Submanifolds of an Almost Paraquaternionic Kähler Product Manifold

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Abstract

In this paper we define an almost paraquaternionic Kähler product manifold and study the geometry of its F -invariant submanifolds.

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1 Introduction

The theory of paraquaternionic manifolds has been developing in recent years ([1], [5], [8], [9], [13]), although its roots date back to 50's, in the work of P. Libermann ([11]).

The concept of quaternionic structure of the second kind has been defined as a triplet of endomorphisms of the tangent bundle $\{J_1, J_2, J_3\}$, in which J_1 is almost complex and J_2 and J_3 are almost product structures satisfying some relations of anti-commutation. S. Ianuș ([6]) showed the existence of such a structure on the tangent bundle of a manifold (without the condition that the dimension of the underlying manifold be a multiple of 4). Under certain conditions on the holonomy group of a metric adapted to such a structure, one arrives to the concept of paraquaternionic Kähler manifold. An important example in this theory is the paraquaternionic projective space as described by Blažić ([3]).

In the present note we show that the properties of F -invariant and F -antiinvariant submanifolds of a Kählerian product manifold ([2]) can be extended in the paraquaternionic geometry. In Section 2 we present the definition

and basic properties of paraquaternionic manifolds. In Section 3 we define the notion of almost paraquaternionic Kähler product manifold and give an example. In Section 4 we research the fundamental properties of F -invariant and F -antiinvariant submanifolds of an almost paraquaternionic Kähler product manifold.

2 Paraquaternionic manifolds

Let M be a differentiable manifold of dimension n and assume that there is a rank 3-subbundle σ of $End(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ of sections of σ exists satisfying:

$$\begin{cases} J_\alpha^2 = -\epsilon_\alpha Id, \forall \alpha = \overline{1, 3} \\ J_1 J_2 = -J_2 J_1 = -J_3 \end{cases} \quad (1)$$

where $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$.

Then the bundle σ is called a paraquaternionic structure on M and $\{J_1, J_2, J_3\}$ is called canonical local basis of σ .

A pseudo-Riemannian metric g is said to be adapted to the paraquaternionic structure σ if it satisfies:

$$g(J_\alpha X, J_\alpha Y) = \epsilon_\alpha g(X, Y), \forall \alpha = \overline{1, 3} \quad (2)$$

for all vector fields X, Y on M and any local basis $\{J_1, J_2, J_3\}$ of σ . Moreover, (M, σ, g) is said to be a paraquaternionic manifold.

It is clear that any paraquaternionic manifold is of dimension $n = 4m$ and any adapted metric is necessarily of neutral signature $(2m, 2m)$.

If the bundle σ is parallel with respect to the Levi-Civita connection ∇ of g , then (M, σ, g) is said to be a paraquaternionic Kähler manifold. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that:

$$\begin{cases} \nabla_X J_1 = -\omega_3(X) J_2 + \omega_2(X) J_3 \\ \nabla_X J_2 = -\omega_3(X) J_1 + \omega_1(X) J_3 \\ \nabla_X J_3 = \omega_2(X) J_1 - \omega_1(X) J_2 \end{cases} \quad (3)$$

for any vector field X on M .

Let (M, σ, g) be a paraquaternionic manifold. If $X \in T_p M, p \in M$, then the 4-plane $PQ(X)$ spanned by $\{X, J_1 X, J_2 X, J_3 X\}$ is called a paraquaternionic 4-plane. A 2-plane in $T_p M$ spanned by $\{X, Y\}$ is called half-paraquaternionic if $PQ(X) = PQ(Y)$.

The sectional curvature for a half-paraquaternionic 2-plane is called paraquaternionic sectional curvature. A paraquaternionic Kähler manifold of constant paraquaternionic sectional curvature is said to be a paraquaternionic space form.

We have the following important fundamental properties concerning paraquaternionic manifolds ([3],[5],[9],[13]):

1. Any paraquaternionic Kähler manifold is an Einstein manifold, provided that $\dim M > 4$.
2. Any paraquaternionic space form is a locally symmetric space.
3. The curvature tensor of a paraquaternionic space form is:

$$R(X, Y)Z = \frac{c}{4} \left\{ g(Z, Y)X - g(X, Z)Y + \sum_{\alpha=1}^3 \epsilon_{\alpha} [g(Z, J_{\alpha}Y)J_{\alpha}X - g(Z, J_{\alpha}X)J_{\alpha}Y + 2g(X, J_{\alpha}Y)J_{\alpha}Z] \right\} \quad (4)$$

for all vector fields X, Y, Z on M and any local basis $\{J_1, J_2, J_3\}$ of σ .

Some recent results concerning the hypersurfaces of paraquaternionic manifolds we find in [7] .

3 Almost Paraquaternionic Kähler product manifolds

Let M be a n -dimensional manifold endowed with a tensor F of type (1,1) such that:

$$F^2 = Id. \quad (5)$$

Then we say that (M, F) is an almost product manifold and F is called an almost product structure on M . Moreover, if (M, F) admits a semi-Riemannian metric g such that:

$$g(FX, FY) = g(X, Y), \quad (6)$$

$\forall X, Y \in \Gamma(TM)$, then we say that (M, F, g) is an almost product semi-Riemannian manifold.

Definition 3.1 *Let (M, F, g) be an almost product semi-Riemannian manifold and assume that there is a rank 3-subbundle σ of $\text{End}(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ exists of sections of σ satisfying (1) and (2). Then (M, F, σ, g) is said to be a paraquaternionic product manifold.*

Definition 3.2 *Let (M, F, σ, g) be a product paraquaternionic manifold. If the Levi-Civita connection ∇ of g satisfies the following conditions:*

$$\begin{cases} (\nabla_X J_1)Y = k[-\omega_3(X)J_2Y + \omega_2(X)J_3Y - \omega_3(FX)J_2(FY) + \omega_2(FX)J_3(FY)] \\ (\nabla_X J_2)Y = k[-\omega_3(X)J_1Y + \omega_1(X)J_3Y - \omega_3(FX)J_1(FY) + \omega_1(FX)J_3(FY)] \\ (\nabla_X J_3)Y = k[\omega_2(X)J_1Y - \omega_1(X)J_2Y + \omega_2(FX)J_1(FY) - \omega_1(FX)J_2(FY)] \end{cases} \quad (7)$$

for some non-zero constant k and any vector field X on M , $\omega_1, \omega_2, \omega_3$ being local 1-forms over the open for which $\{J_1, J_2, J_3\}$ is a local basis of σ , then (M, F, σ, g) is said to be an almost paraquaternionic Kähler product manifold.

Example 3.3

Let (M_1, g_1, σ_1) be a paraquaternionic Kähler manifold. Then, in any coordinate neighborhood U_1 of M_1 , there exists a local bases $\{J_1^{(1)}, J_2^{(1)}, J_3^{(1)}\}$ of σ_1 satisfying (1) and (2). Moreover, the Levi-Civita connection $\nabla^{(1)}$ of g_1 satisfies the following conditions:

$$\begin{cases} \nabla_X^{(1)} J_1^{(1)} = -\omega_3^{(1)}(X)J_2^{(1)} + \omega_2^{(1)}(X)J_3^{(1)} \\ \nabla_X^{(1)} J_2^{(1)} = -\omega_3^{(1)}(X)J_1^{(1)} + \omega_1^{(1)}(X)J_3^{(1)} \\ \nabla_X^{(1)} J_3^{(1)} = \omega_2^{(1)}(X)J_1^{(1)} - \omega_1^{(1)}(X)J_2^{(1)} \end{cases} \quad (8)$$

for any vector field X on M_1 , $\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}$ being local 1-forms over the open for which $\{J_1^{(1)}, J_2^{(1)}, J_3^{(1)}\}$ is a local basis of σ_1 .

Let (M_2, g_2, σ_2) be an another paraquaternionic Kähler manifold. Then, in any coordinate neighborhood U_2 of M_2 , there exists a local basis $\{J_1^{(2)}, J_2^{(2)}, J_3^{(2)}\}$ of σ_2 satisfying (1) and (2). Moreover, the Levi-Civita connection $\nabla^{(2)}$ of g_2 satisfies the following conditions:

$$\begin{cases} \nabla_X^{(2)} J_1^{(2)} = -\omega_3^{(2)}(X)J_2^{(2)} + \omega_2^{(2)}(X)J_3^{(2)} \\ \nabla_X^{(2)} J_2^{(2)} = -\omega_3^{(2)}(X)J_1^{(2)} + \omega_1^{(2)}(X)J_3^{(2)} \\ \nabla_X^{(2)} J_3^{(2)} = \omega_2^{(2)}(X)J_1^{(2)} - \omega_1^{(2)}(X)J_2^{(2)} \end{cases} \quad (9)$$

for any vector field X on M_2 , $\omega_1^{(2)}, \omega_2^{(2)}, \omega_3^{(2)}$ being local 1-forms over the open for which $\{J_1^{(2)}, J_2^{(2)}, J_3^{(2)}\}$ is a local basis of σ_2 .

Let $M = M_1 \times M_2$ be the product manifold of M_1 and M_2 . We define a semi-Riemannian metric g on M by:

$$g(X, Y) = g_1(\Pi_1 X, \Pi_1 Y) + g_2(\Pi_2 X, \Pi_2 Y) \quad (10)$$

for any vector fields X and Y on M , where Π_1 and Π_2 denote the projection mappings of $\Gamma(T(M_1 \times M_2))$ to $\Gamma(T(M_1))$ and $\Gamma(T(M_2))$, respectively, that is:

$$\Pi_1 : \Gamma(TM) \rightarrow \Gamma(TM_1), \Pi_2 : \Gamma(TM) \rightarrow \Gamma(TM_2).$$

We define now:

$$F = \Pi_1 - \Pi_2 \quad (11)$$

and we can easily see that (M, F, g) is an almost product semi-Riemannian manifold.

We define the endomorphisms of tangent space of M by:

$$J_\alpha X = J_\alpha^{(1)} \Pi_1 X + J_\alpha^{(2)} \Pi_2 X, \forall \alpha = \overline{1, 3}, \quad (12)$$

for any vector field X on M , where $\{J_\alpha^{(1)}\}_{\alpha=\overline{1,3}}$ and $\{J_\alpha^{(2)}\}_{\alpha=\overline{1,3}}$ are local basis of σ_1 and σ_2 , respectively.

If we consider the vector bundle σ over M generated by $\{J_\alpha\}_{\alpha=\overline{1,3}}$, then we have that (M, g, σ) is a paraquaternionic manifold (see [12]).

Now, for any local coordinate neighborhood $U_1 \times U_2$ of M we define local 1-form $\omega_1, \omega_2, \omega_3$ by:

$$\omega_\alpha(X) = \omega_\alpha^{(1)}(\Pi_1 X) + \omega_\alpha^{(2)}(\Pi_2 X), \forall \alpha = \overline{1, 3}, \quad (13)$$

for any vector field X on M .

We remark that the Levi-Civita connection ∇ on M is:

$$\nabla_X Y = \nabla_{\Pi_1 X}^{(1)} \Pi_1 Y + \nabla_{\Pi_2 X}^{(2)} \Pi_2 Y, \quad (14)$$

for any vector fields X and Y on M .

We have now:

$$\begin{aligned} (\nabla_X J_1)Y &= \nabla_X J_1 Y - J_1(\nabla_X Y) \\ &= \nabla_{\Pi_1 X}^{(1)} \Pi_1 J_1 Y + \nabla_{\Pi_2 X}^{(2)} \Pi_2 J_1 Y - J_1 \nabla_{\Pi_1 X}^{(1)} \Pi_1 Y - J_1 \nabla_{\Pi_2 X}^{(2)} \Pi_2 Y \\ &= \nabla_{\Pi_1 X}^{(1)} J_1^{(1)} \Pi_1 Y + \nabla_{\Pi_2 X}^{(2)} J_1^{(2)} \Pi_2 Y - J_1^{(1)} \nabla_{\Pi_1 X}^{(1)} \Pi_1 Y - J_1^{(2)} \nabla_{\Pi_2 X}^{(2)} \Pi_2 Y \\ &= (\nabla_{\Pi_1 X}^{(1)} J_1^{(1)})(\Pi_1 Y) + (\nabla_{\Pi_2 X}^{(2)} J_1^{(2)})(\Pi_2 Y). \end{aligned} \quad (15)$$

Thus, taking account of (8) and (9), we obtain:

$$\begin{aligned} (\nabla_X J_1)Y &= -\omega_3^{(1)}(\Pi_1 X) J_2^{(1)}(\Pi_1 Y) + \omega_2^{(1)}(\Pi_1 X) J_3^{(1)}(\Pi_1 Y) \\ &\quad -\omega_3^{(2)}(\Pi_2 X) J_2^{(2)}(\Pi_2 Y) + \omega_2^{(2)}(\Pi_2 X) J_3^{(2)}(\Pi_2 Y). \end{aligned} \quad (16)$$

On the other hand, by using (13) we obtain:

$$\begin{aligned} -\omega_3(X) J_2 Y + \omega_2(X) J_3 Y - \omega_3(FX) J_2(FY) + \omega_2(FX) J_3(FY) \\ = 2\{-\omega_3^{(1)}(\Pi_1 X) J_2^{(1)}(\Pi_1 Y) + \omega_2^{(1)}(\Pi_1 X) J_3^{(1)}(\Pi_1 Y) \\ -\omega_3^{(2)}(\Pi_2 X) J_2^{(2)}(\Pi_2 Y) + \omega_2^{(2)}(\Pi_2 X) J_3^{(2)}(\Pi_2 Y)\}. \end{aligned} \quad (17)$$

From (16) and (17) we deduce:

$$(\nabla_X J_1)Y = \frac{1}{2}[-\omega_3(X) J_2 Y + \omega_2(X) J_3 Y - \omega_3(FX) J_2(FY) + \omega_2(FX) J_3(FY)]$$

and similarly we obtain:

$$(\nabla_X J_2)Y = \frac{1}{2}[-\omega_3(X) J_1 Y + \omega_1(X) J_3 Y - \omega_3(FX) J_1(FY) + \omega_1(FX) J_3(FY)]$$

and

$$(\nabla_X J_3)Y = \frac{1}{2}[\omega_2(X)J_1Y - \omega_1(X)J_2Y + \omega_2(FX)J_1(FY) - \omega_1(FX)J_2(FY)].$$

Thus, we conclude that (M, F, σ, g) is an almost paraquaternionic Kähler product manifold.

We remark that if $M_1(c_1)$ and $M_2(c_2)$ are paraquaternionic space forms with constant paraquaternionic sectional curvatures c_1 and c_2 , respectively, then the curvature tensor R of paraquaternionic Kähler product manifold $M = M_1 \times M_2$ is given by:

$$\begin{aligned} R(X, Y)Z = & \frac{c_1 + c_2}{16} \{g(Z, Y)X - g(X, Z)Y + g(Z, FY)FX - g(X, FZ)FY \\ & + \sum_{\alpha=1}^3 \epsilon_\alpha [g(Z, J_\alpha Y)J_\alpha X - g(Z, J_\alpha X)J_\alpha Y + 2g(X, J_\alpha Y)J_\alpha Z \\ & + g(Z, FJ_\alpha Y)FJ_\alpha X - g(Z, FJ_\alpha X)FJ_\alpha Y + 2g(X, FJ_\alpha Y)FJ_\alpha Z]\} \\ & + \frac{c_1 - c_2}{16} \{g(Z, FY)X - g(X, FZ)Y + g(Z, Y)FX - g(X, Z)FY \\ & + \sum_{\alpha=1}^3 \epsilon_\alpha [g(Z, FJ_\alpha Y)J_\alpha X - g(Z, FJ_\alpha X)J_\alpha Y + 2g(X, FJ_\alpha Y)J_\alpha Z \\ & + g(Z, J_\alpha Y)FJ_\alpha X - g(Z, J_\alpha X)FJ_\alpha Y + 2g(X, J_\alpha Y)FJ_\alpha Z]\} \quad (18) \end{aligned}$$

for all vector fields X, Y, Z on M , where F is the almost product structure on M given by (11) and J_α is given by (12) (see [12]).

4 F -invariant submanifolds of a paraquaternionic product manifold

Let $(\overline{M}, \overline{g})$ be a n -dimensional semi-Riemannian manifold and (M, g) be a m -dimensional semi-Riemannian submanifold of $\overline{M}$. If the curvature tensor R of M satisfies the following condition:

$$R(X, Y)Z \in \Gamma(TM), \forall X, Y, Z \in \Gamma(TM) \quad (19)$$

then M is said to be a curvature-invariant submanifold of $\overline{M}$.

Definition 4.1 Let $(\overline{M}, F, \sigma, \overline{g})$ be a n -dimensional paraquaternionic product manifold and (M, g) be a m -dimensional semi-Riemannian submanifold of $\overline{M}$.

i. If $F(T_p M) \subset T_p M, \forall p \in M$, then M is said to be F -invariant submanifold of paraquaternionic product manifold $\overline{M}$.

ii. If $F(T_p M) \subset T_p M^\perp, \forall p \in M$, where $T_p M^\perp$ is the normal space of $T_p M$ in $T_p \overline{M}$, $p \in M$, then M is said to be F -antiinvariant submanifold of paraquaternionic product manifold $\overline{M}$.

Definition 4.2 Let $(\overline{M}, F, \sigma, \overline{g})$ be a paraquaternionic product manifold. A semi-Riemannian submanifold (M, g) of $\overline{M}$ is called paraquaternionic submanifold if:

$$J(T_p M) \subset T_p M, \forall J \in \sigma_p, \forall p \in M.$$

Theorem 4.3 Let $\overline{M} = \overline{M}_1 \times \overline{M}_2$ be the paraquaternionic product manifold of two paraquaternionic space forms $\overline{M}_1(c)$ and $\overline{M}_2(c)$, $c \neq 0$. If M is a curvature-invariant paraquaternionic submanifold of $\overline{M}$, then M is F -invariant or F -antiinvariant.

Proof. From (18), because $c_1 = c_2 = c$, we obtain that the curvature tensor of $\overline{M}$ is given by:

$$\begin{aligned} \overline{R}(X, Y)Z &= \frac{c}{8} \{ \overline{g}(Z, Y)X - \overline{g}(X, Z)Y + \overline{g}(Z, FY)FX - \overline{g}(X, FZ)FY \\ &\quad + \sum_{\alpha=1}^3 \epsilon_{\alpha} [\overline{g}(Z, J_{\alpha}Y)J_{\alpha}X - \overline{g}(Z, J_{\alpha}X)J_{\alpha}Y + 2\overline{g}(X, J_{\alpha}Y)J_{\alpha}Z \\ &\quad + \overline{g}(Z, FJ_{\alpha}Y)FJ_{\alpha}X - \overline{g}(Z, FJ_{\alpha}X)FJ_{\alpha}Y + 2\overline{g}(X, FJ_{\alpha}Y)FJ_{\alpha}Z] \}. \end{aligned} \quad (20)$$

On the other hand, M is a paraquaternionic submanifold of $\overline{M}$ and thus we have:

$$\begin{aligned} g(Z, FY)FX - g(X, FZ)FY + \sum_{\alpha=1}^3 \epsilon_{\alpha} [g(Z, FJ_{\alpha}Y)FJ_{\alpha}X \\ - g(Z, FJ_{\alpha}X)FJ_{\alpha}Y + 2g(X, FJ_{\alpha}Y)FJ_{\alpha}Z] \in T_p M \end{aligned} \quad (21)$$

$\forall X, Y, Z \in T_p M, \forall p \in M$.

If we take now $Z = X$ in (21) we obtain:

$$\begin{aligned} g(X, FY)FX - g(X, FX)FY + 3[g(FX, J_1Y)FJ_1X \\ - g(FX, J_2Y)FJ_2X - g(FX, J_3Y)FJ_3X] \in T_p M. \end{aligned} \quad (22)$$

Replacing X by J_1X, J_2X and J_3X in (22) we get:

$$\begin{aligned} g(J_1X, FY)FJ_1X - g(X, FX)FY + 3[-g(FX, Y)FX \\ - g(FX, J_3Y)FJ_3X - g(FX, J_2Y)FJ_2X] \in T_p M, \end{aligned} \quad (23)$$

$$\begin{aligned} g(J_2X, FY)FJ_2X + g(X, FX)FY + 3[g(FX, J_3Y)FJ_3X \\ + g(FX, Y)FX - g(FX, J_1Y)FJ_1X] \in T_p M \end{aligned} \quad (24)$$

and

$$\begin{aligned} g(J_3X, FY)FJ_3X + g(X, FX)FY + 3[g(FX, J_2Y)FJ_2X \\ - g(FX, J_1Y)FJ_1X + g(FX, Y)FX] \in T_p M. \end{aligned} \quad (25)$$

Adding (23), (24) and (25) we derive:

$$g(X, FX)FY + 3g(FX, Y)FX - [7g(FX, J_1Y)FJ_1X + g(FX, J_2Y)FJ_2X + g(FX, J_3Y)FJ_3X] \in T_pM. \quad (26)$$

From (23) and (26) we obtain:

$$g(X, FX)FY + 3g(FX, Y)FX - 5g(FX, J_1Y)FJ_1X \in T_pM. \quad (27)$$

On the other hand, from (22) and (23) we have:

$$g(FX, Y)FX + g(FX, J_1Y)FJ_1X \in T_pM. \quad (28)$$

From (27) and (28) it follows that:

$$g(X, FX)FY + 8g(FX, Y)FX \in T_pM \quad (29)$$

and putting now $Y = X$ in (29) we derive:

$$g(X, FX)FX \in T_pM. \quad (30)$$

The relation (30) implies that $FX \in T_pM$ or $g(X, FX) = 0$. In the last situation, from (29) we obtain:

$$g(FX, Y)FX \in T_pM \quad (31)$$

and consequently we have $FX \in T_pM$ or $FX \in T_pM^\perp$ for any vector $X \in T_pM$. Thus we have $F(T_pM) \subset T_pM$ or $F(T_pM) \subset T_pM^\perp$ for each point $p \in M$ and the proof is now complete.

Theorem 4.4 *Let $\overline{M} = \overline{M}_1 \times \overline{M}_2$ be the paraquaternionic product manifold of two paraquaternionic space forms $\overline{M}_1(c)$ and $\overline{M}_2(c)$, $c \neq 0$. If M is a totally geodesic submanifold of $\overline{M}$, then M is F -invariant or F -antiinvariant.*

Proof. Because M is a totally geodesic submanifold of $\overline{M}$ it follows that M is a curvature-invariant and the proof is now complete from Theorem 4.3.

Let us assume that $(\overline{M} = \overline{M}_1 \times \overline{M}_2, F, \sigma, \overline{g})$ be a n -dimensional paraquaternionic product manifold and (M, g) be an m -dimensional semi-Riemannian nondegenerate submanifolds of $\overline{M}$. For any $X \in \Gamma(TM)$ we have decomposition:

$$FX = fX + nX \quad (32)$$

where fX is the tangential part of FX and nX is the normal part of FX . Similarly, for any vector $\xi \in \Gamma(TM^\perp)$ we have:

$$F\xi = t\xi + s\xi \quad (33)$$

where $t\xi$ is the tangential part of $F\xi$ and $s\xi$ is the normal part of $F\xi$. From (32) and (33) we obtain:

$$F^2X = f^2X + n(fX) + t(nX) + s(nX). \quad (34)$$

From (5) and (34) we deduce:

$$f^2X = X - s(nX). \quad (35)$$

On the other hand, for $\forall X, Y \in \Gamma(TM)$, from (31) we derive:

$$\bar{g}(FX, FY) = g(fX, fY) + \bar{g}(nX, nY). \quad (36)$$

From (6) and (36) it follows now that we have:

$$g(fX, fY) = g(X, Y) - \bar{g}(nX, nY). \quad (37)$$

Theorem 4.5 *Let $(\bar{M} = \bar{M}_1 \times \bar{M}_2, F, \sigma, \bar{g})$ be a paraquaternionic Kähler product manifold. If M is a F -invariant nondegenerate submanifold of $\bar{M}$, then $M = M_1 \times M_2$, where M_1 and M_2 are submanifolds of $\bar{M}_1$ and $\bar{M}_2$, respectively. Moreover, M_1 and M_2 are totally geodesic submanifolds of M .*

Proof. Because M is a F -invariant submanifold of $\bar{M}$, we have:

$$nX = 0, \forall X \in \Gamma(TM). \quad (38)$$

From (35), (36) and (38) we obtain:

$$f^2X = X \quad (39)$$

and

$$g(fX, fY) = g(X, Y) \quad (40)$$

for any $X, Y \in \Gamma(TM)$.

Thus we conclude that (f, g) is an almost semi-Riemannian product structure on M and consequently the vertical distribution T_1 and the horizontal distribution T_2 are given by:

$$T_1 = \{X \in \Gamma(TM) | fX = X\} \quad (41)$$

and

$$T_2 = \{X \in \Gamma(TM) | fX = -X\}. \quad (42)$$

Let M_1 and M_2 be the integral manifolds of the vertical distribution T_1 and the horizontal distribution T_2 , respectively.

Because we have:

$$\nabla_Y fX = f\nabla_Y X \quad (43)$$

we deduce:

$$f\nabla_Y X_1 = \nabla_Y fX_1 = \nabla_Y X_1 \quad (44)$$

for all $X_1 \in T_1$ and $Y \in \Gamma(TM)$.

Thus we conclude that the distribution T_1 is parallel.

On the other hand, for any $X_1, Y_1 \in T_1$ we have:

$$\begin{aligned} f[X_1, Y_1] &= f\nabla_{X_1} Y_1 - f\nabla_{Y_1} X_1 = \nabla_{X_1} fY_1 - \nabla_{Y_1} fX_1 \\ &= \nabla_{X_1} Y_1 - \nabla_{Y_1} X_1 = [X_1, Y_1] \end{aligned} \quad (45)$$

and thus we conclude that the distribution T_1 is involutive.

In the same way we obtain that the distribution T_2 is parallel and involutive.

Now, taking into account that T_1 and T_2 are orthogonal distributions and ∇ is the Levi-Civita connection, we obtain:

$$g(\nabla_{X_1} Y_1, Z_2) = -g(Y_1, \nabla_{X_1} Z_2) = 0 \quad (46)$$

for any $X_1, Y_1 \in T_1$ and $Z_2 \in T_2$.

Hence M_1 is a totally geodesic submanifold of M and similarly we have M_2 is a totally geodesic submanifold of M .

On the other hand, for any $X_1 \in T_1$ we have:

$$\begin{aligned} \Pi_1 X_1 &= \frac{1}{2}(\Pi_1 + \Pi_2 + \Pi_1 - \Pi_2)(X_1) \\ &= \frac{1}{2}(Id + F)(X_1) = X_1 \end{aligned} \quad (47)$$

and

$$\begin{aligned} \Pi_2 X_1 &= \frac{1}{2}[\Pi_1 + \Pi_2 - (\Pi_1 - \Pi_2)](X_1) \\ &= \frac{1}{2}(Id - F)(X_1) = 0. \end{aligned} \quad (48)$$

From (47) and (48) we conclude that M_1 is a submanifold of $\overline{M}_1$. In the same way, M_2 is a submanifold of $\overline{M}_2$.

The proof is now complete.

Theorem 4.6 *Let $(\overline{M} = \overline{M}_1 \times \overline{M}_2, F, \sigma, \overline{g})$ be a paraquaternionic Kähler product manifold. If M is a F -invariant paraquaternionic nondegenerate submanifold of $\overline{M}$, then $M = M_1 \times M_2$, where M_1 and M_2 are paraquaternionic submanifolds of $\overline{M}_1$ and $\overline{M}_2$, respectively.*

Proof. From the above Theorem we have that $M = M_1 \times M_2$, where M_1 and M_2 are submanifolds of $\overline{M}_1$ and $\overline{M}_2$, respectively.

Let J_1, J_2, J_3 be a local basis of σ given by (12). If $X \in T_p M_1$, $p \in M_1$, then we have:

$$J_\alpha X = J_\alpha^{(1)} \Pi_1 X + J_\alpha^{(2)} \Pi_2 X = J_\alpha^{(1)} \Pi_1 X \in T_p \overline{M}_1 \cap T_p M = T_p M \quad (49)$$

$\forall \alpha = \overline{1}, \overline{3}$. Consequently we deduce that M_1 is a paraquaternionic submanifolds of $\overline{M}_1$ and similarly M_2 is a paraquaternionic submanifolds of $\overline{M}_2$.

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References

- [1] D. Alekseevsky, Y. Kamishima, Quaternionic and para-quaternionic CR structure on $(4n+3)$ -dimensional manifolds, *Central European J. Math.*, **2**(5) (2004), 732-753.
- [2] M. Atceken, F-invariant submanifolds of Kaehlerian product manifold, *Turk. J. Math.*, **28** (2004), 367-381.
- [3] N. Blažić, Para-quaternionic projective spaces and pseudo Riemannian geometry, *Publ. Inst. Math.*, **60** (1996), 101-107.
- [4] B.Y. Chen, *Geometry of submanifolds and its applications*, Sci. University, Tokio, 1981.
- [5] E. Garcia Rio, Y. Matsushita, R. Vazquez-Lorenzo, Paraquaternionic Kähler manifolds, *Rocky Mountain Journal of Math.*, **31** (2001), 237-260.
- [6] S. Ianuș, Sulle strutture canoniche dello spazio fibrato tangente di una varietà riemanniana, *Rend. Mat.*, **6** (1973), 75-96.
- [7] S. Ianuș, R. Mazzocco, G.E. Vilcu, *Real lightlike hypersurfaces of paraquaternionic Kähler manifolds*, *Mediterr. J. Math.*, **3** (2006), 581-592.
- [8] S. Ivanov, I. Minchev, S. Zamkovoy, Twistor and Reflector Spaces of Almost Para-Quaternionic Manifolds, e-print, math. DG/0511657.
- [9] S. Ivanov, S. Zamkovoy, Para-hermitian and para-quaternionic manifolds, *Differential Geom. Appl.*, **23** (2005), 205-234.
- [10] T.H. Kang, H.C. Nam, Submanifolds of an almost quaternionic Kaehler product manifold, *Bull. Korean Math. Soc.*, **34** (1997), 635-665.

- [11] P. Libermann, Sur les structures presque quaternioniennes de deuxième espèce, *C.R. Acad. Sc. Paris*, **234** (1952), 1030-1032.
- [12] G.E. Vîlcu, Product of paraquaternionic Kähler manifolds, *Bull. Math. Soc. Sc. Math. Roumanie*, **48** (2005), 345-355.
- [13] S. Vukmirović, Para-quaternionic reduction, e-print, math. DG/03044424.
- [14] K. Yano, M. Kon, Submanifolds of Kaehlerian product manifolds, *Memorie Accademia Nazionale dei Lincei*, Vol. CCCLXXVI (1979), 267-292.

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