The Pitch and the Angle of Pitch of a Closed Piece of Ruled Surface in $\mathbb{R}^{3}_{1}$

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Abstract

It is well known that in the Euclidean space $\mathbb{R}^{3}$, the pitch and the angle of pitch on a ruled surface whose directrix curve is periodic can be given [6]. On the other hand, since there is no periodic timelike curve in the Minkowsky space $\mathbb{R}^{3}_{1}$, we give similar formulas on a closed piece of a ruled surface which is obtained by restricting the directrix curve of a ruled surface to a closed interval $[a,b]$ contained in the domain of the directrix curve. Since the directrix curve is periodic spacelike, the formulas are the same in the Euclidean case when the length of the closed interval is equal to the period of the curve.

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1. Introduction

By the aid of the idea of [1], we define the pitch and the angle of pitch of a closed piece of ruled surface (Definition 2.1). In the case which piece of ruled surface is spacelike or timelike, we give these values according to the coefficient of the first fundamental form and the parameter of distribution of the surface.

Second Gaussian curvature of a non-developable ruled surface in $E^{3}$ was given by Blair and Koufogiorgos, [5]. Similar calculation in 3-dimensyonal
Mikowski space was given by [2]. In this work, we classify the conoid and helicoid in the same way of Kobayashi, [7]. We also give the angle of pitch of conoid and helicoid in theorem 3.4-3.6 by using the results of [2].

2. Basic concepts

We give briefly the classical notation of surface theory; for this purpose we have used [9] as a general reference. Let $\varphi(u, v)$ denote the position vector locally, describing a surface $M$ in Minkowsky space $R_3^1 = (R^3, dx^2 + dy^2 - dz^2)$. Then the coefficient of the first fundamental form $E, F$ and $G$ are given by

$$E = \langle \varphi_u, \varphi_u \rangle, \quad F = \langle \varphi_u, \varphi_v \rangle, \quad G = \langle \varphi_v, \varphi_v \rangle$$

(1)

where $\langle, \rangle$ denotes the scalar product of $R_3^1$.

A curve which intersects each ruling of ruled surface orthogonally is said to be an orthogonal trajectory.

Let $M$ be a surface in the Minkowski space $R_3^1$. If the induced metric on the surface $M$ is positive definite (Lorentzian metric), $M$ is said to be spacelike (timelike) in $R_3^1$ [3].

Let $\alpha : I \rightarrow R_3^1$ be a nonnull curve and let $a$ be a fixed point in $I$.

Now, for any point $s$ in $I$, the angle, denoted by $\theta(s)$, between the tangent vectors of $\alpha$ at the points $\alpha(a)$ and $\alpha(s)$ is given by

$$\theta(s) = \int_a^s \kappa(u)du$$

(2)

where $\kappa$ is the curvature function of $\alpha$ as

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}.$$  

(3)

These facts are exactly the same as in the Euclidean case.

3. The Pitch and the Angle of Pitch

Lemma 3.1. Let $M$ be a ruled surface in $R_3^1$. There exists a unique orthogonal trajectory at each point of $M$.

Proof. A ruled surface in $R_3^1$ can be expressed in terms of a directrix curve $\alpha$ and unit vector field $i$ pointing along the ruling as

$$\varphi(u, v) = \alpha(u) + vi(u).$$

(4)
An orthogonal trajectory, if exists, is of the form

\[ \beta : I \rightarrow M, \quad \beta(u) = \alpha(u) + h(u)i(u). \]  \hspace{1cm} (5)

where \( h \) is real function. Using the curve \( \beta \) is orthogonal each ruling, we have

\[ h = -\varepsilon \int <\alpha', i> + c, \quad \varepsilon = <i, i> \quad (c \text{ constant}) \]  \hspace{1cm} (6)

(6) is an unique up-to constant. That is why only one orthogonal trajectory passes through at each point.

\( M \) stands for the closed piece of a non-developable ruled surface which is obtained by restricting the directrix curve of the ruled surface to a closed interval \([a, b]\) in \( \mathbb{R}^3 \). Since the ruled surface is not a developable, we may take \( \alpha \) to be the striction line of the surface.

**Definition 3.2.** Let \( \beta \) be an orthogonal tractory at the point \( \alpha(a) \) of \( M \). The distance between \( \alpha(b) \) and \( \beta(b) \) is called the pitch of \( M \). The angle between the tangent vectors at the points \( \beta(a) \) and \( \beta(b) \) of \( \beta \) is called the angle of pitch of \( M \). (Figure 1)

**Theorem 3.3.** Let \( l \) be the pitch of \( M \). Then

\[ l = \left| -\varepsilon \int_a^b F \right|. \]

**Proof.** From the Definition 3.2, we have

\[ l = |\beta(b) - \alpha(b)| = |h(b)|. \]  \hspace{1cm} (7)

Since \( \beta \) is the orthogonal trajectory at the point \( \alpha(a), \beta(a) = \alpha(a), \) so \( h(a) = 0 \). By using (6) and (7) we get

\[ l = -\varepsilon \int_a^b <\alpha', i> |. \]  \hspace{1cm} (8)

On the other hand, we have from (1), (4) that

\[ F = <\alpha', i>. \]  \hspace{1cm} (9)

The result follows from by substituting equation (9) in equation (8).
**Theorem 3.4.** Let $M$ be a spacelike. If $\theta_{[a,b]}$ is the angle of the pitch of $M$, then

$$\theta_{[a,b]} = \int_a^b \frac{[h^4 + (jQ^2 - QF - hQ' - h^2j)^2 - Q^2h^2]^1/2}{(Q^2 - h^2)^{3/2}}$$

where

$$Q = \langle \alpha', i' \times i \rangle, \quad j = \langle i'', i' \times i \rangle.$$  

(10)

Where $Q$ is the parameter of distribution of the surface [4, p.363].

In particular, if the surface is conoid (the second kind which is given by Kobayashi in [7]), then we have

$$\theta_{[a,b]} = \int_a^b \frac{|h|}{(Q^2 - h^2)^{1/2}} |h^2 + Q'^2 - Q^2|^1/2$$

and if the surface is helicoid (the second kind), we have

$$\theta_{[a,b]} = \int_a^b \frac{|h|}{(Q^2 - h^2)}.$$  

**Proof.** Since $M$ is spacelike, $\alpha$ is a spacelike curve and $i$ is a spacelike vector field. Moreover, we may take $u$ to be the arc length of spherical curve of $i$. In this case we have

$$\langle i, i \rangle = 1, \quad \langle i', i' \rangle = -1, \quad \langle \alpha', i' \rangle = 0.$$  

Note that \{i' \times i, i, i'\} is an orthonormal basis for $R^3_1$. Thus,

$$\alpha' = Qi' \times i + Fi, \quad i'' = ji' \times i + i.$$  

(11)

From (1), (10) we get

$$D = \sqrt{EG - F^2} = \sqrt{Q^2 - v^2}.$$  

(12)

From (6) and (9)

$$h' = -F.$$  

(13)

By using (5), (11) and (13) we have

$$\beta' = Qi' \times i + hi'.$$  

(14)
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then,

\[ \langle \beta', \beta' \rangle = Q^2 - h^2. \]

The condition to be a spacelike curve for \( \beta \) is \( Q^2 - h^2 > 0 \) and this is compatible with (12). From (2) we have

\[ \theta_{[a,b]} = \int_a^b \kappa \]  \hspace{1cm} (15)

where \( \kappa \) is the curvature function of \( \beta \). Now we can calculate \( \kappa \).

By using (14) we get

\[ \beta' \times \beta'' = h \iota' \times i + [Q(jQ - F) - h(Q' + hj)] i + Qh \iota'. \]

From (3), we obtain

\[ \kappa = \left[ |h^4 + (jQ^2 - QF - hQ' - h^2 j)^2 - Q^2 h^2| \right]^{1/2} \]

\[ (Q^2 - h^2)^{3/2} \]

this completes the proof.

If the surface is conoid or helicoid, respectively then \( J = F = 0 \), or \( J = F = Q' = 0 \) by [2]. By using these results in proof of the cases whether the surface is conoid or helicoid can be dealt with easily.

Theorem 3.5. Let \( M \) be a timelike, with a spacelike directrix curve and timelike ruling curve, then the angle of the pitch of \( M \) is

\[ \theta_{[a,b]} = \int_a^b \frac{|h| \left[ |h^2 - Q^2 + Q^2| \right]^{1/2}}{(Q^2 + h^2)^{3/2}}, \]

where

\[ Q = - \langle \alpha', \iota' \times i \rangle, \quad j = - \langle \iota'', \iota' \times i \rangle. \]  \hspace{1cm} (16)

In particular, if the surface is conoid (the third kind is also given by Kobayashi [7], then

\[ \theta_{[a,b]} = \int_a^b \frac{|h|}{(Q^2 + h^2)^3}, \]

and if the surface is helicoid (the third kind), then

\[ \theta_{[a,b]} = \int_a^b \frac{|h|}{(Q^2 + h^2)} \]
Proof. In this case we have 

\[ <i, i> = -1, \quad <i', i'> = 1 \quad \text{and} \quad <\alpha', i'> = 0. \]

Note that \( \{i', i' \times i, i\} \) is an orthonormal basis for \( R^3_1 \). Thus

\[ \alpha' = -Qi' \times i - Fi, \quad i'' = -ji' \times i + i. \quad (17) \]

From (1), (16), we get

\[ D = \sqrt{F^2 - EG} = \sqrt{Q^2 + v^2}. \]

By using (5), (6), (9) and (17) we get

\[ \beta' = hi' - Qi' \times i. \]

It is clear that \( \beta \) is a spacelike curve.

From the similar computation of Theorem 2.3, we get

\[ \kappa = \left[ \frac{|h^4 - (-jq^2 + QF - hQ' - h^2j)^2 + Q^2h^2|}{(Q^2 + h^2)^{3/2}} \right]^{1/2}. \]

This completes the proof.

In this case if the surface is conoid (helicoid) then \( J = F = 0 \) (\( J = F = Q' = 0 \)) too [2].

Theorem 3.6. Let M be a timelike, with a timelike directrix curve and spacelike ruling curve in \( R^3_1 \). Then the angle of the pitch of M is

\[ \theta_{[a,b]} = \int_{a}^{b} \frac{|h^2j + hQ' - Q^2j + QF|^2 + h^2Q^2 - h^4|}{(Q^2 - h^2)^{3/2}}^{1/2} \]

where

\[ Q = <\alpha', i \times i'>, \quad j = <i'', i \times i'>. \quad (18) \]

In particular, if the surface is conoid (the first kind, see [7],then

\[ \theta_{[a,b]} = \int_{a}^{b} \frac{\sqrt{[Q^2 + Q^2 - h^2]}}{(Q^2 - h^2)^{3/2}}^{1/2}, \]
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and if the surface is helicoid (the first kind), then

\[ \theta_{[a,b]} = \int_a^b \frac{|h|}{(Q^2 - h^2)}. \]

**Proof.** In this case, we have

\[ \langle i, i \rangle = 1, \quad \langle i', i' \rangle = 1 \quad \text{and} \quad \langle \alpha', i \rangle = 0. \]

Note that \( \{i, i', i \times i'\} \) is an orthonormal basis for \( R^3_1 \). Thus

\[ \alpha' = Fi - Qi \times i', \quad i'' = -i - ji \times i'. \quad (19) \]

From (1), (18), we get

\[ D = \sqrt{F^2 - EG} = \sqrt{Q^2 - v^2}. \quad (20) \]

By using (5), (6), (9) and (19) we have

\[ \beta' = hi - Qi \times i'. \]

The timelike curve condition for \( \beta \) is \( h^2 - Q^2 < 0 \). This is compatible with (20).

The rest of the proof can be completed by using the same argument with the previous theorem.

**References**


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