The Filtering Problem in Duals of Nuclear Fréchet Spaces

Deng DING

Department of Mathematics
Faculty of Science and Technology
University of Macau, Macao, China
dding@umac.mo

Abstract

The filtering problem, on which the signal process and observation process are both diffusions in dual of nuclear Fréchet space, is considered. Martingale representation theorem and Girsanov's theorem in such space are proved. The innovation process is discussed. The filtering equation for the best estimate of signal process on observation process, is established by the innovation approach.

Mathematics Subject Classifications: 60H20, 60H30, 65H99

Keywords: Filtering problem, Nuclear space, Diffusions, Innovation process, Stochastic evolution equation, Filtering equation

1 Introduction

The filtering problem consists in estimating the state of a stochastic differential system from noisy observations. Let Φ be a separable nuclear Fréchet space with the strong topological dual Φ' . In this paper, we consider the following stochastic system: the state process (signal process) X_t is a Φ' -valued diffusion process given by a stochastic evolution equation (SEE):

$$dX_t = AX_t dt + F(t)dW_t, \quad t \geq 0, \quad (1.1)$$

and the observation process Y_t is another Φ' -valued diffusion process given by a stochastic differential equation (SDE):

$$dY_t = H(X_t)dt + dW_t, \quad t \geq 0 \quad (1.2)$$

where W_t is a Φ' -valued Wiener process, $F(t)$ is an $\mathcal{L}(\Phi', \Phi')$ -valued stochastic process, $A \in \mathcal{L}(\Phi', \Phi')$ and $H(\cdot)$ is a mapping from Φ' into Φ' . Our filtering

problem is to obtain a mathematical expression, more exactly, a stochastic differential equation, for the best estimate $\widehat{X}_t$ of the state X_t based on the observation Y_t . Our main idea lies on the innovation approach which was first used to the nonlinear filtering problem by Kunita et al in [3], and was extended to the Hilbert space valued diffusion processes by Korezioglu in [5], to the distribution valued processes by Korezioglu and Martias in [6].

The organization of the paper is as follows. After this introduce we collect some necessary notations of nuclear spaces as well as some useful results on Φ' -valued diffusion processes, which can be found in [2], in Section 2. In Section 3, we prove the martingale representation theorems for real valued and Φ' -valued martingales, respectively, and establish a Girsanov's theorem for the Φ' -valued diffusion Y_t . In Section 4, we deal with the innovation process corresponding to our filtering problem and show the representation theorems for this innovation process. In Section 5, we show the main result of this paper: the best estimate $\widehat{X}_t = \mathbf{E}[X_t \mid \mathcal{G}_t]$, where $\mathcal{G}_t$ is the σ -algebra generated by the observation $\{Y_s : 0 \leq s \leq t\}$, satisfies a stochastic evolution equation on the space Φ' , which is called the filtering equation.

2 Preliminaries

We recall a locally convex space Ψ is nuclear if there exists a convex circled neighborhood base $\mathcal{U}_h(\Psi)$ such that, for every $U \in \mathcal{U}_h(\Psi)$, $\Psi(U)$ is a Hilbert space and the canonical mapping $k_U : \Psi \rightarrow \Psi(U)$ is nuclear, where $\Psi(U)$ is the completion of quotient space $\Psi / \ker(p_U)$ under the norm $\|k_U \psi\|_U = p_U(\psi)$, $\psi \in \Psi$ ($p_U(\cdot)$ is the gauge functional of U). For any $U, V \in \mathcal{U}_h(\Psi)$ with $U \subset V$, denote $k_{U,V}$ as the canonical mapping from $\Psi(U)$ into $\Psi(V)$ such that $k_V = k_U k_{U,V}$. It is obvious that $k_{U,V}$ is linear and continuous.

Let Ψ' denote the strong topological dual of Ψ . Then for every $U \in \mathcal{U}_h(\Psi)$, the dual specs of $\Psi(U)$ can be identified by the space $\Psi'[U^\circ]$ under the dual operator i_{U° of k_U , where U° is the polar set of U , $\Psi'[U^\circ]$ is the completion of the subspace spanned by U° under Ψ' with the norm p_{U° , and i_{U° is the embedding mapping from $\Psi'[U^\circ]$ into Ψ' . About detail definitions and general properties of nuclear spaces, one can refer to [10].

In this paper, we always assume that Φ be a separable nuclear Fréchet space with the strong topological dual Φ' . Then, Φ is a reflexive, barreled and bornological space, and Φ' is a complete nuclear space ([10, 11]).

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ which satisfies the usual hypothesis. A set $\{X_t^U : U \in \mathcal{U}_h(\Phi')\}$ is said to be a projective system if for every $U \in \mathcal{U}_h(\Phi')$, X_t^U is a $\Phi'(U)$ -valued stochastic process, and for any $U, V \in \mathcal{U}_h(\Phi')$ with $U \subset V$, $k_{V,U} X_t^U = X_t^V$, P -a.s.. For every projective system $\{X_t^U : U \in \mathcal{U}_h(\Phi')\}$ there exists a unique Φ' -valued process $\{X_t\}$, which is called the limit of this projective system, so that for

every $U \in \mathcal{U}_h(\Phi')$, $k_U X_t = X_t^U$, $\mathbf{P}$ -a.s. ([11]).

Assume $T > 0$ is fixed. Let $Q(\cdot, \cdot)$ be a continuous positive definite bilinear form on $\Phi \times \Phi$, and let W_t be a Φ' -valued Wiener process with the covariance $Q(\cdot, \cdot)$. Then, using the nuclear theorem and the regularization technique ([7]), we can show that there exists a $G \in \mathcal{U}_h(\Phi)$ and a $\Phi'[G^\circ]$ -valued Wiener process W_t^G with covariance Q_G , satisfy the following property:

(P1) $i_{G^\circ} W_t^G = W_t$, $0 \leq t \leq T$, P -a.s., and $Q(\cdot, \cdot) = Q_G(k_G \cdot, k_G \cdot)$.

In the following, we introduce the stochastic integrals with respect to the Φ' -valued Wiener process, as well as the Φ' -valued diffusions given by SEE (1.1). For the details we refer to [2].

Let H be a separable Hilbert spaces and let $\|\cdot\|_1$ be the trace norm. Let $\Lambda_Q(\Phi', H)$ be the class of all $\mathcal{L}(\Phi', H)$ -valued processes $F^H(\cdot)$ which satisfies that $F^H(\cdot)\xi$, $\xi \in \Phi'$, is an H -valued predictable process and

$$\mathbf{E} \left[\int_0^T \|F^H(t)QF^{H*}(t)\|_1 dt \right] < \infty,$$

where $\mathcal{L}(H, \Phi)$ -valued process $F^{H*}(\cdot)$ is the adjoint process of $F^H(\cdot)$ given by

$$\xi[F^{H*}(t)h] = \langle h, F^H(t)\xi \rangle_H, \quad \forall (\xi, h) \in \Phi' \times H,$$

and $Q : \Phi \rightarrow \Phi'$ is a symmetric, positive and nuclear operator given by $(Q\varphi_1)[\varphi_2] = Q(\varphi_1, \varphi_2)$, $\varphi_1, \varphi_2 \in \Phi$. Then $\Lambda_Q(\Phi', H)$ is a Hilbert space under the inner product:

$$\langle F_1^H, F_2^H \rangle = \mathbf{E} \left[\int_0^T \|F_1^H(t)QF_2^{H*}(t)\|_1 dt \right],$$

The set of all simple processes is dense in $\Lambda_Q(\Phi', H)$. Hence, there exists a unique, isometric integral, denoted by $\int_0^t F_s^H dW_s$, from $\Lambda_Q(\Phi', H)$ into the space of all H -valued continuous square integrable martingales. Moreover, if $G \in \mathcal{U}_h(\Phi)$ and $\{W_t^G\}$ are given in (P1), and $F_G^H(\cdot) = F^H(\cdot)i_{G^\circ}$, then

$$\int_0^t F^H(s) dW_s = \int_0^t F_G^H(s) dW_s^G, \quad 0 \leq t \leq T. \quad (2.1)$$

Let $\Lambda_Q(\Phi', \Phi')$ be the set of all $\mathcal{L}(\Phi', \Phi')$ -valued predictable processes $F(t)$ so that

$$\mathbf{E} \left[\int_0^T Q(F^*(t)\varphi, F^*(t)\varphi) dt \right] < \infty, \quad \forall \varphi \in \Phi \quad (2.2)$$

Let $F(\cdot) \in \Lambda_Q(\Phi', \Phi')$. Then for each $U \in \mathcal{U}_h(\Phi')$, $F^U(\cdot) = k_U F(\cdot)$ is an $\mathcal{L}(\Phi', \Phi'(U))$ -valued process such that $F^U(\cdot) \in \Lambda_Q(\Phi', \Phi'(U))$. Thus, the integral $\int_0^t F^U(s) dW_s$ exists, and the set $\{\int_0^t F^U(s) dW_s : U \in \mathcal{U}_h(\Phi')\}$ is a projective system. Define the unique limit of this projective system as the stochastic integral of $F(\cdot)$ and denote as $\int_0^t F(s) dW_s$. This integral has the properties:

(P2) $\int_0^t F(s)dW_s$, $0 \leq t \leq T$ is a Φ' -valued continuous square integrable martingale.

(P3) For every $U \in \mathcal{U}_h(\Phi')$,

$$k_U(\int_0^t F(s)dW_s) = \int_0^t F^U(s)dW_s, \quad 0 \leq t \leq T. \quad (2.3)$$

(P4) For every $\varphi \in \Phi$, let $F_t(\varphi)$ be an $\mathcal{L}(\Phi', R)$ -valued process defined by $F_t(\varphi)\xi = F(t)\xi[\varphi]$, $\xi \in \Phi'$, then $F_t(\varphi) \in \Lambda_Q(\Phi', R)$ and

$$(\int_0^t F(s)dW_s)[\varphi] = \int_0^t F_s(\varphi)dW_s, \quad 0 \leq t \leq T. \quad (2.4)$$

(P5) For any $\varphi_1, \varphi_2 \in \Phi$,

$$\langle \int_0^\cdot F_s(\varphi_1)dW_s, \int_0^\cdot F_s(\varphi_2)dW_s \rangle_t = \int_0^t Q(F^*(s)\varphi_1, F^*(s)\varphi_2)ds. \quad (2.5)$$

Now we turn to SEE (1.1). Assume the operator A and the process $S^*(t)$ satisfy the conditions:

(A1) $A^* \in \mathcal{L}(\Phi, \Phi)$, the dual operator of A , is an infinitesimal generator of a $\mathcal{L}(\Phi, \Phi)$ -operator semigroup $S^*(t)$ of class $(C_0, 1)$ with the topology defined by $\mathcal{U}_h(\Phi)$,

(A2) $F(\cdot) \in \Lambda_Q(\Phi', \Phi')$ and

$$\mathbf{E}[\int_0^T \int_0^t Q(F_s(S^*(t-s)\varphi), F_s(S^*(t-s)\varphi))dsdt] < \infty,$$

where the $\mathcal{L}(\Phi', R)$ -valued process $\{F_s(S^*(t-s)\varphi) : 0 \leq s \leq t\}$ is defined as in (P4) and it is regarded as a Φ -valued process since Φ is reflexive.

Then for every $X_0 \in \Phi'$ SEE (1.1) has a weak evolution solution X_t , which is a Φ' -valued diffusion given by

$$X_t = S(t)X_0 + \int_0^t S(t-s)F(s)dW_s, \quad 0 \leq t \leq T, \quad (2.6)$$

and satisfies SEE (1.1) in the sense: for each $\varphi \in \Phi$,

$$X_t[\varphi] = X_0[\varphi] + \int_0^t X_s[A^*\varphi]ds + \int_0^t F_s(\varphi)dW_s, \quad 0 \leq t \leq T, \quad (2.7)$$

where $S(t)$ is denoted the dual operator of $S^*(t)$. Finally, we introduce an Itô's formula for the Φ' -valued diffusion X_t given by (2.7). For every $f \in C_b^2(R)$ and each $\varphi \in D(A^*)$, we have

$$\begin{aligned} df(X_t[\varphi]) &= f'(X_t[\varphi])F_t(\varphi)dW_s + (f'(X_s[\varphi])X_s[A^*\varphi] \\ &\quad + \frac{1}{2}f''(X_s[\varphi])Q(F_s(\varphi), F_s(\varphi)))ds \end{aligned} \quad (2.8)$$

3 Martingale Representation Theorem and Girsanov's Theorem

Let $\mathcal{F}_t^W = \sigma(\sigma\{W_s : 0 \leq s \leq t\} \cup \mathcal{N})$, for every $t \geq 0$, where $\mathcal{N}$ is the collection of all P -null sets. By applying Hahn-Banach extension theorem and using the simple processes approximation we can show the following lemma.

Lemma 3.1 *Let H be a Hilbert space. Assume $G \in \mathcal{U}_h(\Phi)$ and $Z_G^H(\cdot)$ is an $\mathcal{L}(\Phi'[G^\circ], H)$ -valued, $\{\mathcal{F}_t^W\}$ -predictable process such that*

$$\mathbf{E}\left[\int_0^T \|Z_G^H(t)Q_G Z_G^{H*}(t)\|_1 dt\right] < \infty. \quad (3.1)$$

Then there exists an $\{\mathcal{F}_t^W\}$ -predictable process $Z^H(\cdot) \in \Lambda_Q(\Phi', H)$ such that $Z^H(t)i_{G^\circ} = Z_G^H(t)$, P -a.s. for every $0 \leq t \leq T$.

Lemma 3.2 *Let M_t be a Φ' -valued, continuous and square integrable $(\mathcal{F}_t^W)$ -martingale given by the projective system: $\{M_t^U = \int_0^t Z^U(s)dW_s : U \in \mathcal{U}_h(\Phi')\}$. Then there exists an $\{\mathcal{F}_t^W\}$ -predictable process $Z(\cdot) \in \Lambda_Q(\Phi', \Phi')$ such that $k_U Z(\cdot) = Z^U(\cdot)$, P -a.s. and $M_t = \int_0^t Z(s)dW_s$, $0 \leq t \leq T$.*

Proof Define a mapping $\Gamma : \Phi \rightarrow \Lambda_Q(\Phi', R)$ by $(\Gamma\varphi)(\cdot) = (Z^U(\cdot))^*h$, where $U \in \mathcal{U}_h(\Phi')$ is chosen so that there is a $h \in \Phi[U^\circ]$ such that $i_{U^\circ}h = \varphi$. Then it is easy to verify that Γ is linear and continuous. Define $Z(\cdot)\xi = (\Gamma^*\xi)(\cdot)$, $\forall \xi \in \Phi'$. Then $Z(\cdot) \in \Lambda_Q(\Phi', \Phi')$ and so that $k_U Z(\cdot) = Z^U(\cdot)$ for each $U \in \mathcal{U}_h(\Phi')$, and so that $M_t = \int_0^t Z(s)dW_s$, $0 \leq t \leq T$. $\square$

The next result is a Brownian martingale representation theorem for Φ' -valued martingales.

Theorem 3.3 *For every Φ' -valued, continuous and square integrable $(\mathcal{F}_t^W)$ -martingale M_t with $M_0 = 0$, there exists an $\mathcal{L}(\Phi', \Phi')$ -valued, $(\mathcal{F}_t^W)$ -predictable process $Z(\cdot) \in \Lambda_Q(\Phi', \Phi')$ such that $M_t = \int_0^t Z(s)dW_s$, $0 \leq t \leq T$.*

Proof For each $U \in \mathcal{U}_h(\Phi')$, let $M_t^U = k_U M_t$, $0 \leq t \leq T$. Then M_t^U is a $\Phi'(U)$ -valued, continuous and square integrable $\{\mathcal{F}_t^W\}$ -martingale. By Brownian martingale representation theorem in Hilbert space ([9]), there exists an $\mathcal{L}(\Phi'[G^\circ], \Phi'(U))$ -valued $\{\mathcal{F}_t^W\}$ -predictable process $Z_G^U(t)$ which satisfies (3.1) and $M_t^U = \int_0^t Z_G^U(s)dW_s^G$, $0 \leq t \leq T$. By Lemma 3.1, there exists an $\{\mathcal{F}_t^W\}$ -predictable process $Z^U(\cdot) \in \Lambda_Q(\Phi', \Phi'(U))$ such that $Z^U(t)i_{G^\circ} = Z_G^U(t)$, P -a.s., and $M_t^U = \int_0^t Z^U(s)dW_s$, $0 \leq t \leq T$.

It is easy to see that $\{M_t^U : U \in \mathcal{U}_h(\Phi')\}$ is a projective system. Hence, by Lemma 3.2 there exists an $\mathcal{L}(\Phi', \Phi')$ -valued, $\{\mathcal{F}_t^W\}$ -predictable process $Z(\cdot) \in \Lambda_Q(\Phi', \Phi')$ such that $M_t = \int_0^t Z(s)dW_s$, $0 \leq t \leq T$. $\square$

Now we consider Girsanov's theorem for Φ' -valued diffusion. Let $h(\cdot)$ be an $\{\mathcal{F}_t^W\}$ -predictable process such that $h(\cdot) \in \Lambda_Q(\Phi', R)$. Define a Φ' -valued process H_t by $H_t[\varphi] = Q(h(t), \varphi)$ for all $0 \leq t \leq T$ and $\varphi \in \Phi'$, where $h(\cdot)$ is regarded as a Φ -valued process since Φ is reflexive. Let Y_t be a Φ' -valued diffusion defined by

$$Y_t = \int_0^t H_s ds + W_t, \quad 0 \leq t \leq T. \quad (3.2)$$

By the typical Girsanov's approach, we will show this process is also a Φ' -valued Wiener process under a new probability measure. For any $\varphi \in \Phi$ and $x \in \mathbf{R}$, define a real valued process $\theta_t^{\varphi, x}$ by

$$\theta_t^{\varphi, x} = \exp \left\{ - \int_0^t (h(s) + x\varphi) dW_s - \frac{1}{2} \int_0^t Q(h(s) + x\varphi, h(s) + x\varphi) ds \right\},$$

where $x\varphi$ is regarded as a continuous linear functional on Φ' .

Lemma 3.4 Suppose the process $h(\cdot)$ satisfies that, for every $x \in \mathbf{R}$,

$$\mathbf{E} \left[\exp \left\{ \frac{1}{2} (1+x) \int_0^T Q(h(s), h(s)) ds \right\} \right] < \infty. \quad (3.3)$$

Then the real valued process $\theta_t^{\varphi, x}$ is a continuous $(\{\mathcal{F}_t^W\}, \mathbf{P})$ -exponential martingale with $\mathbf{E}[\theta_t^{\varphi, x}] = 1$ for each $0 \leq t \leq T$.

Proof By Novikov's condition on exponential martingales ([4]) we only need to show

$$\mathbf{E} \left[\exp \left\{ \frac{1}{2} \int_0^t Q(h(s) + x\varphi, h(s) + x\varphi) ds \right\} \right] < \infty.$$

Since Q is a continuous positive bilinear form on $\Phi \times \Phi$, we have the inequality

$$\begin{aligned} \int_0^T Q(h(s) + x\varphi, h(s) + x\varphi) ds &\leq (1+x) \int_0^T Q(h(s), h(s)) ds \\ &\quad + x(1+x)TQ(\varphi, \varphi). \end{aligned}$$

Hence, by the condition (3.3) we get

$$\begin{aligned} &\mathbf{E} \left[\exp \left\{ \frac{1}{2} \int_0^t Q(h(s) + x\varphi, h(s) + x\varphi) ds \right\} \right] \\ &\leq \mathbf{E} \left[\exp \left\{ \frac{1}{2} (1+x) \int_0^T Q(h(s), h(s)) ds \right\} \right] \exp \left\{ \frac{1}{2} x(1+x)TQ(\varphi, \varphi) \right\} < \infty. \end{aligned}$$

Thus, the lemma is proved. $\square$

Theorem 3.5 (Girsanov's theorem). Let $\theta_t = \theta_t^{0,0}$, $0 \leq t \leq T$, and assume that there exists some $x_0 > 0$ such that for each $\varphi \in \Phi$ and $0 \leq t \leq T$,

$$\mathbf{E} \left[\exp \{x_0 |Y_t[\varphi]| \} \right] < \infty, \quad (3.4)$$

Then, under (3.3), Y_t is a Φ' -valued $(\{\mathcal{F}_t^W\}, \bar{\mathbf{P}})$ -Wiener process with the same covariance operator Q as W_t , where $\bar{\mathbf{P}}$ is a probability on $(\Omega, \mathcal{F}_T^W)$, defined by $d\bar{\mathbf{P}} = \theta_T d\mathbf{P}$ and being equivalent to $\mathbf{P}$, and so that $d\bar{\mathbf{P}} = \theta_t d\mathbf{P}$ on $\mathcal{F}_t^W$, $0 \leq t \leq T$.

Proof By Lemma 3.4, θ_t is a continuous $(\{\mathcal{F}_t^W\}, \mathbf{P})$ -exponential martingale with $\mathbf{E}[\theta_T] = 1$, and hence, $d\bar{\mathbf{P}} = \theta_T d\mathbf{P}$ define a probability on $\mathcal{F}_T^W$. Again by Lemma 3.4, for every $(\varphi, x) \in \Phi \times \mathbf{R}$, $\theta_t^{\varphi, x}$ is also a continuous $(\{\mathcal{F}_t^W\}, P)$ -exponential martingale, and noting that we have

$$\theta_t^{\varphi, x} = \theta_t \exp \left\{ -xY_t[\varphi] - \frac{1}{2}x^2tQ(\varphi, \varphi) \right\}, \quad 0 \leq t \leq T.$$

Thus, for each $0 \leq t \leq T$ and every $B \in \mathcal{F}_t$, we get

$$\begin{aligned} & \int_B \theta_T \exp \left\{ -xY_T[\varphi] - \frac{1}{2}x^2TQ(\varphi, \varphi) \right\} d\mathbf{P} \\ &= \int_B \theta_t \exp \left\{ -xY_t[\varphi] - \frac{1}{2}x^2tQ(\varphi, \varphi) \right\} d\mathbf{P} \end{aligned}$$

by the martingale property of $\theta_t^{\varphi, x}$. So we get

$$\begin{aligned} & \int_B \exp \left\{ -xY_T[\varphi] - \frac{1}{2}x^2TQ(\varphi, \varphi) \right\} d\bar{\mathbf{P}} \\ &= \int_B \exp \left\{ -xY_t[\varphi] - \frac{1}{2}x^2tQ(\varphi, \varphi) \right\} d\bar{\mathbf{P}} \end{aligned}$$

by the definition of $\bar{\mathbf{P}}$. This expresses the martingale property of the process

$$\exp \left\{ -xY_t[\varphi] - \frac{1}{2}x^2tQ(\varphi, \varphi) \right\}, \quad 0 \leq t \leq T,$$

under $\bar{\mathbf{P}}$. By (3.4) and according to the exponential formula for martingale ([8]), $Y_t[\varphi]$, is an $(\{\mathcal{F}_t^W\}, \bar{\mathbf{P}})$ -square integrable martingale, and $\langle Y[\varphi], Y[\varphi] \rangle_t = tQ(\varphi, \varphi)$. By the linearity of Q , it is easy to verify that for every $\varphi_1, \varphi_2 \in \Phi$, $\langle Y[\varphi_1], Y[\varphi_2] \rangle_t = tQ(\varphi_1, \varphi_2)$. This implies that Y_t is a Φ' -valued $(\mathcal{F}_t^W, \bar{P})$ -Wiener process with the covariance operator Q . $\square$

4 The Innovation Process

Let $\mathcal{G}$ be a sub σ -algebra of $\mathcal{F}$, and let Z be a Φ' -valued random variable satisfies that $\mathbf{E}[||k_U Z||_U] < \infty$ holds for every $U \in \mathcal{U}_h(\Phi')$. Then the conditional expectation of Z given $\mathcal{G}$, denoted by $\mathbf{E}[Z | \mathcal{G}]$, exists and is a Φ' -valued random variable satisfying:

$$\mathbf{E}(Z | \mathcal{G})[\varphi] = \mathbf{E}\{Z[\varphi] | \mathcal{G}\}, \quad \forall \varphi \in \Phi \quad (4.1)$$

Let the observation process Y_t be a Φ' -valued diffusion given in (1.2), more exactly, given by

$$Y_t = \int_0^t H(X_s) ds + W_t, \quad 0 \leq t \leq T, \quad (4.2)$$

and let $\mathcal{G}_t = \sigma(\sigma\{Y_s : 0 \leq s \leq t\} \cup \mathcal{N})$, $0 \leq t \leq T$. Then $\{\mathcal{G}_t\}_{0 \leq t \leq T}$ is a subfiltration such that $\mathcal{G}_t \subset \mathcal{F}_t^W$ for every t . In the following, for every Φ' -valued process Z_t , we always denote: $\hat{Z}_t = \mathbf{E}[Z_t | \mathcal{G}_t]$ if the conditional expectation exists.

Lemma 4.1 Suppose that the Φ' -valued $\{\mathcal{F}_t^W\}$ -predictable process Z_t satisfies the conditions that $\mathbf{E}[\|Z_t^U\|_U] < \infty$, for all $U \in \mathcal{U}_h(\Phi')$ and $0 \leq t \leq T$, and $\int_0^T \mathbf{E}[\|Z_t[\varphi]\|^2] dt < \infty$ for each $\varphi \in \Phi$. Then the Φ' -valued process $\hat{Z}_t$ is $\{\mathcal{G}_t\}$ -predictable and satisfies that $\int_0^T \mathbf{E}[\|\hat{Z}_t[\varphi]\|^2] dt < \infty$ for each $\varphi \in \Phi$.

The proof of this lemma is standard, and we omit it here. Denote $H_t = H(X_t)$ for every $0 \leq t \leq T$. The innovation process N_t is a Φ' -valued, continuous and $\{\mathcal{G}_t\}$ -adapted process defined by

$$N_t = Y_t - \int_0^t \hat{H}_s ds, \quad 0 \leq t \leq T. \quad (4.3)$$

Theorem 4.2 The innovation process N_t is a Φ' -valued $(\{\mathcal{G}_t\}, \mathbf{P})$ -Wiener process with covariance operator Q .

Proof By (4.2) and (4.3), for every $\varphi \in \Phi$ and any $0 \leq s < t \leq T$ we get

$$\mathbf{E}[(N_t - N_s)[\varphi] \mid \mathcal{G}_s] = \mathbf{E}\left[\int_s^t (H_u - \hat{H}_u)[\varphi] du \mid \mathcal{G}_s\right] + \mathbf{E}[(W_t - W_s)[\varphi] \mid \mathcal{G}_s]$$

But we have that $\mathbf{E}[(W_t - W_s)[\varphi] \mid \mathcal{G}_s] = \mathbf{E}\{E[(W_t - W_s)[\varphi] \mid \mathcal{F}_s^W] \mid \mathcal{G}_s\} = 0$, and by using Fubini's theorem, we get

$$\mathbf{E}\left[\int_s^t (H_u - \hat{H}_u)[\varphi] du \mid \mathcal{G}_s\right] = \int_s^t \mathbf{E}[(H_u - \hat{H}_u)[\varphi] \mid \mathcal{G}_s] du = 0$$

Hence, we obtain $\mathbf{E}[(N_t - N_s)[\varphi] \mid \mathcal{G}_s] = 0$, $0 \leq s < t \leq T$, and so N_t is a Φ' -valued $(\{\mathcal{G}_t\}, \mathbf{P})$ -martingale. Now again by (4.2) and (4.3) we get

$$N_t[\varphi] = \int_0^t (H_s - \hat{H}_s)[\varphi] ds + W_t[\varphi], \quad 0 \leq t \leq T, \quad \forall \varphi \in \Phi.$$

Hence, we obtain $\langle N[\varphi_1], N[\varphi_2] \rangle_t = \langle W[\varphi_1], W[\varphi_2] \rangle_t = tQ(\varphi_1, \varphi_2)$ for all $\varphi_1, \varphi_2 \in \Phi$. This implies that the innovation process N_t is a Φ' -valued $(\{\mathcal{G}_t\}, \mathbf{P})$ -Wiener process with covariance operator Q . $\square$

Using the above theorem, and the representation theorem and Girsanov's theorem in Section 3, we can prove a representation theorem for the innovation process. Assume the mapping $H(\cdot)$ satisfies the condition:

(A3) There a bounded and continuous mapping $h(\cdot) : \Phi' \rightarrow \Phi$ such that $H(\xi)[\varphi] = Q(h(\xi), \varphi)$, for all $(\xi, \varphi) \in \Phi' \times \Phi$.

Theorem 4.3 Assume that (A3) and conditions of Theorem 3.5 are satisfied. Then, for every Φ' -valued, continuous and square integrable $(\{\mathcal{G}_t\}, P)$ -martingale M_t with $M_0 = 0$, there exists an $\mathcal{L}(\Phi', \Phi')$ -valued, $\{\mathcal{G}_t\}$ -predictable process $Z(\cdot) \in \Lambda_Q(\Phi', \Phi')$ such that

$$M_t = \int_0^t Z(s) dN_s, \quad 0 \leq t \leq T. \quad (4.4)$$

Proof From (4.3) we have $Y_t = \int_0^t \widehat{H}_s ds + N_t$ for all $0 \leq t \leq T$. Since N_t is a Φ' -valued $(\{\mathcal{G}_t\}, \mathbf{P})$ -Wiener process with covariance operator Q , by Girsanov's theorem Y_t is a Φ' -valued $(\{\mathcal{G}_t\}, \bar{\mathbf{P}})$ -Wiener process with covariance operator Q , where $\bar{\mathbf{P}}$ is a probability on $\mathcal{G}_T$ such that $d\bar{\mathbf{P}} = \theta_t d\mathbf{P}$ on $\mathcal{G}_t$ for each $0 \leq t \leq T$, and θ_T is given by

$$\theta_t = \exp \left\{ - \int_0^t \widehat{h}(s) dN_s - \frac{1}{2} \int_0^t Q(\widehat{h}(s), \widehat{h}(s)) ds \right\}, \quad 0 \leq t \leq T,$$

where $\widehat{h}(t) = \widehat{h(X_t)}$, $0 \leq t \leq T$. Define $\bar{M}_t = (\theta_t)^{-1} M_t$, $0 \leq t \leq T$. Then, it is easy to verify that $\bar{M}_t$ is a Φ' -valued, continuous and square integrable $(\{\mathcal{G}_t\}, \bar{\mathbf{P}})$ -martingale with $\bar{M}_0 = 0$. Since $\mathcal{G}_t = \sigma(\sigma\{Y_s : 0 \leq s \leq t\} \cup \mathcal{N})$ for every $0 \leq t \leq T$, by Theorem 3.3 there exists $(\{\mathcal{G}_t\}, \bar{\mathbf{P}})$ -predictable process $\bar{Z}(\cdot) \in \Lambda_Q(\Phi', \Phi')$ such that

$$\bar{M}_t = \int_0^t \bar{Z}(s) dY_s = \int_0^t \bar{Z}(s) dN_s + \int_0^t \bar{Z}(s) \widehat{H}_s ds, \quad 0 \leq t \leq T$$

or

$$\bar{M}_t[\varphi] = \int_0^t \bar{Z}_s(\varphi) dN_s + \int_0^t Q(\bar{Z}_s(\varphi), \widehat{h}(s)) ds, \quad 0 \leq t \leq T, \quad \forall \varphi \in \Phi$$

where $Q(\bar{Z}_t(\varphi), \widehat{h}(t)) = \bar{Z}_t(\varphi) \widehat{H}_t$ for each $0 \leq t \leq T$. Also, we can verify that the process θ_t satisfies the equation:

$$\theta_t = 1 - \int_0^t \theta_s \widehat{h}(s) dN_s, \quad 0 \leq t \leq T$$

Thus, for each $\varphi \in \Phi$, by using the integration by parts formula we get

$$\begin{aligned} M_t[\varphi] &= \theta_t \bar{M}_t[\varphi] = \int_0^t \theta_s \bar{Z}_s(\varphi) dN_s + \int_0^t \theta_s Q(\bar{Z}_s(\varphi), \widehat{h}_s) ds \\ &\quad - \int_0^t \bar{M}_s[\varphi] \theta_s \widehat{h}(s) dN_s - \int_0^t \theta_s Q(\bar{Z}_s(\varphi), \widehat{h}(s)) ds \\ &= \int_0^t \theta_s [\bar{Z}_s(\varphi) - \bar{M}_s[\varphi] \widehat{h}(s)] dN_s \end{aligned}$$

Now denote $Z(t) = \theta_t [\bar{Z}(t) - (\widehat{h} \otimes \bar{M})(t)]$, $0 \leq t \leq T$, where $(\widehat{h} \otimes \bar{M})(\cdot)$ is an $\mathcal{L}(\Phi', \Phi')$ -valued process defined by $(\widehat{h} \otimes \bar{M})(t)\xi = \xi[\widehat{h}(t)]\bar{M}_t$ for each $\xi \in \Phi'$ and every $0 \leq t \leq T$. Then we can verify that $Z(\cdot)$ is an $\mathcal{L}(\Phi', \Phi')$ -valued, $(\mathcal{G}_t)$ -predictable process such that $Z(\cdot) \in \Lambda_Q(\Phi', \Phi')$ and satisfies (4.4). The theorem is proved, $\square$

5 The Filtering Equation

In this section we denote $h(t) = h(X_t)$, $0 \leq t \leq T$, where $h(\cdot)$ is given in (A3). Define an $\mathcal{L}(\Phi', \Phi')$ -valued process $(h \otimes X)(\cdot)$ by $(h \otimes X)(t)\xi = \xi[h(t)]X_t$ for all $\xi \in \Phi'$ and $0 \leq t \leq T$, and an $\mathcal{L}(\Phi, \Phi)$ -valued process $(X \otimes h)(t)$ by $(X \otimes h)(t)\varphi = X_t[\varphi]h(t)$ for all $\varphi \in \Phi$ and $0 \leq t \leq T$. Using the Hölder inequality, we can prove the following lemma;

Lemma 5.1 *If the processes $h(t)$ and X_t satisfy:*

$$\mathbf{E}\left[\int_0^T (Q(h(s), h(s)))^2 ds\right] < \infty \quad \text{and} \quad \mathbf{E}\left[\int_0^T (X_s[\varphi])^4 ds\right] < \infty, \quad \forall \varphi \in \Phi.$$

Then, the process $(h \otimes X)(\cdot) \in \Lambda_Q(\Phi', \Phi')$ and its adjoint process is given by $(h \otimes X)^(\cdot) = (X \otimes h)(\cdot)$. Moreover, the process $(\widehat{h \otimes X})(\cdot)$ exists, and its adjoint process is given by $(\widehat{h \otimes X})^*(\cdot) = (\widehat{X \otimes h})(\cdot)$, where $(\widehat{h \otimes X})(t) = \mathbf{E}[(h \otimes X)(t) \mid \mathcal{G}_t]$, $0 \leq t \leq T$.*

Now we will prove our main result on the best estimate $\widehat{X}_t$ of the state process X_t based on the observation process Y_t .

Theorem 5.2 *Assume that (A1) – (A3), the conditions of Theorem 3.5 and Lemma 5.1 are satisfied, and Q is strictly positive, i.e. $Q(\varphi, \varphi) > 0$ for all $\varphi \neq 0$. Then $\widehat{X}_t$ satisfies the following SEE:*

$$d\widehat{X}_t = A\widehat{X}_t dt + (\widehat{F}(t) + (\widehat{h \otimes X})(t) - (\widehat{h \otimes \widehat{X}})(t))dN_t \quad (5.1)$$

with the initial condition $\widehat{X}_0 = X_0 \in \Phi'$.

Proof For every $\varphi \in \Phi$, we have

$$X_t[\varphi] = X_0[\varphi] + \int_0^t X_s[A^*\varphi]ds + \int_0^t F_s(\varphi)dW_s, \quad 0 \leq t \leq T,$$

and

$$Y_t[\varphi] = \int_0^t H(X_s)[\varphi]ds + W_t[\varphi], \quad 0 \leq t \leq T.$$

Thus, by Integration by parts formula ([4]), we get

$$\begin{aligned} X_t[\varphi]Y_t[\varphi] &= \int_0^t X_s[\varphi]H(X_s)[\varphi]ds + \int_0^t Y_s[\varphi]X_s[A^*\varphi]ds \\ &\quad + \int_0^t X_s[\varphi]dW_s[\varphi] + \int_0^t Y_s[\varphi]F_s(\varphi)dW_s + \int_0^t Q(F^*(s)\varphi, \varphi)ds \end{aligned}$$

Then, as the real valued stochastic filtering equation ([3]), we can show that there exists a real valued $\{\mathcal{G}_t\}$ -square integrable martingale m_t^φ such that

$$\widehat{X}_t[\varphi]Y_t[\varphi] = m_t^\varphi + \int_0^t (\widehat{a}_s^\varphi + \widehat{b}_s^\varphi)ds + \int_0^t Y_s[\varphi]\widehat{X}_s[{}^*\varphi]ds \quad (5.2)$$

where $a_t^\varphi = X_t[\varphi]H(X_t)[\varphi]$ and $b_t^\varphi = Q(F^*(t)\varphi, \varphi)$, $0 \leq t \leq T$. On the other hand, from (4.3) we have

$$Y_t[\varphi] = \int_0^t \widehat{H}_s[\varphi] ds + N_t[\varphi], \quad 0 \leq t \leq T, \quad (5.3)$$

where we again denote $H_t = H(X_t)$, $0 \leq t \leq T$. Let

$$M_t = \widehat{X}_t - X_0 - \int_0^t A\widehat{X}_s ds, \quad 0 \leq t \leq T.$$

Then, since $\mathcal{G}_t \subset \mathcal{F}_t$, it is easy to verify that M_t is a Φ' -valued $\{\mathcal{G}_t\}$ -martingale. Thus, by Theorem 4.3, there exists a $\{\mathcal{G}_t\}$ -predictable process $Z(\cdot) \in \Lambda_Q(\Phi', \Phi)$ such that $M_t = \int_0^t Z(s) dN_s$, $0 \leq t \leq T$. Hence we have

$$\widehat{X}_t = X_0 + \int_0^t A\widehat{X}_s ds + \int_0^t Z(s) dN_s, \quad 0 \leq t \leq T \quad (5.4)$$

or, we get

$$\widehat{X}_t[\varphi] = X_0[\varphi] + \int_0^t (A\widehat{X}_s)[\varphi] ds + \int_0^t Z_s(\varphi) dN_s \quad (5.5)$$

Applying Integration by parts formula to (5.3) and (5.5), we obtain

$$\widehat{X}_t[\varphi]Y_t[\varphi] = n_t^\varphi + \int_0^t (\alpha_s^\varphi + \beta_s^\varphi) ds + \int_0^t Y_s[\varphi](A\widehat{X}_s)[\varphi] ds \quad (5.6)$$

where n_t^φ is a real valued square integrable $\{\mathcal{G}_t\}$ -martingale, $\alpha_t^\varphi = \widehat{X}_t[\varphi]\widehat{H(X_s)}[\varphi]$ and $\beta_t^\varphi = Q(Z^*(t)\varphi, \varphi)$, $0 \leq t \leq T$. Comparing (5.2) and (5.6), and according to the uniqueness of canonical decomposition of semimartingales, we get

$$\int_0^t (\widehat{a}_s^\varphi + \widehat{b}_s^\varphi) ds = \int_0^t (\alpha_s^\varphi + \beta_s^\varphi) ds, \quad 0 \leq t \leq T,$$

or $Q(Z^*(t)\varphi, \varphi) = \widehat{a}_t^\varphi + \widehat{b}_t^\varphi - \widehat{X}_t[\varphi]\widehat{H(X_s)}[\varphi]$, $0 \leq t \leq T$. Noting that

$$a_t^\varphi = X_t[\varphi]Q(h(t), \varphi) = Q((X \otimes h)(t)\varphi, \varphi), \quad 0 \leq t \leq T,$$

and

$$\widehat{X}_t[\varphi]\widehat{H(X_s)}[\varphi] = \widehat{X}_t[\varphi]Q(\widehat{h}(t), \varphi) = Q((\widehat{X} \otimes \widehat{h})(t)\varphi, \varphi), \quad 0 \leq t \leq T$$

we have

$$Q(Z^*(t)\varphi, \varphi) = Q([\widehat{(X \otimes h)}(t) + \widehat{F}^*(t) - (\widehat{X} \otimes \widehat{h})(t)]\varphi, \varphi), \quad 0 \leq t \leq T.$$

Hence, by Lemma 5.1, we get

$$Z(t) = (\widehat{(h \otimes X)}(t) + \widehat{F}(t) - (\widehat{h} \otimes \widehat{X})(t)), \quad 0 \leq t \leq T$$

since Q is strictly positive and $\varphi \in \Phi$ is arbitrary. Substituting $Z(\cdot)$ to equation (5.4) we obtain the equation (5.1). The proof is completed. $\square$

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Received: July 17, 2006