A Bayesian Analysis of Record Statistics from the Rayleigh Model

M. I. Hendi, S. E. Abu-Youssef^(*) and A. A. Alraddadi

Girls College of Education for Scientific Departments,
P.O. Box 27104, Riyadh 11417, Saudi Arabia

^(*) Department of Statistics & Operations Research,
College of Science, King Saud University,
P.O. Box 2455, Riyadh 11451, Saudi Arabia

Abstract: Based on upper record values, the Bayes estimators for the parameter, reliability function, and failure rate function of Rayleigh distribution are obtained. These estimators are obtained on the basis of square error and LINEX (linear-exponential) loss functions. Comparisons in terms of risks of those under LINEX loss and squared error loss functions with Bayes estimators relative to square error loss function are given. Numerical and simulation examples are included.

Keywords: Bayes estimator; Hazard rate; LINEX loss function; Posterior distribution; Quadratic loss function; Rayleigh distribution; Record values; Reliability function; Risk efficiency; Risk function.

1. Introduction

The Rayleigh distribution is useful in life testing experiments and clinical studies. Dyer and Whisenand (1973) demonstrated the importance of this distribution in communication engineering and Polovko (1968) noted its importance in electrovacuum devices. The origin and other aspects of this distribution can be found in Siddiqui (1962), Hirano (1986) and Howlader and Hossian (1995).

The Rayleigh probability density function (pdf) is

$$f(x|\sigma) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} \quad ; \quad x \geq 0, \sigma > 0. \quad (1.1)$$

The reliability function and the failure rate function (or hazard function) at mission time t , are given respectively by

$$\bar{F}(t) = e^{\frac{-t^2}{2\sigma^2}} \quad ; \quad t \geq 0, \sigma > 0, \quad (1.2)$$

and

$$H(t) = \frac{t}{\sigma^2} \quad ; \quad t \geq 0, \sigma > 0. \quad (1.3)$$

In the estimation of reliability function use of symmetric loss function may be inappropriate as has been recognized by Canfield (1970). Overestimate of reliability

function or average failure time is usually much more serious than underestimate of reliability function or mean failure time. Also, an underestimate of the failure rate results is more serious consequence than an overestimate of the failure rate. For example, in the disaster of a space shuttle (see Feynman (1987)) the management underestimated the failure rate and therefore overestimated the reliability of solid-fuel rocket booster. Varian (1975) and Zellner (1986) proposed an asymmetric loss function known as LINEX loss function which has been found to be appropriate in the situation where overestimation is more serious than underestimation or vice-versa. When estimating a parameter σ by $\hat{\sigma}$, this loss function is given by

$$L(\Delta) = e^{a\Delta} - a\Delta - 1 \quad ; \quad a \neq 0, \quad (1.4)$$

and $\Delta = \frac{\sigma}{\hat{\sigma}} - 1$. The sign and magnitude of " a " represent the direction and degree of asymmetry respectively. The positive value of " a " is used when overestimation is more serious than underestimation while negative value of " a " is used in the reverse situation. For " a " close to zero, this loss function is approximately squared error loss and therefore almost symmetric. Several authors including Basu and Ebrahimi (1991), Rojo (1987), Soliman (2000) and Zellner (1986) have used this loss function in various estimation and prediction problems.

If we define $\Delta_1 = \hat{\sigma} - \sigma$, then $L(\Delta_1)$ is equivalent to the loss function used by Varian (1975) and Zellner (1986). And if we define $\Delta_2 = \left(\frac{\hat{\sigma}}{\sigma}\right)^2 - 1$, then $L(\Delta_2)$ is equivalent to the loss function used by Soliman (2000).

For situations where life tester has no prior information about the parameter σ , we may use the quasi-density prior in the following form:

$$p(\sigma) \propto \frac{1}{\sigma^d} \quad ; \quad \sigma > 0, d > 0.$$

If $d = 1$ we get a non-informative prior (Jeffrey's (1961))

$$p_1(\sigma) \propto \frac{1}{\sigma} \quad . \quad (1.5)$$

Also, if $d = 3$ we get the asymptotically invariant prior, proposed by Hartigan (1964)

$$p_2(\sigma) \propto \frac{1}{\sigma^3} \quad . \quad (1.6)$$

Chandler (1952) introduced the study of record values and documented many of the basic properties of records. Record values can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of observations. In a little over thirty years, a large number of publications devoted to records have appeared. This is possibly due to the fact that we encounter this notion frequently in daily life, especially in singling out record values from a set of others and in registering and recalling record values. Record values can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of observations.

Let $X_1, X_2, \dots$ is a sequence of independent and identically distributed random variables with cdf $F(x)$ and pdf $f(x)$. Set $Y_n = \max(\min)X_1, X_2, \dots, X_n, n \geq 1$. We say X_j is an upper (lower) record of this sequence if $Y_j > (<)Y_{j-1}, j > 1$. By

definition X_1 is an upper as well as a lower record value. One can transform from upper record values to lower records by replacing the original sequence of random variables by $-X_j, j \geq 1$ or (if $P(X_i > 0) = 1$ for all i) by $\frac{1}{X_i}, i \geq 1$; the lower record values of this sequence will correspond to the upper record values of the original sequence. The notations $X_{U(n)}$ and $X_{L(n)}$ are used for the n th upper and lower records, respectively. For more details on record values see Ahsanullah (1995), Arnold et al (1998) and Jaheen (2003), among others .

The plan of the article is as follows: In sections 2, Bayes estimates for the unknown parameter σ of the Rayleigh distribution are obtained based on upper record values. The estimates are given on based the square error and LINEX loss function $L(\Delta_2)$ by using $p_i(\sigma), (i = 1, 2)$ as prior distributions, the risk of estimates have been obtained. Comparison in terms of risk with the estimates of σ under squared error loss and LINEX loss when the prior distributions are $p_i(\sigma), (i = 1, 2)$ have been made. Also, we give a numerical example to compare our results. In section 3 we obtain Bayes estimates of $\bar{F}(t)$ and $H(t)$ under prior distributions $p_i(\sigma), (i = 1, 2)$ when the LINEX loss function $L(\Delta_1)$ is used and compared with those corresponding to the squared error loss using Monte Carlo Simulation.

2. Bayes Estimate of σ .

In this section, we shall be concerned with estimation of the unknown parameter σ of the Rayleigh model based on record values. Suppose we observe n record values $X_{U(1)}, X_{U(2)}, \dots, X_{U(n)}$ from the Rayleigh distribution with pdf given by (1.1). The likelihood function (LF) (see Ahsanullah (1995)) is given by

$$f(\underline{x}|\sigma) = \prod_{i=1}^{n-1} H(x_i) f(x_n) \quad (2.1)$$

where $\underline{x} = (x_1, x_2, \dots, x_n)$ and $H(\cdot)$ is the hazard function corresponding to pdf $f(\cdot)$.

It follows, from (1.1), (1.2) and (1.2), that

$$f(\underline{x}|\sigma) = \left(\prod_{i=1}^n x_i \right) \sigma^{-2n} e^{\frac{-S}{2\sigma^2}} ; \quad \sigma > 0 \quad (2.2)$$

where $S = x_n^2$.

2.1 Maximum likelihood Estimation of σ .

The natural logarithm of the LF (2.2) is

$$\ell = \ln f(\underline{x}|\sigma) = \ln \prod_{i=1}^n x_i - 2n \ln \sigma - \frac{S}{2\sigma^2} \quad (2.3)$$

Assuming that the parameter σ is unknown, the maximum likelihood estimate (MLE) of the parameter σ can be shown to be

$$\hat{\sigma}_{ML} = \left(\frac{S}{2n} \right)^{\frac{1}{2}} \quad (2.4)$$

Note that, when X has one-parameter Rayleigh distribution as in (1.1), then it is easy to see that X^2 has one parameter exponential distribution, which implies $\frac{S}{\sigma^2}$ follows a chi-square distribution with 2 degree of freedom giving the probability density function of S as

$$h(s) = \frac{1}{2\sigma^2} e^{\frac{-s}{2\sigma^2}}, \quad s > 0. \quad (2.5)$$

2.2 Bayes Estimator of σ based on squared error loss function.

Case (i): We assume that the prior density $p_1(\sigma)$, combining the likelihood function (2.2) and the prior $p_1(\sigma)$, the posterior density of σ is

$$\pi_1(\sigma|x) = \frac{s^n \sigma^{-(2n+1)} e^{\frac{-s}{2\sigma^2}}}{2^{n-1} \Gamma(n)}, \quad \sigma > 0 \quad (2.6)$$

By using (2.6) under square error loss ($L(\hat{\sigma}, \sigma) = (\hat{\sigma} - \sigma)^2$), the Bayes estimate of σ denoted by $\hat{\sigma}_{BS1}$ is the posterior mean $E_{\pi_1}(\sigma)$, is

$$\hat{\sigma}_{BS1} = \left(\frac{s}{2}\right)^{\frac{1}{2}} \theta_1, \quad (2.7)$$

$$\text{where } \theta_1 = \frac{\Gamma(n - \frac{1}{2})}{\Gamma(n)}$$

Case (ii): We assume that the prior $p_2(\sigma)$ in (1.6), combining the likelihood function (2.2) and the prior $p_2(\sigma)$, the posterior density of σ is

$$\pi_2(\sigma|x) = \frac{s^{n+1} \sigma^{-(2n+3)} e^{\frac{-s}{2\sigma^2}}}{2^n \Gamma(n+1)}, \quad \sigma > 0 \quad (2.8)$$

Then σ^2 has an inverted gamma distribution ($InGa(\alpha, \beta)$), where in (2.8) $\alpha = n+1$ and $\beta = \frac{s}{2}$ and the Bayes estimate of σ denoted by $\hat{\sigma}_{BS2}$ is the posterior mean $E_{\pi_2}(\sigma)$, hence

$$\hat{\sigma}_{BS2} = \left(\frac{s}{2}\right)^{\frac{1}{2}} \theta_2, \quad (2.9)$$

$$\text{where } \theta_2 = \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n+1)}.$$

2.3 Bayes Estimator of σ based on LINEX loss function.

Under the LINEX loss function (1.4), the posterior expectation of the loss function

$L(\Delta_2)$ with respect to $\pi_i(\sigma|x)$ in (2.6) and (2.8), is

$$E_{\pi_i}[L(\Delta_2)] = \int_0^\infty \left\{ e^{a \left[\left(\frac{\hat{\sigma}}{\sigma} \right)^2 - 1 \right]} - a \left[\left(\frac{\hat{\sigma}}{\sigma} \right)^2 - 1 \right] - 1 \right\} \pi_i(\sigma | \underline{x}) d\sigma, \quad i = 1, 2, \quad (2.10)$$

i.e.

$$E_{\pi_i}[L(\Delta_2)] = e^{-a} E_{\pi_i} \left[\exp \left(a \left(\frac{\hat{\sigma}}{\sigma} \right)^2 \right) \right] - a E_{\pi_i} \left[\left(\frac{\hat{\sigma}}{\sigma} \right)^2 - 1 \right] - 1, \quad i = 1, 2. \quad (2.11)$$

The value of $\hat{\sigma}$ that minimizes the posterior expectation of the loss function $L(\Delta_2)$, denoted by $\hat{\sigma}_{BLi}$, ($i = 1, 2$); are obtained by solving the equations:

$$\frac{dE_{\pi_i}[L(\Delta_2)]}{d\hat{\sigma}} = E_{\pi_i} \left[e^{-a} \frac{\hat{\sigma}}{\sigma^2} \exp \left(a \left(\frac{\hat{\sigma}}{\sigma} \right)^2 \right) \right] - E_{\pi_i} \left(\frac{\hat{\sigma}}{\sigma^2} \right) = 0, \quad i = 1, 2, \quad (2.12)$$

that is, $\hat{\sigma}_{BLi}$ is the solution to the following equations:

$$E_{\pi_i} \left[\frac{\hat{\sigma}_{BLi}}{\sigma^2} \exp \left(a \left(\frac{\hat{\sigma}_{BLi}}{\sigma} \right)^2 \right) \right] = e^a E_{\pi_i} \left(\frac{\hat{\sigma}_{BLi}}{\sigma^2} \right), \quad i = 1, 2, \quad (2.13)$$

provided that all expectations exist and are finite. Using (2.6) and (2.8) in (2.13) respectively when $i = 1, 2$, we get the optimal estimate of σ relative to $L(\Delta_2)$ is:

$$\hat{\sigma}_{BL1} = \left[\frac{s}{2a} \left(1 - e^{\frac{-a}{n+1}} \right) \right]^{\frac{1}{2}}, \quad (2.14)$$

and

$$\hat{\sigma}_{BL2} = \left[\frac{s}{2a} \left(1 - e^{\frac{-a}{n+2}} \right) \right]^{\frac{1}{2}}. \quad (2.15)$$

It may be noted that: both Bayes estimators $\hat{\sigma}_{BLi}$ and $\hat{\sigma}_{BSi}$ ($i = 1, 2$) are asymptotically equal ($n \rightarrow \infty$) to the maximum likelihood estimator $\sigma_{ML}^2 = \left(\frac{S}{2n} \right)^{\frac{1}{2}}$.

2.4.1 The risk efficiency of $\hat{\sigma}_{BLi}$ with respect to $\hat{\sigma}_{BSi}$ ($i = 1, 2$) under LINEX loss $L(\Delta_2)$.

The risk function of estimators $\hat{\sigma}_{BLi}$ and $\hat{\sigma}_{BSi}$ ($i = 1, 2$) are of interest, these risk functions (the expected loss $E_X[L(\Delta_2)]$) are denoted by $R_{L_i}(\hat{\sigma}_{BLi})$ and $R_{L_i}(\hat{\sigma}_{BSi})$, $i = 1, 2$, where the subscript L denote risk relative to $L(\Delta_2)$, and are given by using $h(s)$ in (2.5) as follow:

$$R_{L_i}(\hat{\sigma}_{BLi}) = E_X[L(\Delta_2)] = \int_0^\infty \left\{ e^{a \left[\left(\frac{\hat{\sigma}_{BLi}}{\sigma} \right)^2 - 1 \right]} - a \left[\left(\frac{\hat{\sigma}_{BLi}}{\sigma} \right)^2 - 1 \right] - 1 \right\} h(s) ds, \quad i = 1, 2, \quad (2.16)$$

therefore, after some calculations for (2.16), the risk of $\hat{\sigma}_{BLi}$ under LINEX loss $L(\Delta_2)$ is given by

$$R_{L_1}(\hat{\sigma}_{BL1}) = e^{\frac{-na}{n+1}} + e^{\frac{-a}{n+1}} + a - 2, \quad (2.17)$$

and

$$R_{L_2}(\hat{\sigma}_{BL2}) = e^{\frac{-(n+1)a}{n+2}} + e^{\frac{-a}{n+2}} + a - 2. \quad (2.18)$$

In same manner we get $R_{L_i}(\hat{\sigma}_{BSi})$ as follow:

$$R_{L_i}(\hat{\sigma}_{BSi}) = E_X[L(\Delta_2)] = \int_0^\infty \left\{ e^{a \left[\left(\frac{\hat{\sigma}_{BSi}}{\sigma} \right)^2 - 1 \right]} - a \left[\left(\frac{\hat{\sigma}_{BSi}}{\sigma} \right)^2 - 1 \right] - 1 \right\} h(s) ds, i = 1, 2, \quad (2.19)$$

therefore, the risk function of estimators $\hat{\sigma}_{BSi}$, ($i = 1, 2$) in similarly obtained by

$$R_{L_1}(\hat{\sigma}_{BS1}) = e^{-a} (1 - a\theta_1^2)^{-1} + a(1 - \theta_1^2) - 1, \quad (2.20)$$

and

$$R_{L_2}(\hat{\sigma}_{BS2}) = e^{-a} (1 - a\theta_2^2)^{-1} + a(1 - \theta_2^2) - 1. \quad (2.21)$$

The risk efficiency of $\hat{\sigma}_{BLi}$ with respect to $\hat{\sigma}_{BSi}$ ($i = 1, 2$) under LINEX loss $L(\Delta_2)$ may be defined as the following:

$$RE_{L_i}(\hat{\sigma}_{BLi}, \hat{\sigma}_{BSi}) = \frac{R_{L_i}(\hat{\sigma}_{BSi})}{R_{L_i}(\hat{\sigma}_{BLi})}, i = 1, 2 \quad (2.22)$$

2.4.2 The risk efficiency of estimators $\hat{\sigma}_{BLi}$ and $\hat{\sigma}_{BSi}$ ($i = 1, 2$) under squared error loss $L(\Delta_1)$.

The risk functions (the expected loss $E_X[L(\Delta)]$) are denoted by $R_{S_i}(\hat{\sigma}_{BLi})$ and

$R_{S_i}(\hat{\sigma}_{BSi})$, $i = 1, 2$, with subscript S_i denoting squared error loss and are given by

$$R_{S_i}(\hat{\sigma}_{BLi}) = \int_0^\infty (\hat{\sigma}_{BLi} - \sigma)^2 h(s) ds, \quad i = 1, 2. \quad (2.23)$$

$$\text{Thus } R_{S_1}(\hat{\sigma}_{BL1}) = \int_0^\infty \left\{ \frac{s}{2a} \left(1 - e^{\frac{-a}{n+1}} \right) - 2\sigma \frac{s}{2a} \left(1 - e^{\frac{-a}{n+1}} \right)^{\frac{1}{2}} + \sigma^2 \right\} \frac{1}{2\sigma^2} e^{\frac{-s}{2\sigma^2}} ds, \quad (2.24)$$

i.e.

$$R_{S_1}(\hat{\sigma}_{BL1}) = \sigma^2 (1 + 2\phi_1 - (2\pi\phi_1)^{\frac{1}{2}}), \quad (2.25)$$

$$\text{where } \phi_1 = \frac{1}{2a} (1 - e^{\frac{-a}{n+1}}) \text{ and } \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}.$$

and

$$R_{S_2}(\hat{\sigma}_{BL2}) = \sigma^2 (1 + 2\phi_2 - (2\pi\phi_2)^{\frac{1}{2}}), \quad (2.26)$$

$$\text{where } \phi_2 = \frac{1}{2a} (1 - e^{\frac{-a}{n+2}}).$$

Also, the risk of σ_{BSi}^2 under squared error loss are

$$R_{S_i}(\hat{\sigma}_{BSi}) = \int_0^{\infty} (\hat{\sigma}_{BSi} - \sigma^2)^2 h(s) ds, \quad i = 1, 2. \quad (2.27)$$

Thus

$$R_{S_1}(\hat{\sigma}_{BS1}) = \int_0^{\infty} \left[\frac{s}{2} \theta_1 - 2\sigma\theta_1 \left(\frac{s}{2} \right)^{\frac{1}{2}} + \sigma^2 \right] \frac{1}{2\sigma^2} e^{\frac{-s}{2\sigma^2}} ds, \quad (2.28)$$

i.e.

$$R_{S_1}(\hat{\sigma}_{BS1}) = \sigma^2 (1 - \theta_1 \sqrt{\pi} + \theta_1^2), \quad (2.29)$$

and

$$R_{S_2}(\hat{\sigma}_{BS2}) = \sigma^2 (1 - \theta_2 \sqrt{\pi} + \theta_2^2). \quad (2.30)$$

The efficiency of $\hat{\sigma}_{BLi}$ with respect to $\hat{\sigma}_{BSi}$ under squared error loss is defined

$$RE_{S_i}(\hat{\sigma}_{BLi}, \hat{\sigma}_{BSi}) = \frac{R_{S_i}(\hat{\sigma}_{BSi})}{R_{S_i}(\hat{\sigma}_{BLi})}, \quad i = 1, 2. \quad (2.31)$$

It may be seen from expressions of risks of the estimators that an analytical comparison of these risks may not be possible, therefore, we decide for an empirical comparison.

2.5 Numerical example.

To compare the proposed estimator $\hat{\sigma}_{BLi}$ ($i=1,2$) with the usual estimators $\hat{\sigma}_{BSi}$ ($i=1,2$), the risk function are computed so as to see (i) whether $\hat{\sigma}_{BLi}$ outperforms $\hat{\sigma}_{BSi}$ under LINEX loss $L(\Delta_2)$ and (ii) how $\hat{\sigma}_{BLi}$ performs as compared to $\hat{\sigma}_{BSi}$ when true loss is squared error. A comparison of this type may be needed to check whether an estimator is inadmissible under some loss function. If it is so the estimator would not be used for the losses specified by that loss function. For this purpose the risks of the estimators and risk efficiency have been computed.

Example 2.1: An upper record sample of size 10 is generated from the Rayleigh model given by (1.1) with $\sigma = 1$ and written in order form as:

2.60303, 2.95197, 3.28734, 3.53151, 3.69386, 3.79286, 4.14410, 4.59519, 4.91349, 5.01159, we compute the estimates and their corresponding risk efficiency for σ , in table I and table II as follows:

Table I

The estimators $\hat{\sigma}_{BL1}, \hat{\sigma}_{BS1}$ the risk efficiencies $RE_{L_1}(\hat{\sigma}_{BL1}, \hat{\sigma}_{BS1})$ and $RE_{S_1}(\hat{\sigma}_{BL1}, \hat{\sigma}_{BS1})$ under the prior $p_1(\sigma)$ for the variations in a and an upper record sample $n = 10$.

$\hat{\sigma}_{BS1} = 1.165, R_{S1}(\hat{\sigma}_{BS1}) = 0.525$						
a	$\hat{\sigma}_{BL1}$	$RL_1(\hat{\sigma}_{BL1})$	$RL_1(\hat{\sigma}_{BS1})$	$R_{S1}(\hat{\sigma}_{BL1})$	$RE_{L_1}(\hat{\sigma}_{BL1}, \hat{\sigma}_{BS1})$	$RE_{S_1}(\hat{\sigma}_{BL1}, \hat{\sigma}_{BS1})$
2.0	1.022	0.996	0.957	0.572	0.960	0.918
1.0	1.045	0.316	0.304	0.564	0.963	0.931
0.5	1.056	0.090	0.087	0.560	0.965	0.937
-0.5	1.081	0.122	0.118	0.552	0.969	0.951
-1.0	1.093	0.577	0.561	0.548	0.972	0.958
-2.0	1.119	3.360	3.292	0.540	0.980	0.973

Table II

The estimators $\hat{\sigma}_{BL2}, \hat{\sigma}_{BS2}$ the risk efficiencies $RE_{L_2}(\hat{\sigma}_{BL2}, \hat{\sigma}_{BS2})$ and $RE_{S_2}(\hat{\sigma}_{BL2}, \hat{\sigma}_{BS2})$ under the prior $p_2(\sigma)$ for the variations in a and an upper record sample $n = 10$

$\hat{\sigma}_{BS2} = 1.107, R_{S2}(\hat{\sigma}_{BS2}) = 0.544$						
a	$\hat{\sigma}_{BL2}$	$RL_2(\hat{\sigma}_{BL2})$	$RL_2(\hat{\sigma}_{BS2})$	$R_{S2}(\hat{\sigma}_{BL2})$	$RE_{L_2}(\hat{\sigma}_{BL2}, \hat{\sigma}_{BS2})$	$RE_{S_2}(\hat{\sigma}_{BL2}, \hat{\sigma}_{BS2})$
2.0	0.982	1.006	0.973	0.586	0.967	0.929
1.0	1.002	0.320	0.310	0.579	0.969	0.940
0.5	1.012	0.092	0.089	0.575	0.971	0.946
-0.5	1.034	0.124	0.121	0.568	0.975	0.958
-1.0	1.045	0.588	0.574	0.564	0.977	0.964
-2.0	1.067	3.436	3.378	0.557	0.983	0.977

We compute and report the estimates and their corresponding risk efficiencies for σ . Table I and II show Bayes LINEX estimators $\hat{\sigma}_{BLi} (i=1,2)$, risk efficiency $RE_{L_i}(\hat{\sigma}_{BLi}, \hat{\sigma}_{BSi})$, $(i=1,2)$ of $\hat{\sigma}_{BLi} (i=1,2)$ with respect to $\hat{\sigma}_{BSi} (i=1,2)$ under LINEX loss ($L(\Delta_2)$), risk efficiency $RE_{S_i}(\hat{\sigma}_{BLi}, \hat{\sigma}_{BSi})$ under squared error loss. From table I and II (estimators based on prior density functions, $p_i(\sigma), (i=1,2)$) we note:

- i- The risk efficiency $RE_{L_i}(\hat{\sigma}_{BLi}, \hat{\sigma}_{BSi})$, $(i=1,2)$ is smaller than one for all value of " a ", $(a = \pm 0.5, \pm 1, \pm 2)$ which indicates that the proposed estimators $\hat{\sigma}_{BLi} (i=1,2)$ is not preferable to $\hat{\sigma}_{BSi} (i=1,2)$.
- ii- The risk efficiency $RE_{L_i}(\hat{\sigma}_{BLi}, \hat{\sigma}_{BSi})$ is greater than the risk efficiency $RE_{S_i}(\hat{\sigma}_{BLi}, \hat{\sigma}_{BSi})$, $(i=1,2)$ for all value " a ", $(a = \pm 0.5, \pm 1, \pm 2)$.

3. Bayes estimate of $\bar{F}(t)$ and $H(t)$.

Let $\gamma = \bar{F}(t)$ be the probability that a system will survive a specified mission time t . By substituting $\sigma^2 = t^2 / (-2 \ln \gamma)$ in (2.6) and (2.8), we obtain the posterior pdf of γ (under $p_1(\sigma)$ and $p_2(\sigma)$ which given in (1.5) and (1.6) respectively), as the following:

$$\xi_1(\gamma|x) = \frac{1}{\Gamma(n)} \left(\frac{s}{t^2} \right)^n \gamma^{\frac{s}{t^2}-1} (-\ln \gamma)^{n-1}, \quad 0 < \gamma < 1, \quad (3.1)$$

and

$$\xi_2(\gamma|x) = \frac{1}{\Gamma(n+1)} \left(\frac{s}{t^2} \right)^{n+1} \gamma^{\frac{s}{t^2}-1} (-\ln \gamma)^n, \quad 0 < \gamma < 1. \quad (3.2)$$

Using the convex loss function $L(\Delta_1)$, $\Delta_1 = \hat{\gamma} - \gamma$. It is seen that this loss function is quite asymmetric when $a=1$ with overestimation being more costly than underestimation. Also, when $a < 0$, $L(\Delta_1)$ rises almost exponentially when $\Delta_1 < 0$, and almost linearly when $\Delta_1 > 0$. For small values of $|a|$ the loss function $L(\Delta_1)$ is approximately squared error loss and therefore almost symmetric. For more details

about $L(\Delta_1)$ see Zellner (1986). The posterior expectation of the LINEX loss function $L(\Delta_1)$ is

$$E_{\gamma_{BLi}}[L(\Delta_1)] = \int_0^1 [e^{a(\hat{\gamma}-\gamma)} - a(\hat{\gamma}-\gamma) - 1] \xi_i(\gamma|\underline{x}) d\gamma, \quad i=1,2, \quad (3.3)$$

i.e.

$$E_{\gamma_{BLi}}[L(\Delta_1)] = e^{a\hat{\gamma}} E_{\gamma_{BLi}}(e^{-a\gamma}) + E_{\gamma_{BLi}}(a\gamma) - a\gamma - 1, \quad i=1,2. \quad (3.4)$$

The Bays estimate of γ relative to $L(\Delta_1)$, denoted by $\hat{\gamma}_{BLi}$, $i=1,2$ is obtained by minimizing (3.4), then

$$\frac{dE_{\gamma_{BLi}}[L(\Delta_1)]}{d\hat{\gamma}} = ae^{a\hat{\gamma}} E_{\gamma_{BLi}}(e^{-a\gamma}) - a = 0, \quad i=1,2,$$

hence

$$E_{\gamma_{BLi}}(e^{-a\gamma}) = e^{-a\hat{\gamma}},$$

i.e.

$$\int_0^1 e^{-a\gamma} \xi_i(\gamma|\underline{x}) d\gamma = e^{-a\hat{\gamma}}, \quad i=1,2 \quad (3.5)$$

when $i=1$, in (3.5), we obtain

$$\hat{\gamma}_{BL1} = \frac{-1}{a} \ln \sum_{j=1}^{\infty} \frac{(-a)^j}{j!} \left(1 + \frac{jt^2}{s}\right)^{-n}, \quad (3.6)$$

and when $i=2$, in (3.5), we obtain

$$\hat{\gamma}_{BL2} = \frac{-1}{a} \ln \sum_{j=1}^{\infty} \frac{(-a)^j}{j!} \left(1 + \frac{jt^2}{s}\right)^{-(n+1)} \quad (3.7)$$

Also, using (3.1) and (3.2) under squared error loss ($L(\hat{\gamma}, \gamma) = (\hat{\gamma} - \gamma)^2$) the Bayes estimate of γ , denoted by $\hat{\gamma}_{BSi}$, ($i=1,2$) is the posterior mean $E_{\xi_i}(\gamma)$, are

$$\hat{\gamma}_{BS1} = \left(1 + \frac{t^2}{s}\right)^{-n}, \quad (3.8)$$

and

$$\hat{\gamma}_{BS2} = \left(1 + \frac{t^2}{s}\right)^{-(n+1)}. \quad (3.9)$$

It can be seen that the risk functions $R_L(\hat{\gamma}_{BLi}), R_L(\hat{\gamma}_{BSi}), R_S(\hat{\gamma}_{BLi})$ and $R_S(\hat{\gamma}_{BSi}), (i=1,2)$ do not exist, where L denoted risk relative to $L(\Delta_1)$.

In life testing, the failure-time distributions are often specified by choosing an appropriate hazard rate function, the Rayleigh distribution is a member of class of life-time distributions characterized by a linear hazard rate. Let $\alpha = H(t)$ be the failure rate function (hazard rate function), by substituting $\sigma^2 = \frac{t}{\alpha}$ in equation (2.6) and (2.8), we obtain the posterior pdf of α (under $p_1(\sigma)$ and $p_2(\sigma)$ in (1.5) and (1.6) respectively), as:

$$\eta_1(\alpha | \underline{x}) = \frac{1}{\Gamma(n)} \left(\frac{s}{2t} \right)^n \alpha^{n-1} e^{\frac{-\alpha s}{2t}}, \quad \alpha > 0, \quad (3.10)$$

and

$$\eta_2(\alpha | \underline{x}) = \frac{1}{\Gamma(n+1)} \left(\frac{s}{2t} \right)^{n+1} \alpha^n e^{\frac{-\alpha s}{2t}}, \quad \alpha > 0. \quad (3.11)$$

The posterior expectation of LINEX loss function $L(\Delta_1)$ is

$$E_{\alpha BLi}[L(\Delta_1)] = \int_0^\infty [e^{a(\hat{\alpha}-\alpha)} - a(\hat{\alpha}-\alpha) - 1] \eta_i(\alpha | \underline{x}) d\alpha; i=1,2, \quad (3.12)$$

i.e.

$$E_{\alpha BLi}[L(\Delta_1)] = e^{a\hat{\alpha}} E_{\alpha BLi}(e^{-a\alpha}) + E_{\alpha BLi}(a\alpha) - a\hat{\alpha} - 1. \quad (3.13)$$

The Bayes estimate of α relative to $L(\Delta_1)$, denoted by $\hat{\alpha}_{BLi}$ are obtained by minimizing (3.13), therefore

$$\frac{dE_{\alpha BLi}[L(\Delta_1)]}{d\hat{\alpha}} = -a e^{a\hat{\alpha}} E_{\alpha BLi}(e^{-a\alpha}) + a = 0, i=1,2,$$

then

$$E_{\alpha BLi}(e^{-a\alpha}) = e^{-a\hat{\alpha}}, i=1,2,$$

i.e.

$$\int_0^\infty e^{-a\alpha} \eta_i(\alpha | \underline{x}) d\alpha = e^{-a\hat{\alpha}}, i=1,2, \quad (3.14)$$

when $i=1$, in (3.14), we obtain

$$\hat{\alpha}_{BL1} = \frac{1}{a} \ln \left(1 + \frac{2at}{s} \right)^n, \quad (3.15)$$

and when $i=2$, in (3.14), we obtain:

$$\hat{\alpha}_{BL2} = \frac{1}{a} \ln \left(1 + \frac{2at}{s} \right)^{n+1}. \quad (3.16)$$

Also, using (3.10) and (3.11) under squared error loss ($L(\hat{\alpha}, \alpha) = (\hat{\alpha} - \alpha)^2$), the Bayes estimate of α , denoted by $\hat{\alpha}_{BSi}$ ($i=1,2$) is the posterior mean $E_{\eta_i}(\alpha)$, are when $i=1$, is

$$\hat{\alpha}_{BS1} = \frac{2nt}{s}, \quad (3.17)$$

and when $i=2$, is

$$\hat{\alpha}_{BS2} = \frac{2(n+1)t}{s}. \quad (3.18)$$

Example 3.1: A gain let us consider the set of data a generated in example 2.1, let $t=1$, we compute $\hat{\gamma}_{BLi}, \hat{\gamma}_{BSi}, \hat{\alpha}_{BLi}$ and $\hat{\alpha}_{BSi}$, ($i=1,2$) where subscript i denote estimator based on the prior density function $p_i(\alpha)$, $i=1,2$ the results with the corresponding values of " a " and n are given in tables III and IV. It can be seen the Bayes estimators of reliability function and hazard functions relative to $L(\Delta_1)$ for positive value of a are lower than the Bayes estimate under squared error loss.

Example 3.2: (Simulation)

One sample does not tell us much. We generated $N=100$ samples of sizes $n=5, 7, 10$ from (1.1) with $\sigma=1$. Define the Monte Carlo estimator of a parameter

θ by: $\hat{\theta} = \frac{\sum_{i=1}^N \hat{\theta}_i}{N}$ and the root-mean-square error $RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta_0)^2}{N}}$, $\theta_0 =$

true value of θ . Compute $\hat{\gamma}_{BLi}, \hat{\gamma}_{BSi}, \hat{\alpha}_{BLi}$ and $\hat{\alpha}_{BSi}, (i=1,2)$ at $t=1, a=3$ the results are in Table V entries within parentheses represent the corresponding RMSE. From table V (simulation Example) we note :

i- The Bayes estimators $\hat{\gamma}_{BLi} (i=1,2)$ of reliability function relative to, when prior distributions of σ are $p_i(\sigma), (i=1,2)$ have greater RMSE than that of other estimators $\hat{\gamma}_{BSi} (i=1,2)$.

ii- The Bayes estimators $\hat{\alpha}_{BLi} (i=1,2)$ of hazard function relative to $L(\Delta_1)$, when the prior distributions are $p_i(\sigma), (i=1,2)$ have smaller RMSE than that of other estimators $\hat{\alpha}_{BSi} (i=1,2)$. From the above results we conclude that in situations involving estimation of failure rate function, asymmetric loss functions are more appropriate than squared error loss functions.

Table III

Bayes estimation of reliability function based on LINEX loss ($\hat{\gamma}_{BLi}$)
and squared error loss ($\hat{\gamma}_{BSi}$), ($i=1,2$) for variation in "a"
and an upper record sample $n=10$, ($t=1$, the true value $\gamma_{t=1} = 0.607$)

	$\hat{\gamma}_{BS1} = 0.677$	$\hat{\gamma}_{BS2} = 0.651$
a	$\hat{\gamma}_{BL1}$	$\hat{\gamma}_{BL2}$
3	0.6662	0.6402
2	0.6699	0.6439
1	0.6734	0.6474
-1	0.6801	0.6543
-2	0.6834	0.6577
-3	0.6867	0.6610

Table IV

Bayes estimation of hazard function based on LINEX loss ($\hat{\alpha}_{BLi}$)
 and squared error loss ($\hat{\alpha}_{BSi}$), ($i = 1, 2$) for variation in "a"
 and an upper record sample $n = 10$, ($t = 1$, the true value $\alpha_{t=1} = 1$)

	$\hat{\alpha}_{BS1} = 0.796$	$\hat{\alpha}_{BS2} = 0.876$
a	$\hat{\alpha}_{BL1}$	$\hat{\alpha}_{BL2}$
3	0.7141	0.7855
2	0.7389	0.8128
1	0.7662	0.8428
-1	0.8298	0.9128
-2	0.8674	0.9541
-3	0.9099	1.0009

Table V

Simulation estimates of the mean value of
 Bayes estimators of reliability function γ and hazard function α

n	true value $\gamma_{t=1} = 0.607$				true value $\alpha_{t=1} = 1$			
	$\hat{\gamma}_{BL1}$	$\hat{\gamma}_{BL2}$	$\hat{\gamma}_{BS1}$	$\hat{\gamma}_{BS2}$	$\hat{\alpha}_{BL1}$	$\hat{\alpha}_{BL2}$	$\hat{\alpha}_{BS1}$	$\hat{\alpha}_{BS2}$
5	0.5671 (0.1425)	0.5133 (0.1701)	0.5923 (0.1316)	0.5367 (0.1551)	0.8639 (0.3449)	1.0367 (0.3820)	1.1867 (0.6313)	1.4240 (0.8388)
7	0.5535 (0.1298)	0.5125 (0.1540)	0.5731 (0.1191)	0.5312 (0.1410)	0.9586 (0.3118)	1.0955 (0.3658)	1.2178 (0.5498)	1.3918 (0.6974)
10	0.5579 (0.0982)	0.5279 (0.1177)	0.5723 (0.0895)	0.5419 (0.1072)	0.9977 (0.2380)	1.0975 (0.2794)	1.1748 (0.3755)	1.2923 (0.4681)

REFERENCE

- [1] M. Ahsanullah. *Record Statistics*, Nova Science Publishers, Commack, NY. (1995).
- [2] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja. *Records*, John Wiley and Sons, New York, (1998).
- [3] A.P. Basu, and N.Ebrahimi. Bayesian Approach to life Testing and Reliability Estimation Using Asymmetric Loss-Function, Jour.Stat. Plann., Infer., 29, (1991), 21-31.
- [4] R. V. Canfield. A Bayesian Approach to Reliability Estimation Using a Loss Function, IEEE Transaction on Reliability, R-19, (1970), 13-16.
- [5] K. N. Chandler. The distribution and Frequency of record values, J. Roy. Stat. Soc. B 14. (1952).
- [6] D. D. Dyer and Whisenand. Best Linear Unbiased Estimator of the Parameter of the Rayleigh Distribution, IEEE Transaction on Reliability, R-22, (1973), 27-34; and 455-466.

- [7] J. A. Hartigan. Invariant Prior Distribution, *Ann. Math. Statist.* 34, (1964), 836-845.
- [8] K. Hirano. *Rayleigh Distributions*, New York: Wiley. (1986):
- [9] H. A. Howlader, and A. Hossain. On Bayesian Estimation and Prediction From Rayleigh Distribution Based on Type II Censored Data, *Commun. Statist. Theor. Meth*, 24(9), (1995), 2249-2259.
- [10] H. Jeffreys. *Theory of Probability*, Oxford: Clarendon Press. (1961).
- [11] B. N. Pandey. Testimator of the Scale Parameter of the Exponential Distribution Using Linex Loss Function, *Commun. Statist. Theor. Meth.*, 26(9), (1997) 2191-2202.
- [12] A. M. Polovko. (1968): *Fundamentals of Reliability Theory*, Academic Press, New York. (1997).
- [13] O. Rai. A some-times pool Estimation of Mean life under Linex Loss Function, *Commun. Statist. Theor. Meth.*, 25, (1996), 2057-2067.
- [14] J. Rojo. On the Admissibility of CX+d with respect to the LINEX Loss Function, *Commun. Statist. Theor. Meth.*, 16, (1987), 3745-3748.
- [15] M. M. Siddiqui. Some Problems Connected with Rayleigh Distributions, *J. Res Nat. Bur. Stand.*, 66D, (1962), 167-174.
- [16] A. Ahmed Soliman,. Comparison of linex and quadratic Bayes estimators for the Rayleigh distribution, *commun. statist.-Theory Meth.*, 29(1) (2000), 95-107.
- [17] H. R. Varian.. *A Bayesian Approach to Real Estate Assessment*, Amsterdam: North Holland. (1975), 195-208.
- [18] A. Zellner.. Bayesian Estimation and Prediction using Asymmetric Loss Functions, *Jour. Amer. Statist. Assoc.* 81, (1986), 446-451.

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