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Some Formulas for Sums of Binomial Coefficients and Gamma Functions

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Abstract

In several mathematical problems formulas involving the evaluation of sums of binomial coefficients and ratios of gamma functions are sometimes encountered. In this paper we collect some results concerning the exact evaluation of some sums of binomial coefficients and an asymptotic expansion for the sum of some ratios of gamma functions.

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1 Introduction

In several mathematical problems, formulas involving binomial coefficients and gamma functions are encountered. For example, the analysis of convergence of numerical methods for solving differential equations of fractional order [4, 5] requires the evaluation of sums of binomial coefficients and ratios of gamma functions. Sometimes the knowledge of an asymptotic behavior of these sums, instead of their exact value, is required.

Lots of papers and books are devoted to collect relationships of this kind (e.g. [1] and [3]).

Nevertheless, for some summations we encountered in our analysis, we were not able to find in the literature any useful results. So we were forced to evaluate them. In this paper we give a survey of the obtained relationships with the main aim of collecting and making them available for further use.

We are not sure about originality of results in this paper. Perhaps they could have been already published in some other papers even though we were not able to find them.

2 Summations of binomial coefficients

In [4] a wide quantity of useful results concerning binomial coefficients and their summations are referred in the context of studying differential operators of fractional order. The following two results can also be given.

Proposition 2.1. *Let k and m be nonnegative integers, $0 \le m < k$. Then*

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} i^m = 0.$$

Proof. Note that for $k \ge 1$ and $m = 0$ the proof is immediate. Assume now the thesis for m and, in order to prove it for $m+1$, with $m+1 < k$, observe that $\binom{k}{i} = \frac{k}{i}\binom{k-1}{i-1}$ and consequently

$$\begin{aligned} \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^{m+1} &= \sum_{i=1}^{k} (-1)^i \binom{k}{i} i^{m+1} = -k \sum_{i=1}^{k} (-1)^{i-1} \binom{k-1}{i-1} i^m \\ &= -k \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} (i+1)^m. \end{aligned}$$

Expand now $(i+1)^m = i^m + m i^{m-1} + \ldots$ and the proof follows thanks to the hypothesis of induction. □

Proposition 2.2. *Let k and m be nonnegative integers, $0 \le m \le k$. Then*

$$\sum_{i=m}^{k} (-1)^i \binom{k}{i} \binom{i}{m} = \begin{cases} (-1)^k & \text{if } m = k \\ 0 & \text{if } m < k \end{cases}$$

Proof. When $m = k$ the proof is immediate. For $m < k$ observe that [1]

$$\binom{i}{m} = \frac{(-1)^m}{m!} \sum_{\ell=0}^{m} S_m^{(\ell)} i^\ell,$$

where $S_m^{(\ell)}$ are Stirling numbers of the first kind, and the thesis is a direct consequence of the application of Proposition 2.1. □

In the manipulation of binomial coefficients it is often useful to introduce the gamma function, which in the complex plain is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

A more general definition, not restrict to values with positive real part, can also be given in form of the Euler limit

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)}.$$

Thanks to the important property $\Gamma(z+1) = z\Gamma(z)$, the gamma function can be considered as an extension, to non–integer arguments, of the factorial function (indeed, since $\Gamma(1) = 1$, it is easy to show that for a positive integer n it holds $\Gamma(n+1) = n!$). As a consequence, the gamma function is strictly related to binomial coefficients and the extension of binomial coefficients for non–integer values is usually defined as

$$\frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)} = (-1)^j \binom{\beta}{j}.$$

We can see that the following relationships between sums of binomial coefficients and gamma functions hold.

Proposition 2.3. *Let β be any real value, $\beta \neq 0, -1, -2, \ldots,$ and k a nonnegative integer. Then*

$$\sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{i-\beta} = \frac{k!\,\Gamma(-\beta)}{\Gamma(k+1-\beta)}.$$

Proof. We proceed by induction on k. As $k = 0$, it is $-\beta\Gamma(-\beta) = \Gamma(1-\beta)$ and the thesis immediately follows. Assume now the thesis for k and observe that, since $\binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}$, for $k+1$ it holds

$$\sum_{i=0}^{k+1} \binom{k+1}{i} \frac{(-1)^i}{i-\beta} = \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{i-\beta} - \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{i+1-\beta}.$$

The hypothesis of induction yields to

$$\sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{i-\beta} = \frac{k!\,\Gamma(-\beta)}{\Gamma(k+1-\beta)}$$

and

$$\sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{i+1-\beta} = \frac{k!\,\Gamma(1-\beta)}{\Gamma(k+2-\beta)} = -\beta \frac{k!\,\Gamma(-\beta)}{\Gamma(k+2-\beta)}.$$

Therefore we have

$$\begin{aligned}
\sum_{i=0}^{k+1} \binom{k+1}{i} \frac{(-1)^i}{i-\beta} &= \frac{k!\,\Gamma(-\beta)}{\Gamma(k+1-\beta)} + \beta \frac{k!\,\Gamma(-\beta)}{\Gamma(k+2-\beta)} \\
&= \frac{k!\,\Gamma(-\beta)}{\Gamma(k+2-\beta)} \Big(k+1-\beta+\beta\Big) = \frac{(k+1)!\Gamma(-\beta)}{\Gamma(k+2-\beta)}
\end{aligned}$$

and the thesis is proved. □

Proposition 2.4. *Let k and m be nonnegative integers, $0 \leq m \leq k$. Then*

$$\sum_{i=m}^{k} (-1)^i \binom{k}{i} \binom{i}{m} \frac{1}{i-\beta} = (-1)^m \frac{k!\,\Gamma(m-\beta)}{m!\,\Gamma(k+1-\beta)}. \tag{1}$$

Proof. As $m = 0$ we have from Proposition 2.3

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{i}{0} \frac{1}{i-\beta} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{1}{i-\beta} = \frac{k!\,\Gamma(-\beta)}{\Gamma(k+1-\beta)}.$$

Assume now the thesis true for m and observe that for $m+1$ it holds

$$\begin{aligned} \sum_{i=m+1}^{k} (-1)^i \binom{k}{i} \binom{i}{m+1} \frac{1}{i-\beta} &= \sum_{i=m+1}^{k} (-1)^i \binom{k}{i} \binom{i}{m} \frac{i-m}{(m+1)(i-\beta)} \\ &= \sum_{i=m+1}^{k} (-1)^i \binom{k}{i} \binom{i}{m} \Big(\frac{i-\beta}{m+1} - \frac{m-\beta}{m+1} \Big) \frac{1}{i-\beta} \\ &= \frac{1}{m+1} \sum_{i=m}^{k} (-1)^i \binom{k}{i} \binom{i}{m} - \frac{m-\beta}{m+1} \sum_{i=m+1}^{k} (-1)^i \binom{k}{i} \binom{i}{m} \frac{1}{i-\beta} \end{aligned}$$

Observe now that, since Proposition 2.2, the first of the two terms in the above sum is zero and hence, by applying the hypothesis of induction

$$\sum_{i=m+1}^{k} (-1)^i \binom{k}{i} \binom{i}{m+1} \frac{1}{i-\beta} = -\frac{m-\beta}{m+1} (-1)^m \frac{k!\,\Gamma(m-\beta)}{m!\,\Gamma(k+1-\beta)}$$

from which the thesis immediately follows. □

3 Summation of ratios of gamma functions

In [6] it has been proved that for large values of z, and real β, the quotient $\Gamma(z-\beta)/\Gamma(z+\mu)$ behaves asymptotically as

$$\frac{\Gamma(z-\beta)}{\Gamma(z+\mu)} = \sum_{s=0}^{\infty} T_s(\mu) z^{-\beta-\mu-s}, \quad \text{as } z \to \infty, \tag{2}$$

where $T_s(\mu)$ are some real coefficients depending on β and μ. In particular, the first three coefficients $T_s(\mu)$ are given by

$$T_0(\mu) = 1, \quad T_1(\mu) = \frac{1}{2}(-\beta-\mu)(-\beta+\mu-1)$$

$$T_2(\mu) = \frac{1}{12}\binom{-\beta-\mu}{2}\Big(3(-\beta+\mu-1)^2+\beta+\mu-1\Big)$$

and the remaining coefficients can be recursively evaluated as

$$T_s(\mu) = \frac{1}{s}\sum_{\ell=0}^{s-1}\Big[\binom{-\beta-\mu-\ell}{s-\ell+1} - (-1)^{s-\ell+1}(-\beta-\mu)\mu^{s-\ell}\Big]T_\ell(\mu).$$

In the context of numerical solution of differential equations of fractional order $0 < \beta < 1$ by means of methods based on the discretization of the fractional differential operator with a Grünwald–Letnikov scheme, the analysis of the local truncation error requires [2] asymptotic expansions, similar to (2), also for the summations

$$\sum_{j=0}^{n-1}\frac{\Gamma(j-\beta)}{\Gamma(j+1)}j^i,$$

for any integer $i \geq 0$. In [4] results of this kind are been given for the case $i = 0$, for which it holds

$$\sum_{j=0}^{n-1}\frac{\Gamma(j-\beta)}{\Gamma(j+1)} = -\frac{\Gamma(n-\beta)}{\beta\Gamma(n)}, \tag{3}$$

and for the case $i = 1$ for which, by substituting $\frac{j}{\Gamma(j+1)} = \frac{1}{\Gamma(j)}$, it holds

$$\sum_{j=0}^{n-1}\frac{\Gamma(j-\beta)}{\Gamma(j+1)}j = \sum_{j=0}^{n-1}\frac{\Gamma(j-\beta)}{\Gamma(j)} = \frac{\Gamma(n-\beta)}{(1-\beta)\Gamma(n-1)}. \tag{4}$$

In order to generalize the above results, we recall here Stirling numbers of the second kind $S_i^{[\ell]}$, which are defined as the number of ways of partitioning a set of m elements into ℓ nonempty sets, and are given by

$$S_m^{[\ell]} = \frac{1}{\ell!}\sum_{k=0}^{\ell}(-1)^{\ell-k}\binom{\ell}{k}k^m.$$

The above Stirling number of the second kind can also be evaluated by means of the recursive formula

$$S_{m+1}^{[\ell]} = S_m^{[\ell-1]} + \ell S_m^{[\ell]}, \quad S_0^{[0]} = 1, \quad S_0^{[\ell]} = S_m^{[0]} = 0. \tag{5}$$

A well–known result states that $S_{i-1}^{[i-2]} = \frac{(i-1)(i-2)}{2}$. Furthermore we can prove the following lemma.

Lemma 3.1. *Let $i \geq 3$. Then $S_i^{[i-2]} = \frac{3i-5}{4}\binom{i}{3}$.*

Proof. Since $S_2^{[0]} = 0$, from (5) we can write the recurrence relation

$$S_i^{[i-2]} = S_{i-1}^{[i-3]} + \frac{(i-1)(i-2)^2}{2}$$

which is solved by

$$S_i^{[i-2]} = \sum_{\ell=3}^{i} \frac{(\ell-1)(\ell-2)^2}{2} = \sum_{j=1}^{i-2} \frac{j^2(j+1)}{2} = \frac{1}{2}\sum_{j=1}^{i-2} j^2 + \frac{1}{2}\sum_{j=1}^{i-2} j^3.$$

Moreover, since it holds

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4},$$

the thesis follows after some simple computation. □

We are now able to give the main result of this section.

Theorem 3.2. *Let $n \geq 1$, $i \geq 0$ and $\beta \in \mathbb{R}$, with β not an integer value. The following asymptotic expansion holds*

$$\sum_{j=0}^{n-1} \frac{\Gamma(j-\beta)}{\Gamma(j+1)} j^i = \sum_{s=0}^{\infty} F_s(i) n^{i-s-\beta}, \quad n \to \infty, \tag{6}$$

where coefficients $F_s(i)$ depend on β, are independent on n, and are given by

$$F_s(i) = \sum_{\ell=\max(0,s-i)}^{s} \frac{S_i^{[i-s+\ell]} T_\ell(-i+s-\ell)}{i-s+\ell-\beta}.$$

Proof. We can see [1] that for any integer j, the power j^i can be written as

$$j^i = \sum_{\ell=0}^{\min(i,j)} S_i^{[\ell]} \frac{\Gamma(j+1)}{\Gamma(j-\ell+1)}$$

and, consequentially,

$$\sum_{j=0}^{n-1} \frac{\Gamma(j-\beta)}{\Gamma(j+1)} j^i = \sum_{\ell=0}^{i} S_i^{[\ell]} \sum_{j=\ell}^{n-1} \frac{\Gamma(j-\beta)}{\Gamma(j-\ell+1)}.$$

Moreover it is easy to see that

$$\sum_{j=\ell}^{n-1}\frac{\Gamma(j-\beta)}{\Gamma(j-\ell+1)}=\sum_{j=\ell}^{n-1}\frac{\Gamma(j-\ell-(\beta-\ell))}{\Gamma(j-\ell+1)}=\sum_{j=0}^{n-\ell-1}\frac{\Gamma(j-\beta+\ell)}{\Gamma(j+1)}=\frac{\Gamma(n-\beta)}{(\ell-\beta)\Gamma(n-\ell)},$$

where last equality holds by (3). So we can write

$$\sum_{j=0}^{n-1}\frac{\Gamma(j-\beta)}{\Gamma(j+1)}j^i=\sum_{\ell=0}^{i}\frac{S_i^{[\ell]}}{\ell-\beta}\frac{\Gamma(n-\beta)}{\Gamma(n-\ell)}$$

and, by using the asymptotic expansion (2), we have

$$\sum_{j=0}^{n-1}\frac{\Gamma(j-\beta)}{\Gamma(j+1)}j^i=\sum_{s=0}^{\infty}\sum_{\ell=0}^{i}\frac{S_i^{[\ell]}}{\ell-\beta}T_s(-\ell)n^{-\beta+\ell-s},\qquad \text{as } n\to\infty.$$

Hence, by making some changes in the indexes, we easily obtain

$$\sum_{j=0}^{n-1}\frac{\Gamma(j-\beta)}{\Gamma(j+1)}j^i=\sum_{s=0}^{\infty}\sum_{\ell=\max(0,s-i)}^{s}\frac{S_i^{[i-s+\ell]}T_\ell(-i+s-\ell)}{i-s+\ell-\beta}n^{i-s-\beta},\qquad \text{as } n\to\infty$$

from which the proof immediately follows. □

Expansion (6) can be better exploited by reformulating its first terms in an explicit and more easily evaluable form.

Proposition 3.3. *Let $i\geq 0$ and $\beta\in\mathbb{R}$, with β not an integer value. The first three coefficients in (6) are given by*

$$F_0(i)=\frac{1}{(i-\beta)},\quad F_1(i)=\frac{-i+(\beta+1)^2}{2(i-1-\beta)},$$

$$F_2(i)=\frac{2i(i-3)-2i\beta(5+3\beta)+(\beta+1)(3\beta+1)(2+\beta)^2}{24(i-2-\beta)}.$$

Proof. From Theorem 3.2, and since $S_i^{[i]}=1$, for $s=0$ we have

$$F_0(i)=\frac{S_i^{[i]}}{i-\beta}T_0(-i)=\frac{1}{i-\beta}.$$

Moreover for $s=1$, when $i=0$ it holds

$$F_1(0)=\frac{S_0^{[0]}}{-\beta}T_1(0)=\frac{(-\beta)(-1-\beta)}{2(-\beta)}=-\frac{1+\beta}{2}$$

and when $i \geq 1$, since $S_i^{[i-1]} = \binom{i}{2}$, it holds

$$F_1(i) = \frac{S_i^{[i-1]}}{i-1-\beta}T_0(-i+1) + \frac{S_i^{[i]}}{i-\beta}T_1(-i) = \frac{i(i-1)}{2(i-1-\beta)} + \frac{(i-\beta)(-i-1-\beta)}{2(i-\beta)}.$$

By means of some simple computation the thesis now immediately descends for $F_1(i)$. Consider now the case for $s = 2$ and first observe that by (6)

$$F_2(i) = \begin{cases} \dfrac{S_0^{[0]}T_2(0)}{-\beta} & \text{if } i = 0 \\ \dfrac{S_1^{[0]}T_1(0)}{-\beta} + \dfrac{S_1^{[1]}T_2(-1)}{1-\beta} & \text{if } i = 1 \\ \dfrac{S_i^{[i-2]}T_0(-i+2)}{i-2-\beta} + \dfrac{S_i^{[i-1]}T_1(-i+1)}{i-1-\beta} + \dfrac{S_i^{[i]}T_2(-i)}{i-\beta} & \text{if } i \geq 2 \end{cases}$$

Since $T_1(-i) = -\frac{1}{2}(i-\beta)(i+1+\beta)$ and $T_2(-i) = \frac{(i-\beta)(i-\beta-1)}{24}(3(i+1+\beta)^2 + \beta - i - 1)$ we have

$$F_2(0) = -\frac{\beta+1}{24}(3\beta^2 + 7\beta + 2) \text{ and } F_2(1) = -\frac{\beta}{24}(3\beta^2 + 13\beta + 10).$$

For $i \geq 2$ apply Lemma 3.1 thank to which

$$F_2(i)\frac{3i-5}{4(i-2-\beta)}\binom{i}{3} - \frac{i(i-1)(i+\beta)}{4} + \frac{(i-\beta-1)}{24}\Big(3(i+1+\beta)^2 + \beta - i - 1\Big)$$

and the thesis follows by means of some standard computation. □

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