $E \to D$ nor the $D \to H$ properties.

For convenience, we present this information in the following alternate form:

$G$ is an $E-H$ group $\iff \text{End}(G) = D(E(G)) = E(G)$;
$G$ is an $E-D$ group $\iff \text{End}(G) \subsetneq D(E(G)) = E(G)$;
$G$ is a $D-H$ group $\iff \text{End}(G) = D(E(G)) \subsetneq E(G)$;
$G$ is a $D-D$ group $\iff \text{End}(G) \subsetneq D(E(G)) \subsetneq E(G)$.

Recently, in the dissertation of Diener [7], a similar problem was considered. He studied distributive elements in centralizer near-rings. In particular, let $S$ be a semigroup of endomorphisms of $G$ and let

$$M_S(G) = \{f : G \to G | f(0) = 0, f \sigma = \sigma f, \forall \sigma \in S \}.$$  

Let $\text{End}_S(G) = \text{End}(G) \cap M_S(G)$. Then $\text{End}_S(G) \subseteq D(M_S(G)) \subseteq M_S(G)$. Diener considered when the containments were proper or nonproper, for certain classes of $S$ and $G$. This extends previous work done in this area by Maxson and Meldrum [17].

Now we characterize $E-H$, $E-D$, $D-H$ and $D-D$ groups in terms of the orbits of $G$. In this paper, the orbits of $G$ are the fully invariant subgroups of $G$ of the form $E(G)(a) := \{f(a) : f \in E(G)\}$, where $a \in G$. The following result of Fröhlich [9, Theorem 4.4.3] will be useful.

Proposition 2.7 (Fröhlich). Let $(N, +, \cdot)$ be a distributively generated near-ring with a left or right unity, generated by $(S_d, \cdot)$, a semigroup of distributive elements. The following are equivalent:

(i) $N$ is a ring;

(ii) $N$ is distributive;
(iii) For all \( a, b \in S_d \), \( a + b = b + a \);

(iv) \((N^2, +)\) is abelian.

**Theorem 2.8.** Let \( R \) be the property that all orbits are abelian. Let \( S \) be the property that if \( f \in \mathcal{E}(G) \) is a homomorphism when restricted to each orbit, then \( f \in \text{End}(G) \). Then

(i) \( E\text{-}H \) groups are exactly the groups satisfying \( R \) and \( S \);

(ii) \( E\text{-}D \) groups are exactly the groups satisfying \( R \) and not \( S \);

(iii) \( D\text{-}H \) groups are exactly the groups satisfying not \( R \) and \( S \);

(iv) \( D\text{-}D \) groups are exactly the groups satisfying not \( R \) and not \( S \).

**Proof.** To prove the theorem we will show that, for a group \( G \), property \( R \) is equivalent to the \( E \rightarrow D \) property, and property \( S \) is equivalent to the \( D \rightarrow H \) property. By Proposition 2.7, \( \mathcal{E}(G) \) is a ring if and only if \((\mathcal{E}(G), +)\) is abelian if and only if \( \mathcal{E}(G) = \mathcal{D}(\mathcal{E}(G)) \). Thus, \( G \) has the \( E \rightarrow D \) property if and only if for each \( a \in G \), \((f + h)(a) = (h + f)(a)\) for each \( f, g \in \mathcal{E}(G) \) if and only if \( \mathcal{E}(G)(a) \) is abelian for each \( a \in G \) if and only if \( G \) has property \( R \).

To show \( G \) has property \( S \) if and only if \( G \) has the \( D \rightarrow H \) property, it suffices to show that if \( f \in \mathcal{E}(G) \), then \( f \in \mathcal{D}(\mathcal{E}(G)) \) if and only if \( f \big|_{\mathcal{E}(G)(a)} \) is a homomorphism for each \( a \in G \). Let \( f \in \mathcal{D}(\mathcal{E}(G)) \), \( c \in G \), and \( x, y \in \mathcal{E}(G)(c) \). Then for some \( g, h \in \mathcal{E}(G) \),

\[
f(x + y) = f(g(c) + h(c)) = f(g + h)(c) = (fg + fh)(c) = f(x) + f(y);
\]

hence, \( f \big|_{\mathcal{E}(G)(a)} \) is a homomorphism for each \( a \in G \). Now let \( f, g, h \in \mathcal{E}(G) \), \( c \in G \), and assume \( f \big|_{\mathcal{E}(G)(a)} \) is a homomorphism for each \( a \in G \). We have that

\[
f(g + h)(c) = f \big|_{\mathcal{E}(G)(c)} (g(c) + h(c)) = f \big|_{\mathcal{E}(G)(c)} (g(c)) + f \big|_{\mathcal{E}(G)(c)} (h(c)) = (fg + fh)(c),
\]

implying that \( f \in \mathcal{D}(\mathcal{E}(G)) \).

For \( E\text{-}H \) and \( E\text{-}D \) groups, we present better characterizations, which will lead us to identify an enormous variety of examples of these two classes of groups. We then find some sufficient conditions for a group to be a \( D\text{-}H \) group, and use these to find many examples belonging to this subclass. Finally, we use properties of \( E\text{-}D \) and \( D\text{-}H \) groups to construct examples of \( D\text{-}D \) groups.
3  \textit{E-H} Groups

Let \((T, +)\) be a groupoid and let \(A\) and \(B\) be sets. We say \(\{A, B\}\) is a \((T, +)\)-commutative pair if \(A, B \subseteq T\) and \(a + b = b + a\), for each \(a \in A\) and \(b \in B\). If \(A = B\), we say \(A\) is a \((T, +)\)-commutative set.

\textbf{Lemma 3.1.} Let \(\alpha, \beta \in \text{End}(G)\). The following are equivalent:

(i) \(\alpha + \beta \in \text{End}(G)\);

(ii) \(\{\alpha(G), \beta(G)\}\) is a \(G\)-commutative pair.

\textbf{Proof.} This follows from a routine calculation. \hfill \Box

\textbf{Proposition 3.2.} The following are equivalent:

(i) \(G\) is an \(E-H\) group (\(G\) has both the \(E \rightarrow D\) and \(D \rightarrow H\) properties);

(ii) \(\mathcal{E}(G) \subseteq \text{End}(G)\);

(iii) \(\{f(G), h(G)\}\) is a \(G\)-commutative pair for each \(f, h \in \mathcal{E}(G)\);

(iv) \(\{\alpha(G), \beta(G)\}\) is a \(G\)-commutative pair for each \(\alpha, \beta \in \text{End}(G)\);

(v) \(G\) is abelian;

(vi) \(1_G + 1_G \in \text{End}(G)\);

(vii) \(\gamma + \xi \in \text{End}(G)\), for some \(\gamma, \xi \in \text{Aut}(G)\).

\textbf{Proof.} \(G\) is an \(E-H\) group if and only if \(G\) has both the \(E \rightarrow D\) and \(D \rightarrow H\) properties, if and only if \(\mathcal{E}(G) \subseteq \text{End}(G)\). This establishes that (i) \(\Leftrightarrow\) (ii). It is clear that (iii) \(\Rightarrow\) (iv). Assume (iv) and note that since \(1_G \in \text{End}(G)\), we have that \(1_G(G) = G\) is a \(G\)-commutative set. In other words, \(G\) is abelian. The implication (v) \(\Rightarrow\) (iii) is immediate. Thus, we have established the equivalence of (iii), (iv) and (v). Since (v) \(\Rightarrow\) (ii) follows by a routine calculation, we only need to show (ii) \(\Rightarrow\) (iv) to establish the equivalence of (i)–(v). Assume (ii) and let \(\alpha, \beta \in \text{End}(G)\). Then \(\alpha + \beta \in \mathcal{E}(G)\) implies that \(\alpha + \beta \in \text{End}(G)\). By Lemma 3.1, we have that \(\{\alpha(G), \beta(G)\}\) is a \(G\)-commutative pair. Now we show that (v), (vi) and (vii) are equivalent. Clearly, we have that (v) \(\Rightarrow\) (vi) \(\Rightarrow\) (vii). Assume (vii). By Lemma 3.1, we have that \(\{\gamma(G), \xi(G)\}\) is a \(G\)-commutative pair. But \(\gamma, \xi \in \text{Aut}(G)\), so we have that \(G\) is abelian. Thus (vii) \(\Rightarrow\) (v); so we are done. \hfill \Box

\textbf{Remark.} Much of Proposition 3.2 is already known. It is provided here for completeness and to exhibit some not so well-known characterizations using new techniques.

By Proposition 3.2, a group has both the \(E \rightarrow D\) and \(D \rightarrow H\) properties if and only if it is an abelian group. This allows us to establish the following three corollaries:
Corollary 3.3. A group $G$ is an $E$-$D$ group if an only if $G$ is a nonabelian group with the $E \rightarrow D$ property.

Corollary 3.4. A group $G$ is a $D$-$H$ group if and only if $G$ is a nonabelian group with the $D \rightarrow H$ property.

Corollary 3.5. A group $G$ is a $D$-$D$ group if and only if $G$ is a nonabelian group that is not an $E$-$D$ or $D$-$H$ group.

4 $E$-$D$ Groups

Just as $E$-$H$ groups have already been characterized, so too have $E$-$D$ groups. In 1958, Fröhlich (see Proposition 2.7) gave several characterizations of $E$-groups. So if we include the hypothesis that $G$ is nonabelian, we get several characterizations of $E$-$D$ groups from his results. Nonetheless, it was conjectured that nonabelian $E$-groups did not exist. It was Faudree [8] who proved the conjecture false in 1971, by constructing the first example of a nonabelian $E$-group—the first $E$-$D$ group. (See Malone [15] for a lively account of the events leading to this discovery.)

In this section, we give new characterizations of $E$-groups and different proofs of previously known characterizations (as mentioned, see Fröhlich and also Malone [13]). We begin by determining exactly when the sum of two distributive elements is distributive.

Lemma 4.1. Let $f, h \in D(E(G)).$ The following are equivalent:

(i) $f + h \in D(E(G));$

(ii) $\{f(E(G)(a)), h(E(G)(a))\}$ is a $G$-commutative pair, for each $a \in G.$

Proof. This follows from a routine calculation. 

We use Proposition 4.1 to establish parts of the following theorem:

Theorem 4.2. The following are equivalent:

(i) $G$ is either an $E$-$H$ or $E$-$D$ group;

(ii) $E(G) \subseteq D(E(G))$ (i.e., $G$ has the $E \rightarrow D$ property);

(iii) $\{f(E(G)(a)), h(E(G)(a))\}$ is a $G$-commutative pair, for each $a \in G$ and $f, h \in E(G);$

(iv) $\{\alpha(E(G)(a)), \beta(E(G)(a))\}$ is a $G$-commutative pair, for each $a \in G$ and $\alpha, \beta \in \text{End}(G);$
(v) **every orbit of** $G$ **is abelian;**

(vi) $\text{End}(G)(a)$ **is a** $G$-**commutative set for each** $a \in G$;

(vii) $(\mathcal{E}(G), +)$ **is abelian;**

(viii) $G$ **is an** $E$-**group (i.e.,** $\mathcal{E}(G)$ **is a ring);**

(ix) $\{1_G, \text{End}(G)\}$ **is an** $(\mathcal{E}(G), +)$-**commutative pair;**

(x) $1_G + \alpha \in \mathcal{A}(1_G, \text{End}(G))$, **for each** $\alpha \in \text{End}(G)$;

(xi) $\text{End}(G)$ **is an** $(\mathcal{E}(G), +)$-**commutative set;**

(xii) $1_G + 1_G \in \mathcal{D}(\mathcal{E}(G));$

(xiii) $\gamma + \xi \in \mathcal{D}(\mathcal{E}(G))$, **for some** $\gamma, \xi \in \text{Aut}(G)$;

(xiv) $(\text{End}(G))^2$ **is an** $(\mathcal{E}(G), +)$-**commutative set;**

(xv) $f \mid_{\mathcal{E}(G)(a)}$ **is a homomorphism for each** $a \in G$ **and** $f \in \mathcal{E}(G)$.

**Proof.** The equivalence of (ii), (vii), (viii), (xi) and (xiv) are established in Proposition 2.7. By definition, $G$ has the $E \to D$ property if and only if $G$ is either an $E$-$H$ or $E$-$D$ group. Thus (i)$\iff$(ii). Assume (iii) and note that (iii)$\Rightarrow$(iv) is obvious. Assume (iv). Since $1_G \in \text{End}(G)$ and $1_G(\mathcal{E}(G)(a)) = \mathcal{E}(G)(a)$ **is a** $G$-**commutative set, for each** $a \in G$, we have that **every orbit of** $G$ **is abelian; hence, (iv)$\Rightarrow$(v). The implication (v)$\Rightarrow$(vi) is clear. Since $\mathcal{E}(G) = \langle \text{End}(G) \rangle_{(M_0(G), +)}$, we obtain that (vi)$\Rightarrow$(vii). Assume (vii) and let $f, h, s, t \in \mathcal{E}(G)$ **and** $a \in G$. Then we have that $f(s(a)) + h(t(a)) = (fs + ht)(a) = (ht + fs)(a) = h(t(a)) + f(s(a))$. Thus, (vii)$\Rightarrow$(iii) and we have established the equivalence of (i)–(viii). Assume (ix). Let $\alpha, \beta \in \text{End}(G)$. Then we have that

$$(1_G + \alpha)(1_G + \beta) = (\alpha + 1_G)(1_G + \beta) = \alpha + \alpha\beta + 1_G + \beta = 1_G + \alpha + \alpha\beta + \beta$$

$$= 1_G + \alpha + (\alpha + 1_G)\beta = 1_G + \alpha + (1_G + \alpha)\beta$$

$$= 1_G + \alpha + \beta + \alpha\beta = (1_G + \alpha)1_G + (1_G + \alpha)\beta.$$ 

Thus (ix)$\Rightarrow$(x). Assume (x) and let $\alpha, \beta \in \text{End}(G)$. We have that

$$(1_G + \alpha)(1_G + \beta) = (1_G + \alpha)1_G + (1_G + \alpha)\beta = 1_G + \alpha + \beta + \alpha\beta,$$

and

$$(1_G + \alpha)(1_G + \beta) = 1_G(1_G + \beta) + \alpha(1_G + \beta) = 1_G + \beta + \alpha + \alpha\beta.$$
Therefore, \( \alpha + \beta = \beta + \alpha \), establishing that \((x) \Rightarrow (xi)\). It is clear that \((xi) \Rightarrow (ix)\), which establishes the equivalence of \((i) - (xii)\). Assume \((xiii)\). By Lemma 4.1, \( \gamma + \xi \in \mathcal{D}(\mathcal{E}(G)) \) implies that, for each \( a \in G \), \( \{\gamma(\mathcal{E}(G)(a)), \xi(\mathcal{E}(G)(a))\} \) is a \( G \)-commutative pair. But \( \gamma(\mathcal{E}(G)(a)) = \mathcal{E}(G)(a) \), for each \( \gamma \in \text{Aut}(G) \), since \( \gamma(\gamma^{-1}f)(a) = f(a) \), for each \( f \in \mathcal{E}(G) \). Thus, \( \mathcal{E}(G)(a) \) is abelian for each \( a \in G \). Hence, \((xiii) \Rightarrow (v)\). Assume \((v)\). Then \( 1_G(\mathcal{E}(G)(a)) \) is a \( G \)-commutative set, which implies that \( 1_G + 1_G \in \mathcal{D}(\mathcal{E}(G)) \) by Lemma 4.1. This establishes \((v) \Rightarrow (xii)\) and the equivalence of \((i) - (xiv)\). We complete the logical circuit by showing that \((ii) \Leftrightarrow (xv)\). Assume \((ii)\) and let \( f \in \mathcal{E}(G) \) and \( a \in G \). Let \( g(a), h(a) \in \mathcal{E}(G)(a) \). Then \( f(g(a) + h(a)) = f(g + h)(a) = (fg + fh)(a) = f(g(a)) + g(h(a)) \), which implies that \( f \rvert_{\mathcal{E}(G)(a)} \) is a homomorphism. Assume \((xv)\) and let \( f \in \mathcal{E}(G) \) and \( a \in G \). For \( g, h \in \mathcal{E}(G) \), we have that

\[
f(g + h)(a) = f(g(a) + h(a)) = f \rvert_{\mathcal{E}(G)(a)} (g(a) + h(a)) = f \rvert_{\mathcal{E}(G)(a)} (g(a)) + f \rvert_{\mathcal{E}(G)(a)} (h(a)) = (fg + fh)(a);
\]

hence \( f \in \mathcal{D}(\mathcal{E}(G)) \). \(\square\)

**Corollary 4.3.** \( G \) is an \( E-D \) group if and only if \( G \) is a nonabelian \( E \)-group.

**Proof.** This follows directly from Theorem 4.2 and Corollary 3.3. \(\square\)

**Remark.** We have established that \( G \) is an \( E-D \) group if and only if \( G \) is nonabelian and satisfies any one of the properties in Theorem 4.2. Furthermore, we have that

\[
\{E\text{-groups}\} = \{E-H \text{ groups}\} \cup \{E-D \text{ groups}\} = \{\text{groups with the } E \to D \text{ property}\},
\]

where the braces denote classes. Also note that Malone [13, Theorem 6] has established the equivalence of \((ix)\) and \((xii)\) (from Theorem 4.2), using a different proof.

**Example 4.4.** If \( p \) is an odd prime, the Faudree groups \( F_{p^k} \), introduced in 1971 [8], are \( E-D \) groups. This is an infinite family of finite \( p \)-groups. It was Malone [14] who pointed out that \( p \) must be odd for Faudree’s construction to work.

**Example 4.5.** Many of the groups constructed by Jonah and Konvisser [12] are \( E-D \) groups. All of their groups have the property that \( (\text{Aut}(G), \circ) \) is abelian. Malone [14] designates the ones that are \( E-D \) groups as \( JK \) groups, and notes that, like the \( F_{p^k} \) groups, the \( JK \) groups have order \( p^8 \). Since \( p \) can be any prime natural number, this construction yielded the first published example of an \( E-D \) group that is a 2-group.

Example 4.7. Using homological algebra techniques, Caranti, Franciosi and de Giovanni [5] constructed the first published examples of infinite $E$-$D$ groups. These include both torsion and torsion-free examples.

5 $D$-$H$ Groups

In 1971, the same year that Faudree exhibited the first $E$-$D$ group, Maxson considered the problem of determining what additional conditions are needed on a group $G$ to insure that $G$ is abelian when $\mathcal{E}(G)$ is a ring. (It had just been determined by Faudree [8] that the conjecture “$\mathcal{E}(G)$ a ring implies $G$ abelian” was false.) In particular, he gives a complete answer [16, Theorem 2.4] in the restricted setting of finitely generated groups. This theorem will be proved at the end of this section as a corollary of results in this paper.

Theorem 5.1 (Maxson). Let $G$ be finitely generated and let $\mathcal{E}(G)$ be a ring. The following are equivalent:

(i) $G$ is abelian;

(ii) $G$ is a cyclic $\mathcal{E}(G)$-module.

Following Maxson, a group $G$ is a cyclic $\mathcal{E}(G)$-module if there exists an element $c$ in $G$ such that $\mathcal{E}(G)(c) = G$. The element $c \in G$ is called the $\mathcal{E}(G)$-generator of $G$. (The Quaternion group is an example of a nonabelian cyclic $\mathcal{E}(G)$-module.) Note that Johnson [11] has shown that a group $G$ is a cyclic $\mathcal{E}(G)$-module if and only if $G$ is invariantly simple. (Recall that a group is invariantly simple if it has no proper nontrivial fully invariant subgroups.)

Using results established here, we easily obtain the following generalization of Maxson’s result without any restriction on $G$.

Proposition 5.2. Let $\mathcal{E}(G)$ be a ring. The following are equivalent:

(i) $G$ is abelian;

(ii) $G$ has the $D \rightarrow H$ property.

Proof. By Proposition 3.2, we have that (i)$\Rightarrow$(ii). Assume (ii). Since $\mathcal{E}(G)$ a ring implies that $G$ has the $E \rightarrow D$ property by Theorem 4.2, $G$ has both the $E \rightarrow D$ and $D \rightarrow H$ properties. Thus $G$ is an $E$-$H$ group; hence it is abelian by Proposition 3.2. □
In this section we identify and study the nonabelian members of the class of groups with the $D \to H$ property, i.e., the $D$-$H$ groups. See Diener [7, Lemma II.2] and Maxson and Meldrum [17, Theorem 1.1] for similar results in the area of centralizer near-rings.

**Proposition 5.3.**

(i) Let $G$ be a group such that for each $a, b \in G$, $\{a, b\} \subseteq \mathcal{E}(G)(c)$, for some $c \in G$. Then $G$ has the $D \to H$ property.

(ii) Cyclic $\mathcal{E}(G)$-modules have the $D \to H$ property.

(iii) nonabelian groups satisfying the hypothesis of (i) or (ii) are $D$-$H$ groups.

**Proof.** To prove (i), let $f \in \mathcal{D}(\mathcal{E}(G))$ and $a, b \in G$. Then for some $g, h \in \mathcal{E}(G)$,

$$f(a + b) = f(g(c) + h(c)) = f(g + h)(c) = (fg + fh)(c)$$

$$= f(g(c)) + f(h(c)) = f(a) + f(b).$$

This establishes that $G$ has the $D \to H$ property. It is clear that a cyclic $\mathcal{E}(G)$-module satisfies the hypothesis of (i); so (ii) holds. Part (iii) follows from Corollary 3.4.

**Corollary 5.4.** Nonabelian invariantly simple groups are $D$-$H$ groups.

**Proposition 5.5.** Let $X$ be a set. Let $G = F \langle X \rangle$ denote the free group on $X$.

(i) $G$ is a cyclic $\mathcal{E}(G)$-module;

(ii) if $|X| > 1$, $G$ is $D$-$H$ group;

(iii) every nontrivial group is the quotient of a $D$-$H$ group.

**Proof.** To prove (i), we define for each $a \in G$, a mapping $f_a : X \to G$ via $f_a(x) = a$, for each $x \in X$. By the Universal Mapping Theorem, there exists $\overline{f_a} \in \text{End}(G)$, for each $a \in G$, such that $\overline{f_a}|_X = f_a$. So for each $a \in G$ and any $x \in X$, $\overline{f_a}(x) = a$. Thus, $\text{End}(G)(x) = G$, for each $x \in X$. Therefore, $G$ is a cyclic $\mathcal{E}(G)$-module. To prove (ii), note that if $|X| > 1$, then $G = F \langle X \rangle$ is nonabelian. By (i) and Proposition 5.3 (iii), $G$ is a $D$-$H$ group. To prove (iii), note that every group is the quotient of a free group, and use (ii).

**Proposition 5.6.** Any nonabelian orbit of any group is a $D$-$H$ group.
Proof. We first establish that any orbit $S$ of a group $G$ is a cyclic $\mathcal{E}(S)$-module. Let $a \in G$, and let $S = \mathcal{E}(G)(a)$. If $\mathcal{E}(G)(a) = G$ then $S = G$ is a cyclic $\mathcal{E}(S)$-module. Suppose $S \leq G$. Since $S$ is a fully invariant subgroup of $G$, $f |_S \in \mathcal{E}(S)$, for each $f \in \mathcal{E}(G)$. Since $a \in \mathcal{E}(G)(a) = S$, we have that $S = \mathcal{E}(G)(a) = \mathcal{E}(G)|_S a \subseteq \mathcal{E}(S)(a)$. Thus $\mathcal{E}(S)(a) = S$ and $a \in S$. Therefore $S$ is a cyclic $\mathcal{E}(S)$-module. It follows by Proposition 5.3 (iii), that a nonabelian orbit is a $D$-$H$ group.

Corollary 5.7.

(i) Every $D$-$H$ group $G$ is either a cyclic $\mathcal{E}(G)$-module or contains a proper nontrivial $D$-$H$ subgroup.

(ii) Every $D$-$D$ group contains a proper nontrivial $D$-$H$ subgroup.

Proof. By Theorem 4.2, $D$-$H$ and $D$-$D$ groups must contain a nonabelian orbit $S$, which is a cyclic $\mathcal{E}(G)$-module by Proposition 5.6. Furthermore, a $D$-$D$ group $G$ cannot be a cyclic $\mathcal{E}(G)$-module by Proposition 5.3 (ii); so $S$ is proper.

Proposition 5.8. Let $G$ be a nonabelian group such that each proper orbit is abelian. Then $G$ is either a $D$-$H$ or $E$-$D$ group.

Proof. If $\mathcal{E}(G)(a) \leq G$, for each $a \in G$, then $G$ is an $E$-$D$ group by Theorem 4.2 and Corollary 3.3. Otherwise, $\mathcal{E}(G)(a) = G$, for some $a \in G$. So $G$ is a $D$-$H$ group by Proposition 5.3 (iii).

Proposition 5.8 provides an excellent “sieve” technique for locating $D$-$H$ groups via the computer. While it theoretically will collect all $E$-$D$ groups, it identifies many $D$-$H$ groups as well, especially in groups of small order.

Example 5.9. All nonabelian groups of order at most 30 are $D$-$H$ groups. This was established using the SONATA [19] package, which is built on top of the GAP [10] group analysis program.

Corollary 5.10. A group which is not an $E$-group and has all proper orbits abelian is a $D$-$H$ group.

Proof. This follows since a group which is not an $E$-group is nonabelian and not an $E$-$D$ group. See Theorem 4.2 (i) and (viii).

Example 5.11. Let $p$ be an odd prime. Then $D_p$, the dihedral group of order $2p$, is a $D$-$H$ group. Since Chandy [6] has shown that all $E$-groups are nilpotent of class at most 3, we have that $D_p$ is not an $E$-$H$ or $E$-$D$ group. Since all subgroups of $D_p$ are abelian, $D_p$ is a $D$-$H$ group by Corollary 5.10.
Example 5.12. The Tarski groups (infinite groups all of whose proper nontrivial subgroups have order a fixed prime) are $D$-$H$ groups. These are some of the most pathological examples of nonsolvable $p$-groups. (See Ol’šanskiĭ [18] for examples of Tarski groups.) The fact that these are $D$-$H$ groups follows from the Corollary 5.10 (since every proper subgroup is abelian) or from Corollary 5.4 (since these groups are also simple).

Up to now, all examples of $D$-$H$ groups have been nonabelian cyclic $E(G)$-modules. But before we can give an example of a $D$-$H$ group that is not a cyclic $E(G)$-module, we must establish some properties of cyclic $E(G)$-modules. We begin by introducing some notation and definitions.

The following definitions are, essentially, those of Weinstein [20], but altered slightly to reflect our use of additive notation for groups. Let \( \{A_i\}_{i \in I} \) be an indexed set of groups with indexing set \( I \). Let \( 0_{A_i} \) denote the identity element of \( A_i \). Let \( O_{A_i,A_j} \) denote the homomorphism that sends each element of \( A_i \) to \( 0_{A_j} \). If \( i = j \), we write \( O_{A_i} \) instead of \( O_{A_i,A_i} \), for the identity element of \( (E(A_i),+) \). Define

\[
\sum \{A_i\}_{i \in I} := \{ g | g : I \to \bigcup \{A_i\}_{i \in I}, g(i) \in A_i \forall i \in I \}.
\]

We call \( \sum \{A_i\}_{i \in I} \) the unrestricted direct sum of \( \{A_i\}_{i \in I} \). The group operation on \( \sum \{A_i\}_{i \in I} \) is defined componentwise by the equation \( (g + h)(i) = g(i) + h(i) \), where \( g, h \in \sum \{A_i\}_{i \in I} \) and \( i \in I \). If \( g \in \sum \{A_i\}_{i \in I} \), let \( S(g) = \{i \in I | g(i) \neq 0_{A_i} \} \). Define

\[
\sum \bigoplus \{A_i\}_{i \in I} := \{ g | g \in \sum \{A_i\}_{i \in I}, |S(g)| < \infty \}.
\]

We call \( \sum \bigoplus \{A_i\}_{i \in I} \) the restricted direct sum of \( \{A_i\}_{i \in I} \).

Let \( G = \sum \{A_i\}_{i \in I} \) and let \( \omega_i \in E(A_i) \), for each \( i \in I \). Let \( \omega : G \to G \) be defined via \( \omega(g)(i) = \omega_i(g(i)) \), for each \( i \in I \). We write \( \omega = (\omega_i)_{i \in I} \). It is clear that \( \omega \in E(G) \). In some settings, all elements of \( E(G) \) are of this form.

Lemma 5.13. Let \( \{A_i\}_{i \in I} \) be an indexed set of groups such that \( \text{Hom}(A_i,A_j) = \{O_{A_i,A_j}\} \), for each \( i,j \in I, i \neq j \). Let \( G = \sum \bigoplus \{A_i\}_{i \in I} \). Then for each \( \omega \in \text{End}(G) \), \( \omega = (\omega_i)_{i \in I} \), where \( \omega_i \in \text{End}(A_i) \).

Proof. Let \( \{\Pi_i\}_{i \in I} \) be the coordinate projections and let \( \{\Gamma_i\}_{i \in I} \) be the coordinate injections. First we show that \( \omega \Gamma_i(a) \in \Gamma_i(A_i) \), for each \( i \in I, a \in A_i \), and \( \omega \in \text{End}(G) \). Suppose \( \omega \Gamma_i(a) \notin \Gamma_i(A_i) \), for some \( i \in I, a \in A_i \), \( \omega \in \text{End}(G) \). Then there exists \( A_j, j \neq i \), such that \( 0_{A_j} \neq \Pi_j \omega \Gamma_i(a) \in A_j \). But by hypothesis, \( \Pi_j \omega \Gamma_i \in \text{Hom}(A_i,A_j) = \{O_{A_i,A_j}\} \), which is a contradiction.

Now let \( \omega \in \text{End}(G) \) and \( g \in G \). Since \( |S(g)| < \infty \), we have that \( g = \sum_{i \in S(g)} \Gamma_i g(i) \). Thus \( \omega(g) = \sum_{i \in S(g)} \omega \Gamma_i g(i) \). Since we have shown that
$\omega \Gamma_i(A_i) \subseteq \Gamma_i(A_i)$, for each $i \in I$, we obtain that

$$\omega(g)(i) = \Pi_i \omega(g) = \Pi_i \sum_{i \in S(g)} \omega \Gamma_i g(i) = \sum_{i \in S(g)} \Pi_i \omega \Gamma_i g(i)$$

$$= (\Pi_i \omega \Gamma_i)(g(i)).$$

Therefore, $\omega = (\Pi_i \omega \Gamma_i)_{i \in I}$. Since $\Pi_i \omega \Gamma_i \in \text{End}(A_i)$, we are done. \hfill \Box

**Proposition 5.14.** Let $\{A_i\}_{i \in I}$ be an indexed set of groups such that $A_i$ is a cyclic $E(A_i)$-module, for each $i \in I$. Then $G = \sum \{A_i\}_{i \in I}$ is a cyclic $E(G)$-module.

**Proof.** Let $c \in G$ be defined via $c(i)$ = any $E(A_i)$-generator of $A_i$, for each $i \in I$. Then if $g \in G$, there exists $w_i \in E(A_i)$, such that $g(i) = \omega_i(c(i))$, for each $i \in I$. Thus $g = (\omega_i)_{i \in I}(c)$. Since $(\omega_i)_{i \in I} \in E(G)$, $G$ is a cyclic $E(G)$-module. \hfill \Box

**Theorem 5.15.** Let $\{A_i\}_{i \in I}$ be an indexed set of groups such that $A_i$ is a cyclic $E(A_i)$-module for each $i \in I$. Then

(i) $G = \sum \bigoplus \{A_i\}_{i \in I}$ has the $D \rightarrow H$ property;

(ii) Let $\text{Hom}(A_i, A_j) = \{O_{A_i, A_j}\}$ for each $i, j \in I, i \neq j$. Then $G$ is a cyclic $E(G)$-module if and only if at most finitely many $A_i$ are nontrivial.

**Proof.** To prove (i), let $g, h \in G$. We show that there exists $c \in G$ such that $\{g, h\} \subseteq E(G)(c)$. Then we are done by Propositions 5.2 and 5.3 (i). Let $J = S(g) \cup S(h)$ and note that $|J| < \infty$. Let $c \in G$ be defined by

$$c(i) = \begin{cases} \\ \text{any } E(A_i)\text{-generator of } A_i, & \text{if } i \in J; \\ O_{A_i}, & \text{if } i \notin J. \\ \end{cases}$$

Let $\omega = (\omega_i)_{i \in I}$, where

$$\omega_i = \begin{cases} \\ f \in E(A_i) \text{ such that } g(i) = f(c(i)), & \text{if } i \in S(g); \\ O_{A_i}, & \text{if } i \notin S(g). \\ \end{cases}$$

Then $\omega \in E(G)$. Similarly, define $\nu = (\nu_i)_{i \in I} \in E(G)$ by

$$\nu_i = \begin{cases} \\ f \in E(A_i) \text{ such that } g(i) = f(c(i)), & \text{if } i \in S(h); \\ O_{A_i}, & \text{if } i \notin S(h). \\ \end{cases}$$

Thus, $g = \omega(c)$ and $h = \nu(c)$, which implies that $\{g, h\} \subseteq E(G)(c)$. Therefore, $G$ has the $D \rightarrow H$ property.

To prove (ii), assume that infinitely many $A_i$ are nontrivial. We show that an $E(G)$-generator of $G$ cannot exist. Suppose $c$ is such a generator. Let
Let $j \in I \setminus S(c)$ (which is nonempty since infinitely many $A_i$ are nontrivial) and let $\omega \in \mathcal{E}(G)$. Then by Proposition 5.13, $\omega = (\omega_i)_{i \in I}$, where $\omega_i \in \mathcal{E}(A_i)$. Pick $g \in G$ such that $g(j) \neq 0_{A_j}$ (there is such a $g$ since infinitely many $A_i$ are nontrivial). Then we have that $\omega_j(c(j)) = \omega_j(0_{A_j}) = 0_{A_j}$; hence $g \notin \mathcal{E}(G)(c)$, a contradiction. The converse follows from Proposition 5.14.

Before we give our example of $D$-$H$ group that is not a cyclic $\mathcal{E}(G)$-module, we prove the following:

**Lemma 5.16.** A cyclic group $G$ is a cyclic $\mathcal{E}(G)$-module.

**Proof.** Let $G$ be a cyclic group with generator $a$. Let $S = \{n1_G : n \in \mathbb{Z}\}$. Then $S \subseteq \text{End}(G)$ and $S(a) = G$. Thus, $\mathcal{E}(G)(a) = G$. 

**Example 5.17.** Let $q$ be an odd prime and let $\mathcal{P}$ be any infinite set of prime natural numbers not including $q$. Let $C_p$ denote a cyclic group of order $p$. Then $G = D_q \oplus \bigoplus \{C_p\}_{p \in \mathcal{P}}$ is a $D$-$H$ group but not a cyclic $\mathcal{E}(G)$-module. To ascertain this, let $I = \mathcal{P} \cup \{q\}$ and

$$A_i = \begin{cases} D_i, & \text{if } i = q; \\ C_i, & \text{if } i \neq q. \end{cases}$$

Then, we have that $G = \bigoplus \bigoplus \{A_i\}_{i \in I}$. Note that $G = C_p$ is a cyclic $\mathcal{E}(G)$-module by the lemma above. That $G = D_q$ is a cyclic $\mathcal{E}(G)$-module is established in the proof of Proposition 5.8. By Theorem 5.15 (i), $G$ has the $D \to H$ property. Since $G$ is nonabelian, $G$ is a $D$-$H$ group. However, by Theorem 5.15 (ii), $G$ is not a cyclic $\mathcal{E}(G)$-module.

Recall that Maxson proved that if $G$ is finitely generated and $\mathcal{E}(G)$ is a ring, then $G$ is abelian if and only if $G$ is a cyclic $\mathcal{E}(G)$-module. This fact is easy to prove using results established in this paper: First assume that $G$ is abelian. Since $G$ is also finitely generated, it is the finite direct sum of cyclic groups. By Lemma 5.16 and Proposition 5.14, $G$ is a cyclic $\mathcal{E}(G)$-module. Now assume that $G$ is a cyclic $\mathcal{E}(G)$-module. Then it has the $D \to H$ property by Proposition 5.3 (ii). Since $G$ is an $E$-group, it also has the $E \to D$ property. Thus, by Proposition 3.2, $G$ must be abelian.

6 \textbf{\textit{D-D Groups}}

It is possible to construct a $D$-$D$ group as a direct sum of a $D$-$H$ group and an $E$-$D$ group using the following two theorems. The first result [13, Theorem 4] is due to Malone.

**Theorem 6.1 (Malone).** Let $G$ be a group with the $E \to D$ property and let $\gamma \in \text{End}(G)$. Then $\gamma(G)$ has the $E \to D$ property.
We will prove an analogous result with respect to the \( D \to H \) property. But first we introduce another substructure of \( E \): Let \( G = (G, +) \) be a group, not necessarily abelian, with identity element 0, and let \( K \) be a nonzero subgroup of \( G \). Let \( \text{Hom}(G, K) \) denote the set of homomorphisms from \( G \) into \( K \) and let \( \mathcal{H}(G, K) \) denote the subgroup of \( (\mathcal{E}(G), +) \), generated by \( \text{Hom}(G, K) \). Since \( \text{Hom}(G, K) \) is a semigroup under function composition, then \( \mathcal{H} := \mathcal{H}(G, K) \) is a near-ring under pointwise addition and function composition. Note that an element \( f \in \mathcal{H} \) can be written as \( f = \sum_{j \in J} \varepsilon_j \sigma_j \) where \( \sigma_j \in \text{Hom}(G, K) \), \( \varepsilon_j = \pm 1 \), and \( J \) is some linearly ordered, finite indexing set. We will let \( O_G \) denote the zero element of the near-ring \( (\mathcal{H}(G, K), +, \circ) \) and we will assume \( \mathcal{H} \) is non-trivial and let \( \mathcal{H}^\# \) denote the set \( \mathcal{H} \setminus \{O_G\} \). Note that Birkenmeier, Heatherly and Pilz [1], [2] initiated the study of the \( \mathcal{H}(G, K) \) substructure.

The following result [3, Proposition 2.7] identifies a useful connection between the structure of a group \( G \) and \( \mathcal{H}(G, K) \) having a one-sided unity. It is useful to assign \( \text{Ker} \mathcal{H} := \bigcap \{ h^{-1}(0) : h \in \mathcal{H} \} \). The justification for the use of the symbol \( \text{Ker} \mathcal{H} \) follows from the observation that \( \bigcap \{ h^{-1}(0) : h \in \mathcal{H} \} = \bigcap \{ \ker h : h \in \text{Hom}(G, K) \} \). Thus \( \text{Ker} \mathcal{H} \) is a normal subgroup of \( G \).

**Proposition 6.2.** The following are equivalent:

(i) \( \mathcal{H}(G, K) \) contains a right unity \( r \);

(ii) there exists an idempotent \( r \in \mathcal{H}^\# \) such that \( G = \text{Ker} \mathcal{H} + r(G) \);

(iii) there exists an idempotent \( r \in \mathcal{H}^\# \) such that \( \rho_r : \mathcal{H} \to \mathcal{E}(\overline{K}) |_{r(G)} \), defined via \( \rho_r(f) = f |_{r(G)} \), for each \( f \in \mathcal{H} \), is an injective group homomorphism.

This structural result will help us establish the first of two lemmas that will assist us in proving the main theorem of this section. Recall that \( A \) is the set of functions that left distribute over \( A \).

**Lemma 6.3.** Let \( K \leq G \). If \( f \in A \mathcal{E}(K) \) and \( \mathcal{H}(G, K) \) has a right unity, then \( f \in A \mathcal{H}(G, K) \).

**Proof.** Let \( f \in A \mathcal{E}(K) \), \( a \in G \) and \( h, g \in \mathcal{H}(G, K) \). Let \( r \) be a right unity of \( \mathcal{H}(G, K) \). By Proposition 6.2, \( G = \text{Ker} \mathcal{H} + r(G) \); hence \( a = n + r(b) \), for some \( n \in \text{Ker} \mathcal{H} \) and \( b \in G \). Thus, we have that

\[
(f(h + g))(a) = f(h(n + r(b)) + g(n + r(b))) = f(hr(b) + gr(b)) = f(h |_\mathcal{K} |_\mathcal{K})(r(b)) = (f(h |_\mathcal{K}) + f(g |_\mathcal{K}))(r(b)) = (fh + fg)(a).
\]

\( \Box \)
Then the result follows from Proposition 2.7, since \( H \) is a ring by the definition of an \( E \)-group. Thus \( f \gamma \) is a right unity in \( H \), then \( f \gamma \) has the \( D \to H \) property.

**Proof.** Let \( f \in \mathcal{D}(\mathcal{H}(G, K)) \), \( a \in G \) and \( h, g \in \mathcal{E}(G) \). Let \( r \) be a right unity of \( \mathcal{H}(G, K) \) in \( \text{Hom}(G, K) \). Then

\[
\begin{align*}
    f(h + g)(a) &= fr(h(a) + g(a)) = f(rh(a) + rg(a)) = f(rh + rg)(a) \\
          &= (frh + frg)(a) = (fh + fg)(a).
\end{align*}
\]

As promised, we obtain the following analog to Theorem 6.1

**Theorem 6.5.** Let \( G \) be a group with the \( D \to H \) property and let \( \gamma \in \text{End}(G) \). If \( \gamma \) is a right unity of \( \mathcal{H}(G, \gamma(G)) \), then \( \gamma(G) \) has the \( D \to H \) property.

**Proof.** Let \( f \in \mathcal{D}(\mathcal{E}(\gamma(G))) \). By Lemma 6.3, \( f \in \mathcal{H}(G, \gamma(G)) \). Since \( \gamma \in \mathcal{H}(G, \gamma(G)) \) and \( f \gamma \in \mathcal{H}(G, \gamma(G)) \), we have that \( f \gamma \in \mathcal{D}(\mathcal{H}(G, \gamma(G))) \). By Lemma 6.4, \( f \gamma \in \mathcal{D}(\mathcal{E}(G)) \), which implies that \( f \gamma \in \text{End}(G) \), since \( G \) has the \( D \to H \) property. If we let \( \gamma(a), \gamma(b) \in \gamma(G) \), we have that \( f(\gamma(a) + \gamma(b)) = f\gamma(a + b) = f\gamma(a) + f\gamma(b) \). Therefore, \( f \in \text{End}(\gamma(G)) \).

We next establish two additional properties of \( \mathcal{H}(G, K) \).

**Lemma 6.6.** If \( G = A \oplus B \) and \( \text{Hom}(A, B) = \{ O_{A, B} \} \), then \( \pi_B \), the projection of \( G \) onto \( B \), is a right unity of \( \mathcal{H}(G, B) \).

**Proof.** Let \( f \in \mathcal{H}(G, B) \) and let \( x \in G = A \oplus B \). Then \( f\pi_B(x) = f\pi_B(a + b) = f(b) \). Since \( \text{Hom}(G, B) |_{A} \subseteq \text{Hom}(A, B) = \{ O_{A, B} \} \), we have that

\[
    f(x) = f(a + b) = \sum_{j \in J} \varepsilon_j \sigma_j(a + b) = \sum_{j \in J} \varepsilon_j (\sigma_j(a) + \sigma_j(b)) \\
    = \sum_{j \in J} \varepsilon_j \sigma_j(b) = f(b).
\]

Thus \( \pi_B \) is a right unity of \( \mathcal{H}(G, B) \).

**Lemma 6.7.** If \( G = A \oplus B \), \( \pi_B \) is a right unity of \( \mathcal{H}(G, B) \), and \( B \) is an \( E \)-group, then \( \mathcal{H}(G, B) \) is a ring.

**Proof.** We only need to show that \( f + g = g + f \) for all \( f, g \in \text{Hom}(G, B) \). Then the result follows from Proposition 2.7, since \( \mathcal{H}(G, B) \) is distributively generated by \( \text{Hom}(G, B) \). Let \( f, g \in \text{Hom}(G, B) \). Noting that \( f\pi_B, g\pi_B \in \mathcal{E}(B) \), we obtain that \( f + g = f\pi_B + g\pi_B = g\pi_B + f\pi_B = g + f \), since \( \mathcal{E}(B) \) is a ring by the definition of an \( E \)-group.
Armed with Theorems 6.1 and 6.5 and the previous two lemmas, we are finally ready to build our first $D$-$D$ group.

**Proposition 6.8.** Let $G = A \oplus B$, where $A$ is a $D$-$H$ group, $B$ is an $E$-$D$ group, and $\text{Hom}(A, B) = \{O_{A,B}\}$. Then $G$ is a $D$-$D$ group.

We present three methods of proof: The first one is indirect, the second direct, and the third constructive. The purpose is to illustrate various aspects of the theory developed.

**Proof.** By Lemma 6.6, $\pi_B$ is a right unity of $\mathcal{H}(G, B)$. 

*Method I:* If $G$ is an $E$-$D$ group, then $\pi_A(G) = A$ is an $E$-$D$ group by Theorem 6.1, a contradiction. If $G$ is a $D$-$H$ group, then $\pi_B(G) = B$ is a $D$-$H$ group by Theorem 6.5, a contradiction. Since $G$ is not abelian, it is not an $E$-$H$ group; hence $G$ is a $D$-$D$ group.

*Method II:* Note that $\text{Hom}(G, B) \subseteq \mathcal{H}(G, B)$, since $\overline{B} = B$ is nonabelian. By Lemma 6.7, $\mathcal{H}(G, B)$ is a ring, so $\mathcal{H}(G, B) = \mathcal{D}(\mathcal{H}(G, B))$. By Lemma 6.4, $\mathcal{D}(\mathcal{H}(G, B)) \subseteq \mathcal{D}(\mathcal{E}(G))$. By Theorem 6.1, $G$ is not an $E$-group. Thus, $\mathcal{D}(\mathcal{E}(G)) \subsetneq \mathcal{E}(G)$, by Theorem 4.2. It follows that $\text{End}(G) \subsetneq \mathcal{D}(\mathcal{E}(G)) \subsetneq \mathcal{E}(G)$. Therefore, $G$ is a $D$-$D$ group.

*Method III:* We show that $1_G + 1_G$ is an element in $\mathcal{E}(G)$ not in $\mathcal{D}(\mathcal{E}(G))$, and that $\pi_B + \pi_B$ is an element in $\mathcal{D}(\mathcal{E}(G))$ not in $\text{End}(G)$. Then $G$ is a $D$-$D$ group by definition. Clearly, $1_G + 1_G \in \mathcal{E}(G)$. But $1_G + 1_G \notin \mathcal{D}(\mathcal{E}(G))$ by Theorem 4.2 (xii), since $G$ is not an $E$-group by Theorem 6.1. By Proposition 3.1, $\pi_B + \pi_B \in \text{End}(G)$ if and only if $\pi_B(G) = B$ is abelian. Since $B$ is nonabelian, $\pi_B + \pi_B \notin \text{End}(G)$. By Proposition 4.1, $\pi_B + \pi_B \in \mathcal{D}(\mathcal{E}(G))$ if and only if $\pi_B(\mathcal{E}(G)(a))$ is abelian for each $a \in G$. Since $\mathcal{H}(G, B)$ is a ring by Lemma 6.7, and $\pi_B \alpha \in \mathcal{H}(G, B)$, for any $\alpha \in \mathcal{E}(G)$, we are done.  

**Example 6.9.** Let $p, q$ be distinct odd primes. Let $G = D_q \oplus B$, where $B$ is an $E$-$D$ group of order $p^8$ from Example 4.4 or 4.5. It is clear that all hypotheses of Proposition 6.8 are satisfied.

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