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# Ideal Amenability of Second Duals of Banach Algebras

**M. Eshaghi Gordji[(1)] , F. Habibian[(2)] and B. Hayati[(3)]**

(1) Department of Mathematics, Faculty of Sciences, Semnan University, Semnan, Iran
(2) Department of Mathematics, Isfahan University, Isfahan, Iran
(3) Department of Mathematics, Shahid Beheshti University, Tehran, Iran
eimail:madjideg@walla.com & fhabibian@ui.ac.ir & bahmanhayati@yahoo.com

**Abstract**

In this paper we study the ideal amenability of second duals of Banach algebras. We investigate relations between ideal amenability of the second dual of a Banach algebra with the first and the second Arens products.

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## 1 Introduction

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{A}^{**}$ be the second dual algebra of $\mathcal{A}$ endowed with the first or the second Arens product. Throughout this paper, the first and the second Arens product are respectively denoted by $\Box$ and $\Diamond$. These products can be defined by

$$F\Box G = w^* \lim_i \lim_j \hat{f}_i \hat{g}_j \qquad and \qquad F\Diamond G = w^* \lim_j \lim_i \hat{f}_i \hat{g}_j$$

where $(f_i)$ and $(g_j)$ are nets of elements of $\mathcal{A}$ such that $f_i \longrightarrow F$ and $g_j \longrightarrow G$ in $w^*$-topology (see [A] and [D-H]). If $X$ is a Banach $\mathcal{A}$-bimodule, then a derivation from $\mathcal{A}$ into $X$ is a linear map $D$ such that for every $a, b \in \mathcal{A}$, $D(ab) = D(a).b + a.D(b)$. If $x \in X$ and we define $\delta_x : \mathcal{A} \longrightarrow X$ by $\delta_x(a) =$

$a.x - x.a \quad (a \in \mathcal{A})$, then $\delta_x$ is a derivation. Derivations of this form are called inner derivations. Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule, then $X^*$ is a Banach $\mathcal{A}$-bimodule if for each $a \in \mathcal{A}$ and $x \in X$ and $x^* \in X^*$ we define

$$\langle ax^*, x\rangle = \langle x^*, xa\rangle, \qquad \langle x^*a, x\rangle = \langle x^*, ax\rangle.$$

A Banach algebra $\mathcal{A}$ is amenable if every derivation from $\mathcal{A}$ into every dual $\mathcal{A}$-bimodule is inner, equivalently if $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule $X$, where $H^1(\mathcal{A}, X^*)$ is the first cohomology group of $\mathcal{A}$ with coefficients in $X^*$[J1], [R], [H].
A Banach algebra $\mathcal{A}$ is weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$ (the special case of $X = \mathcal{A}$) [B-C-D], [J2]. A Banach algebra $\mathcal{A}$ is ideally amenable if $H^1(\mathcal{A}, I^*) = \{0\}$ for every closed two-sided ideal $I$ in $\mathcal{A}$. Eshaghi-Gordji and Yazdanpanah have introduced the concept of ideal amenability of Banach algebras in [G-Y], (see [E-H]).

## 2 Some examples

In this section, let $\mathcal{A}$ be a Banach algebra, $X$ a Banach $\mathcal{A}-$bimodule and let $Y$ be a Banach $\mathcal{A}-$submodule of $X$. We give some examples to show that $H^1(\mathcal{A}, X^*) = \{0\}$ ( $H^1(\mathcal{A}, X) = \{0\}$) is not sufficient condition for $H^1(\mathcal{A}, Y^*) = \{0\}$ ( $H^1(\mathcal{A}, Y) = \{0\}$). So, there are some examples to show that weak amenability dose not imply ideal amenability.

**Example 1.** Let $\mathcal{A}$ be a non weakly amenable Banach algebra with bounded left(or right) approximate identity, such that $H^2(\mathcal{A}, \mathcal{A}^*) = \{0\}$. We know that

$$\{0\} \longrightarrow \ker\Delta \xrightarrow{\ i\ } \mathcal{A}\hat{\otimes}\mathcal{A} \xrightarrow{\ \Delta\ } \mathcal{A} \longrightarrow \{0\}$$

is an exact sequence of Banach $\mathcal{A}-$bimodules, and

$$\{0\} \longrightarrow \mathcal{A}^* \longrightarrow (\mathcal{A}\hat{\otimes}\mathcal{A})^* \longrightarrow (\ker\Delta)^* \longrightarrow \{0\}$$

is an admissible exact sequence. $\mathcal{A}^*$ is a Banach $\mathcal{A}-$submodule of $(\mathcal{A}\hat{\otimes}\mathcal{A})^*$ and by Theorem 2.4.6 of [R] we have $H^1(\mathcal{A}, (\mathcal{A}\hat{\otimes}\mathcal{A})^*) \simeq H^2(\mathcal{A}, \mathcal{A}^*) = \{0\}$. Now, let $X = (\mathcal{A}\hat{\otimes}\mathcal{A})^*$ and $Y = \mathcal{A}^*$, then $H^1(\mathcal{A}, X) = \{0\}$ but $H^1(\mathcal{A}, Y) \neq \{0\}$.

**Example 2.** Let $G = PS(2, \mathbb{R})$, $\mathcal{A} = X = L^1(G)$ and let $Y$ be the augmentation ideal of $\mathcal{A}$. Then we have $H^1(\mathcal{A}, Y^*) \neq \{0\}$ [J-W]. On the other hand $\mathcal{A}$ is weakly amenable and so $H^1(\mathcal{A}, X^*) = \{0\}$. $\mathcal{A}$ is also an example of a weakly amenable Banach algebra which is not ideally amenable.

**Example 3.** Let $G = PS(2, \mathbb{R})$ and let $I$ be the augmentation ideal of $L^1(G)$. Suppose that $X = \mathcal{A} = I^{\#}$ and $Y = I$. Then $H^1(\mathcal{A}, X^*) = \{0\} \neq H^1(\mathcal{A}, Y^*)$ [G-Y].

**Example 4.** Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A} \times \mathcal{A}$ is a Banach algebra by the following product and norm:

$$(a, x)(b, y) = (ab, ay + xb) \quad (a, b, x, y \in \mathcal{A}),$$

$$\|(a, b)\| = \|a\| + \|b\| \qquad (a, b \in \mathcal{A}).$$

Let $\mathcal{A}$ be a weakly amenable Banach algebra with a bounded approximate identity such that span $\{ab-ba : a, b \in \mathcal{A}\}$ is dense in $\mathcal{A}$. Then by Theorem 5.4 of [Zh], $\mathcal{A} \times \mathcal{A}$ is weakly amenable. Now let $0 \neq \varphi \in \Omega_{\mathcal{A}}$, then $D : \mathcal{A} \times \mathcal{A} \longrightarrow \{0\} \times \mathcal{A}^*$ defined by $D(a, x) = (0, \varphi(a)\varphi)$ is a derivation. Since

$$\begin{aligned} D((a, x)(b, y)) &= D(ab, ay + xt) \\ &= (0, \varphi(ay + xb)\varphi) \\ &= (0, (\varphi(a)\varphi(y) + \varphi(x)\varphi(t))\varphi) \\ &= (0, (\varphi(x)\varphi)b) + (0, (a\varphi(y))\varphi) \\ &= (0, \varphi(x)\varphi)(b, y) + (a, x)(0, \varphi(y)\varphi) \\ &= D(a, x)(b, y) + (a, x)D(b, y). \end{aligned}$$

If $D = \delta_{(0,f)}$ for some $f \in \mathcal{A}^*$, then we have

$$\begin{aligned} (0, \varphi(x)\varphi) &= D(a, x) \\ &= (a, x)(0, f) - (0, f)(a, x) \\ &= (xf, af) - (xf, fa) \\ &= (xf - fx, af - fa). \end{aligned}$$

It shows that $(0, \varphi(x)\varphi) = D(a, x) = D(0, x) = (0, 0)$, which is a contradiction with $\varphi \neq 0$. Now let $B = X = \mathcal{A} \times \mathcal{A}$ and $Y = \{0\} \times \mathcal{A}$. We have $H^1(B, X^*) = \{0\}$ and $H^1(B, Y^*) \neq \{0\}$.

# 3 Ideal Amenability of Second dual of a Banach algebra

In [Gh-L-W] Ghahramani, Loy and Willis studied some implications of amenability and weak amenability of $\mathcal{A}^{**}$. In this paper we study the ideal amenability of the second dual of a Banach algebra.

**Theorem 3.1.** Let $\mathcal{A}$ be a commutative Banach algebra. Suppose that $\mathcal{A}^{**}$ is ideally amenable. If one of the following assertions holds:

(i) $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$.

(ii) $\mathcal{A}$ is a dual algebra.

(iii) $\mathcal{A}$ is Arens regular and every derivation from $\mathcal{A}$ into its dual is weakly compact.

Then $\mathcal{A}$ is ideally amenable.

**Proof.** Let $\mathcal{A}^{**}$ be ideally amenable, then it is weakly amenable. If one of the (i) , (ii) or (iii) holds, then by Theorem 3.2 of [Gh-L-W], Theorem 2.2 of [Gh-L] and Corollary 7.5 of [D-G-V], $\mathcal{A}$ is weakly amenable. Since $\mathcal{A}$ is commutative, then $\mathcal{A}$ is ideally amenable [G-Y].

**Theorem 3.2.** Let $\mathcal{A}$ be a Banach algebra. Consider $\mathcal{A}^{**}$ endowed with the first Arens product. If $\mathcal{A}$ is an ideal of $\mathcal{A}^{**}$ with a bounded approximate identity, then ideal amenability of $\mathcal{A}^{**}$ implies ideal amenability of $\mathcal{A}$.

**Proof.** Let $I$ be a closed two sided ideal of $\mathcal{A}$. Then $I$ is an ideal in $\mathcal{A}^{**}$. Now let $D : \mathcal{A} \longrightarrow I^*$ be a derivation, then there is a bounded derivation $\bar{D} : \mathcal{A}^{**} \longrightarrow I^*$ which is an extension of $D$ [R]. Since $\mathcal{A}^{**}$ is ideally amenable, then there exists $f \in I^*$ such that $\bar{D} = \delta_f$. Thus $D = \delta_f$ and $\mathcal{A}$ is ideally amenable.

For a Banach algebra $\mathcal{A}$, let $\mathcal{A}^{op}$ be the Banach algebra with underlying Banach space same as $\mathcal{A}$ and with product $\circ$ given by $a \circ b = ba$.

**Lemma 3.3.** Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is ideally amenable if and only if $\mathcal{A}^{op}$ is ideally amenable.

**Proof.** It is clear that $I$ is an ideal of $\mathcal{A}$ if and only if $I$ is an ideal of $\mathcal{A}^{op}$. Now let $\mathcal{A}$ be ideally amenable and let $D : \mathcal{A}^{op} \longrightarrow I^*$ be a continuous derivation, where $I$ is a closed ideal of $\mathcal{A}^{op}$. Since for every $a, b \in \mathcal{A}$, $D(ab) = D(b \circ a)$ and $I$ is a closed ideal of $\mathcal{A}$, then $D : \mathcal{A} \longrightarrow I^*$ is a continuous derivation. But $\mathcal{A}$ is ideally amenable, so there is $\xi \in I^*$ such that $D(a) = a.\xi - \xi.a = -(a \circ \xi - \xi \circ a)$. This means that $\mathcal{A}^{op}$ is ideally amenable. The converse is similar.

**Lemma 3.4.** Let $\mathcal{A}$, $\mathcal{B}$ be Banach algebras and let $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous isomorphism. If $\mathcal{A}$ is ideally amenable, then $\mathcal{B}$ is ideally amenable.

**Proof.** It is evident.

**Theorem 3.5.** Let $\mathcal{A}$ be a Banach algebra with a continuous anti-isomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$. Then $(\mathcal{A}^{**}, \Box)$ is ideally amenable if and only if $(\mathcal{A}^{**}, \Diamond)$ is ideally amenable.

**Proof.** Let $(\mathcal{A}^{**}, \Box)$ be ideally amenable and let $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous anti-isomorphism. Take $F, G \in \mathcal{A}$ and let $(f_i), (g_j)$ be nets in $\mathcal{A}$ such that

$$weak^* - \lim_i \hat{f}_i = F \qquad and \qquad w^* - \lim_j \hat{g}_j = G.$$

Let $\varphi^{**} : (\mathcal{A}^{**}, \Box) \longrightarrow (\mathcal{A}^{**}, \Diamond)$ be the second adjoint of $\varphi$. We have

$$\begin{aligned} \varphi^{**}(F \Box G) &= w^* - \lim_i w^* - \lim_j \varphi^{**}(\hat{f}_i \hat{g}_j) \\ &= w^* - \lim_i w^* - \lim_j (\varphi(\widehat{f_i)\varphi(g_j})) \\ &= w^* - \lim_i w^* - \lim_j \varphi^{**}(\hat{g}_j)\varphi^{**}(\hat{f}_i) \\ &= \varphi^{**}(G) \Diamond \varphi^{**}(F). \end{aligned}$$

Then $\varphi^{**}$ is a continuous isomorphism from $(\mathcal{A}^{**}, \Box)$ onto $(\mathcal{A}^{**}, \Diamond)^{op}$ and so by lemma 3.4, $(\mathcal{A}^{**}, \Diamond)^{op}$ is ideally amenable. Now, by lemma 3.3, $(\mathcal{A}^{**}, \Diamond)$ is ideally amenable. The converse is similar.

**Corollary 3.6.** Let $\mathcal{A}$ be a commutative Banach algebra. Then $(\mathcal{A}^{**}, \Box)$ is ideally amenable if and only if $(\mathcal{A}^{**}, \Diamond)$ is ideally amenable.

**Proof.** If $\mathcal{A}$ is a commutative Banach algebra then the identity map is a continuous anti-isomorphism on $\mathcal{A}$. Now apply theorem 3.5.

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