Best Approximation in Reflexive Subspaces of $C(Q)$

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Abstract

The purpose of this paper is to introduce and discuss the concept of best approximation for subspaces of $C(Q)$. We shall characterize polynomials of best approximation for given $f \in C(Q)$ in the reflexive subspaces of $C(Q)$. We also show that any commutative C^* -algebra with identity has not any quasi-Chebyshev maximal ideal.

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1 Introduction and preliminaries

Let M be a subset of the normed linear space X . For any $x_0 \in X$, the (possibly empty) set of best approximations to x from M is defined by

$$P_M(x) = \{y \in M : \|x - y\| = d(x, M)\},$$

where $d(x, M) = \inf\{\|x - y\| : y \in M\}$.

The subset M is said to be proximal (resp., Chebyshev) if $P_M(x)$ is nonempty (resp., is a singleton) for each $x \in X$.

A necessary condition for proximality of a subset M of a normed linear space X is closeness (see, [3]). Also a sufficient condition for this, is compactness (i.e., each sequence of elements of M has a subsequence convergent to an element of M). Indeed, for each $x \in X$, the functional $e_x: M \rightarrow \mathbf{R}$ defined by $e_x(m) = \|m - x\|$ is continuous (see, [7, p. 391]) and hence must achieve its minimum on any compact set M .

For example every finite-dimensional subspace of a normed linear space or any

closed ideal of a C^* -algebra, is proximal. Also any proximal subset of a strictly convex space is Chebyshev. (see, [1])

2 Best approximation in reflexive subspaces of $C(Q)$

Let $C(Q)$ denotes the space of all continuous, complex valued functions defined on a compact set Q with the supremum norm. In [7] Singer proved Kolmogorov's type for every finite-dimensional subspace M of $C(Q)$. In this section we shall prove kolmogorov's type characterization of best approximation for given $f \in C(Q)$ in a reflexive subspace M of $C(Q)$.

Let X be a normed linear space, X^* its dual space and $\mathbf{B}_{X^*}$ the closed unit ball in X^* . A set $E \subseteq \mathbf{B}_{X^*}$ is said to be a norming set if

$$\|x\| = \sup\{|\phi(x)| : \phi \in E\}$$

for all $x \in X$. It is well known that for each x in X ,

$$\|x\| = \sup\{|\phi(x)| : \phi \in \mathbf{B}_{X^*}\} = \sup\{|\phi(x)| : \phi \in \text{ext}(\mathbf{B}_{X^*})\}$$

where, $\text{ext}(S)$, denotes the extreme points of the set S .

For a reference we give the following lemma which was introduced by [5].

Lemma 2.1. *Let X be a Banach space, M be a reflexive subspace of X , $x \in X \setminus M$ and $y_0 \in M$. Moreover, assume that $E \subseteq \mathbf{B}_{X^*}$ is an arbitrary norming set. Then y_0 belongs to $P_M(x)$ if and only if for every $\varepsilon > 0$ there exist $\phi_1, \phi_2, \dots, \phi_n \in E$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbf{C}$ such that $\sum_{i=1}^n |\lambda_i| \leq 1$,*

$$\left| \sum_{i=1}^n \lambda_i \phi_i(y) \right| < \varepsilon \|y\|$$

for all $y \in M$, and

$$\left| \|x - y_0\| - \sum_{i=1}^n \lambda_i \phi_i(x - y_0) \right| < \varepsilon.$$

Theorem 2.2. *Let M be a reflexive subspace of $C(Q)$. Assume that $f \in C(Q) \setminus M$ and $g_0 \in M$. Then $g_0 \in P_M(f)$ if and only if for every $\varepsilon > 0$, there exist $q_1, q_2, \dots, q_n \in Q$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbf{C}$ such that $\sum_{i=1}^n |\lambda_i| \leq 1$,*

$$\left| \sum_{i=1}^n \lambda_i g(q_i) \right| < \varepsilon \|g\|$$

for all $g \in M$, and

$$\left| \|f - g_0\| - \sum_{i=1}^n \lambda_i (f - g_0)(q_i) \right| < \varepsilon.$$

Proof. We set $E = \text{ext}(B_{X^*})$ (where, $X = C(Q)$). By the fact mentioned just before the Lemma 2.1, it is clear, that E is a norming set in B_{X^*} . By Lemma 2.1, hence, $g_0 \in P_M(f)$ if and only if for every $\varepsilon > 0$ there exist $\phi_1, \phi_2, \dots, \phi_n \in E$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbf{C}$, such that $\sum_{i=1}^n |\alpha_i| \leq 1$,

$$\left| \sum_{i=1}^n \alpha_i \phi_i(g) \right| < \varepsilon \|g\|$$

for all $g \in M$, and

$$\left| \|f - g_0\| - \sum_{i=1}^n \alpha_i \phi_i(f - g_0) \right| < \varepsilon.$$

It is known that $f \in X^*$ is an extremal point of B_{X^*} if and if there exists an element $q \in Q$, and a scalar $\beta \in \mathbf{C}$ with $|\beta| = 1$, such that

$$f(y) = \beta y(q) \quad (y \in C(Q)).$$

For each $1 \leq i \leq n$, $\phi_i \in \text{ext}(B_{X^*})$, therefore there exists a q_i in Q and a scalar β_i with $|\beta_i| = 1$, such that

$$\phi_i(g) = \beta_i g(q_i) \quad (y \in C(Q)).$$

Putting $\lambda_i = \alpha_i \beta_i$, then, we have

$$\begin{aligned} \left| \sum_{i=1}^n \lambda_i g(q_i) \right| &= \left| \sum_{i=1}^n \alpha_i \beta_i g(q_i) \right| \\ &= \left| \sum_{i=1}^n \alpha_i \phi_i(g) \right| < \varepsilon \|g\| \end{aligned}$$

for all $g \in M$, and

$$\left| \|f - g_0\| - \sum_{i=1}^n \lambda_i (f - g_0)(q_i) \right| = \left| \|f - g_0\| - \sum_{i=1}^n \alpha_i \phi_i(f - g_0) \right| < \varepsilon. \blacksquare$$

The following corollary is an immediate consequence of Theorem 2.2 and the next one (Corollary 2.4) is a consequence of a result from [5] and Corollary 2.2.

Corollary 2.3. *Let M be a reflexive subspace of $C(Q)$. Assume that $f \in C(Q) \setminus M$ and $F \subseteq M$. Then $F \subseteq P_M(f)$ if and only if for every $\varepsilon > 0$ there exist $q_1, q_2, \dots, q_n \in Q$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbf{C}$ such that $\sum_{i=1}^n |\lambda_i| \leq 1$,*

$$\left| \sum_{i=1}^n \lambda_i g(q_i) \right| < \varepsilon \|g\|$$

for all $g \in M$, and

$$\left| \|f - g_0\| - \sum_{i=1}^n \lambda_i (f - g)(q_i) \right| < \varepsilon$$

for all $g \in F$.

Corollary 2.4. *Let M be a reflexive subspace of $C(Q)$. Then*

(a) *M is a non-Chebyshev subspace of X .*

(b) *for every $\varepsilon > 0$ there exist $q_1, q_2, \dots, q_n \in Q$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbf{C}$ such that $\sum_{i=1}^n |\lambda_i| \leq 1$,*

$$\left| \sum_{i=1}^n \lambda_i g(q_i) \right| < \varepsilon \|g\|$$

for all $g \in M$, and

$$\left| \|f - g_0\| - \sum_{i=1}^n \lambda_i (f - g_0)(q_i) \right| < \varepsilon.$$

and

$$\left| \|f\| - \sum_{i=1}^n \lambda_i f(q_i) \right| < \varepsilon.$$

3 Best approximation by ideals of $C(Q)$

To each closed ideal I of $C(Q)$ there is associated a closed subset F of Q such that

$$I = \{f \in C(Q) : f(x) = 0 \text{ for all } x \text{ in } F\}$$

(see [2]).

To show that the closed subset F of Q is associated to the ideal I of $C(Q)$, we use the expression "the closed ideal (I, F) of $C(Q)$ " rather than "the closed ideal I of $C(Q)$ ".

Lemma 3.1. *Let (I, F) be a closed ideal of $C(Q)$, then*

(a) *$F = \emptyset$ if and only if $I = C(Q)$.*

(b) *$F = Q$ if and only if $I = \{0\}$.*

(c) *F is a singleton if and only if I is a maximal ideal of $C(Q)$.*

(d) If F is a finite set then I is a finite codimensional ideal of $C(Q)$.

proof. (a) Given $g \in C(Q)$. Since $F = \emptyset$, for each $x \in Q$, there exists $f_x \in I$ such that $f_x(x) \neq 0$. By continuity of f_x , therefore, there is an open set U_x containing x such that $f_x(t) \neq 0$ for all $t \in U_x$. Compactness of Q , now implies, the existence of a finite number of elements $x_1, x_2, \dots, x_n$ in Q such that for each $1 \leq i \leq n$, $f_{x_i}(t) \neq 0$ for all $t \in U_{x_i}$. Putting $f = f_{x_1}^2 + f_{x_2}^2 \cdots + f_{x_n}^2$ it follows that $f(x) \neq 0$ for all $x \in C(Q)$. Since $f \in I$ and I is an ideal, $g = f \cdot (\frac{g}{f}) \in I$. Therefore $I = C(Q)$.

The implication " $I = C(Q) \Rightarrow F = \emptyset$ " and parts (b) and (c) are trivial.

If $F = \{x_1, x_2, \dots, x_n\} \subseteq Q$, for $i = 1, 2, \dots, n$, define

$$\Lambda_i(f) = f(x_i) \quad (f \in C(Q)).$$

Then, for each $1 \leq i \leq n$, Λ_i is a linear functional on $C(Q)$. Since

$$I = \{f \in C(Q) : \Lambda_i(f) = 0, \text{ for all } i = 1, 2, \dots, n\}$$

it follows that I is of finite codimension in $C(Q)$ which completes the proof. ■

Following notations of section 1, the set M is called a quasi-Chebyshev subset of X if $P_M(x)$ is a nonempty compact subset of M , for all x in X .

For example any nonempty closed subspace of a finite dimensional space X is quasi-Chebyshev. (see, [4])

We call a nonzero ideal I of $C(Q)$, which is not $C(Q)$, a nontrivial ideal of $C(Q)$.

Theorem 3.2. Let (I, F) be a closed nontrivial ideal of $C(Q)$ such that $\text{int}(I) = \emptyset$. Then I is not a quasi-Chebyshev ideal of $C(Q)$.

proof. There exists $x_0 \in F$ such that $g(x_0) = 0$, for all $g \in I$. Choose $f \in C(Q)$ such that $f(x_0) = 1$ and $0 \leq f \leq 1$. We want to show that $P_I(f)$ is not compact in I . By Ascoli's Theorem (see, for example, [6, p. 277]), it suffices to show that $P_I(f)$ is not equicontinuous at x_0 . To do this end, we must find an $\varepsilon > 0$ such that for every neighborhood U of x_0 there exist $x_1 \in U_x$ and $h \in P_M(f)$ such that $|h(x_0) - h(x_1)| \geq \varepsilon$.

Put $\varepsilon = 1/2$, and let U be a neighborhood of x_0 . Since U is open and $\text{int}(F) = \emptyset$, U is not a subset of F and hence there exists an element x_1 in U which is not in F . Since x is not in F , there exists $g \in I$ such that $g(x_1) \neq 0$. Define

$$h(x) = \frac{g^2(x)}{\max\{g^2(x), g^2(x_1)\}} \quad (x \in Q)$$

Then h belongs to I , $0 \neq h$ and $0 \leq h \leq 1$. Since also $0 \leq f \leq 1$, $\|f - h\| \leq 1$. On the other hand $|f(x_0) - h(x_0)| = 1$, so that $\|f - h\| = 1$.

For all $g \in I$, however, $\|f - g\| \geq |f(x_0) - g(x_0)| = 1$. Since h belongs to I and $\|f - h\| = 1$, it follows that $h \in P_M(f)$. However, $|h(x_0) - h(x_1)| = 1 \geq \varepsilon$. ■

Theorem 3.2 in combination with Lemma 3.1 gives the following corollary.

Corollary 3.3. *No maximal ideal of $C(Q)$ is a quasi-Chebyshev ideal.*

Let A be a commutative C^* -algebra with identity and let Σ be its maximal ideal space. Then the Gelfand transform $\varphi: A \rightarrow C(\Sigma)$ is an isometric $*$ -isomorphism of A onto $C(\Sigma)$ (see, e. g., [2, p. 236]).

We use here, the Greek letter φ for the Gelfand transform on a C^* -algebra. The following Lemma is an straightforward consequence of the facts that the Gelfand transform is both onto and isometric.

Lemma 3.4. Let A be a commutative C^* -algebra then,

- (a) I is a maximal ideal of A if and only if $\varphi(I)$ is a maximal ideal of $C(\Sigma)$.
- (b) I is a quasi-Chebyshev ideal of A if and only if $\varphi(I)$ is a quasi-Chebyshev ideal of $C(\Sigma)$.

Corollary 3.5. *No C^* -algebra with identity has a quasi-Chebyshev maximal ideal.*

proof. Let A be a commutative C^* -algebra with identity, and let I be a maximal ideal of A which is quasi-Chebyshev. Then from Lemma 3.4, $\varphi(I)$ is a quasi-Chebyshev maximal ideal of $C(\Sigma)$, which contradicts Corollary 3.3. ■

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