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Regular Polygonal Numbers and Generalized Pell Equations

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Abstract. In the eighteenth century, both square numbers and triangular numbers were investigated by Euler and Goldbach (1742), who determined the recurrence relations satisfied by the sequence and established the general formulae explicitly. It seems to the author that the topics around this subject have not been touched in mathematical literature. As the first attempt to explore it, this work will present a systematic procedure to deal with the problem. For the regular (λ, μ)-polygonal numbers, the corresponding Diophantine equations will be reduced to the generalized Pell equations. Then solutions of the associated Pell equations will essentially enable us to resolve the problem. By means of *Computer Algebra*, the recurrence relations and generating functions satisfied by (λ, μ)-polygonal numbers can be recovered systematically. As exemplification, the results on the first twenty regular (λ, μ)-polygonal sequences will be presented in details.

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1. Introduction and Notation

In the treatise **Polygonal Numbers**, Diophantus quoted the definition of polygonal numbers (cf. [1, pp. 1-3] for reference) due to Hypsicles (around 175 B.C.): "If there are as many numbers as we please beginning with one and increasing by the same common difference, then when the common difference is one, the sum of all the terms is a triangular number; when 2, a square; when 3, a pentagonal number. And the number of the angles is called after the number exceeding the common difference by 2, and the side after the number of terms including 1". Given therefore an arithemetical progression with the first term 1 and the common difference $\lambda - 2$, the sum of n terms is the regular n-th λ-gonal number $p_\lambda(n)$.

From this definition, we can compute the regular n-th λ-gonal number $p_\lambda(n)$ explicitly as follows:

$$p_\lambda(n) = \sum_{k=0}^{n-1}\{1+(\lambda-2)k\} = n+(\lambda-2)\binom{n}{2} \tag{1.1}$$

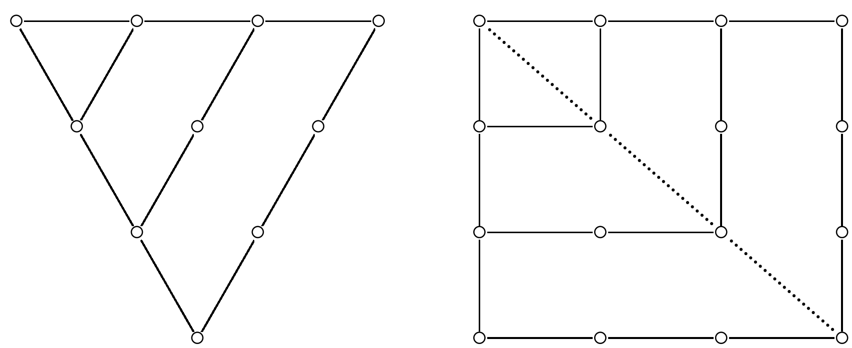
where $n=1,2,\cdots$. Hence the set of all the regular λ-polygonal numbers is determined by

$$\mathcal{P}_\lambda = \Big\{n+(\lambda-2)\binom{n}{2} \mid n=1,2,\cdots\Big\}. \tag{1.2}$$

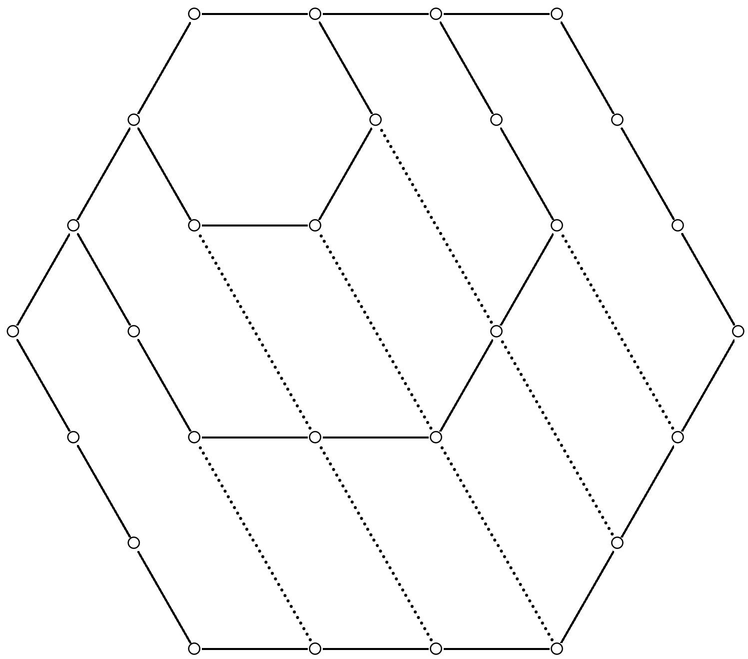
Triangular numbers and square numbers are respectively given by

$$\mathcal{P}_3 = \Big\{\binom{n+1}{2} \mid n=1,2,\cdots\Big\} \quad \text{and} \quad \mathcal{P}_4 = \Big\{n^2 \mid n=1,2,\cdots\Big\}.$$

They can be geometrically represented by the following figures:



In general, the regular polygonal numbers can be generated recursively as follows. For $p_\lambda(n)$, fix with λ the number of sides of the polygons, called λ-polygons, and with n the side-length minus one. Conventionally one fixes $p_\lambda(1)=1$ because the regular λ-polygon with side length equal to zero reduces to one point. Based on a λ-polygon with side-length equal to $n-1$, we can construct the next polygon with the side-length equal to n, extending by unit the two *base* sides which cross at the starting point and then adding $\lambda-2$ new sides parallel with the remaining sides. During this construction, we have added $1+n(\lambda-2)$ new points to the polygon on the new sides. This procedure can be illustrated by the following figure of hexagons:



Therefore we have the following recurrence relation

$$p_\lambda(1+n) \;=\; 1 + n(\lambda - 2) + p_\lambda(n) \tag{1.3a}$$

$$\text{with} \quad p_\lambda(1) = 1 \quad \text{and} \quad n = 1, 2, \cdots \tag{1.3b}$$

which is consistent with (1.1), derived from the arithemetical setting.

For $\lambda, \mu \in \mathbb{N}$ with $\lambda \neq \mu$ and $\lambda, \mu \geq 3$, we define the regular (λ, μ)-polygonal numbers to be the natural numbers in the intersection set $\mathcal{P}_\lambda \cap \mathcal{P}_\mu$, which are both regular λ-polygonal and μ-polygonal. They are characterized by natural number solutions (x, y) of the Diophantine equation $p_\lambda(x) = p_\mu(y)$, written explicitly as

$$x + (\lambda - 2)\binom{x}{2} \;=\; y + (\mu - 2)\binom{y}{2}. \tag{1.4}$$

It is not hard to check that there is bijective correspondence between $\mathcal{P}_\lambda \cap \mathcal{P}_\mu$ and the solutions of $p_\lambda(x) = p_\mu(y)$ in natural numbers. In fact, for any $z \in \mathcal{P}_\lambda \cap \mathcal{P}_\mu$, there exist exactly two natural numbers x and y such that $z = p_\lambda(x) = p_\mu(y)$. Throughout the paper, the regular (λ, μ)-polygonal numbers will be represented by the triples $(x, y; z)$, where z is both regular λ-gonal and μ-gonal number with side-lengths equal to x and y respectively.

For $\lambda = 3$ and $\mu = 4$, the first triangular-square numbers can be displayed in the following table:

n	x_n	y_n	z_n
0	1	1	1
1	8	6	36
2	49	35	1225
3	288	204	41616
4	1681	1189	1413721
5	9800	6930	48024900

Euler observed in 1730 that the triangle-square numbers have the following form (cf. Dickson [1, p. 10]):

$$x_n = \frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n}{4} - \frac{1}{2} \quad (n \in \mathbb{N})$$
$$y_n = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{4\sqrt{2}} \quad (n \in \mathbb{N}).$$

Almost half century later, Euler (1778) proved that these are all the regular $(3,4)$-polygonal numbers (cf. Dickson [1, p. 16]) which also satisfy the following crossing recurrence relations of the first degree:

$$\left.\begin{aligned} x_{1+m} &= 3x_m + 4y_m + 1 \\ y_{1+m} &= 3y_m + 2x_m + 1 \end{aligned}\right\} \quad x_0 = y_0 = 1$$

and the independent recurrence relations of the second degree

$$\begin{aligned} x_{1+n} &= 6x_n - x_{n-1} + 2: \quad x_0 = 1, \qquad x_1 = 8 \\ y_{1+n} &= 6y_n - y_{n-1}: \qquad\quad y_0 = 1, \qquad y_1 = 6 \end{aligned}$$

as well as the respective generating functions:

$$f(x) := \sum_{n=0}^{\infty} x^n x_n = \frac{1+x}{(1-x)\big(1-6x+x^2\big)}$$
$$g(y) := \sum_{n=0}^{\infty} y^n y_n = \frac{1}{1-6y+y^2}.$$

Moreover, in 1742, there was a communication (see Dickson [1, pp. 10-11]) between Euler and Goldbach on the pentagon numbers

$$\mathcal{P}_5 = \left\{ \frac{3n^2-n}{2} \,\Big|\, n = 1, 2, \cdots \right\} \tag{1.5}$$

for which there holds the following celebrated pentagon number theorem due to Euler (cf. [2, §19.9]):

$$\prod_{m=1}^{\infty} (1-q^m) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{\frac{3n^2-n}{2}} \quad \text{where} \quad |q| < 1.$$

The object of the paper is to determine all the regular (λ,μ)-polygonal numbers. As preliminaries, we shall review basic facts about the Pell equations and generalized Pell equations in the next section. Then the Diophantine equation (1.4) will be reduced to generalized Pell equation in the third section. The classification, recurrence relations and generating functions of the regular (λ,μ)-polygonal numbers will be investigated in the fourth section. By means of computer algebra, the fifth and the last section collects twenty examples of the regular (λ,μ)-polygonal numbers, which presents a full coverage for the cases $3 \le \lambda \ne \mu \le 9$.

2. Generalized Pell Equations

In order to investigate the Diophantine equations on the regular (λ,μ)-polygonal numbers, we review some basic results about Pell equations and generalized Pell equations. For details, the reader can refer to the books [3, §10.9 and §11.5], [6, §6.2] and [8, §7.8].

2.1. **Pell equation.** With D being a non-perfect-square natural number, Pell equation

$$u^2 - Dv^2 = 1 \tag{2.1}$$

admits always infinite solutions. They can be determined through continued fraction expansion of $\sqrt{D}$. If (u,v) is the minimal positive solution, then all the non-negative solutions are given by

$$u_n \pm v_n\sqrt{D} = \left\{u \pm v\sqrt{D}\right\}^n, \qquad (n \in \mathbb{N}_0)$$

which leads us to the following explicit formulas:

$$u_n = \frac{1}{2}\left\{\left(u+v\sqrt{D}\right)^n + \left(u-v\sqrt{D}\right)^n\right\} \tag{2.2a}$$

$$v_n = \frac{1}{2\sqrt{D}}\left\{\left(u+v\sqrt{D}\right)^n - \left(u-v\sqrt{D}\right)^n\right\}. \tag{2.2b}$$

In addition, the solutions $\{u_n, v_n\}$ satisfy the recurrence relations

$$\left.\begin{aligned} u_{1+n} &= 2u\,u_n - u_{n-1} \\ v_{1+n} &= 2u\,v_n - v_{n-1} \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} u_{1+n} &= u\,u_n + v\,v_n D \\ v_{1+n} &= u\,v_n + v\,u_n. \end{aligned}\right.$$

2.2. **Generalized Pell equation.** For two integers D and N with D being positive and non-perfect-square, the generalized Pell equation and the associated Pell equation are displayed as

$$U^2 - DV^2 = N \qquad \text{and} \qquad u^2 - Dv^2 = 1. \tag{2.3}$$

If (U,V) and (u,v) are solutions respectively corresponding to the generalized Pell equation and the associated Pell equation, then

$$(uU + vVD, uV + vU)$$

derived from the product (cf. [7, §58])

$$(u+v\sqrt{D}) \times (U+V\sqrt{D}) = (uU+vVD) + (uV+vU)\sqrt{D}$$

forms a new solution of the generalized Pell equation thanks to the relation

$$(uU+vVD)^2 - D(uV+vU)^2 = (u^2-Dv^2) \times (U^2-DV^2).$$

Two solutions (U, V) and (U', V') of the generalized Pell equation are said to be equivalent if there exists one solution (u, v) of the associated Pell equation such that

$$(U', V') \quad = \quad (uU + vVD, uV + vU).$$

Therefore all the solutions of the generalized Pell equation can be divided into equivalent classes, among which each class of solutions consist of double-sequences subject to the crossing recurrence relation of the first degree:

$$U_{1+n} = uU_n + vV_nD \tag{2.4a}$$
$$V_{1+n} = uV_n + vU_n \tag{2.4b}$$

where (u, v) is a solution of the associated Pell equation.

Now replacing n by $n-1$, we can restate the crossing recurrences (2.4a-2.4b) as

$$\begin{aligned} U_n &= uU_{n-1} + vV_{n-1}D \\ V_n &= uV_{n-1} + vU_{n-1}. \end{aligned}$$

Eliminating V_{n-1} and U_{n-1} from these two equations and noting that $u^2 - Dv^2 = 1$, we obtain the following expressions

$$\begin{aligned} DvV_n &= uU_n - U_{n-1} \\ vU_n &= uV_n - V_{n-1}. \end{aligned}$$

Substituting them into (2.4a) and (2.4b), we get the independent recurrence relations of the second degree

$$U_{1+n} = 2uU_n - U_{n-1} \tag{2.5a}$$
$$V_{1+n} = 2uV_n - V_{n-1} \tag{2.5b}$$

which are satisfied by the equivalent class of solutions of the generalized Pell equation corresponding to the crossing recursions (2.4a-2.4b).

3. Diophantine Equation $p_\lambda(x) = p_\mu(y)$

For the Diophantine Equation $p_\lambda(x) = p_\mu(y)$, given explicitly by (1.4), this section will show how to reduce it canonically to the generalized Pell equation.

Firstly, define two integer parameters:

$$d(\lambda) = \gcd(\lambda-4, 2\lambda-4) = \begin{cases} 1, & \lambda \equiv 1 \pmod 2 \\ 2, & \lambda \equiv 2 \pmod 4 \\ 4, & \lambda \equiv 0 \pmod 4 \end{cases} \tag{3.1a}$$

$$c(\lambda,\mu) = \gcd\left\{d^2(\lambda)(\mu-2), d^2(\mu)(\lambda-2)\right\}. \tag{3.1b}$$

They allow us to rewrite the regular λ-polygonal numbers as

$$\begin{aligned} 8\,p_\lambda(x) &= 4\,(\lambda-2)\left\{x - \frac{\lambda-4}{2\lambda-4}\right\}^2 - \frac{(\lambda-4)^2}{\lambda-2} \\ &= \frac{1}{\lambda-2}\Big\{(2\lambda-4)x - (\lambda-4)\Big\}^2 - \frac{(\lambda-4)^2}{\lambda-2} \\ &= \frac{d^2(\lambda)}{\lambda-2}\left\{\frac{2\lambda-4}{d(\lambda)}x - \frac{\lambda-4}{d(\lambda)}\right\}^2 - \frac{(\lambda-4)^2}{\lambda-2} \end{aligned}$$

and the Diophantine equation $p_\lambda(x) = p_\mu(y)$ as

$$\frac{(\lambda-\mu)(\lambda\mu-2\lambda-2\mu)}{c(\lambda,\mu)} = \frac{(\mu-2)d^2(\lambda)}{c(\lambda,\mu)}\left\{\frac{2\lambda-4}{d(\lambda)}x - \frac{\lambda-4}{d(\lambda)}\right\}^2 \tag{3.2a}$$

$$- \frac{(\lambda-2)d^2(\mu)}{c(\lambda,\mu)}\left\{\frac{2\mu-4}{d(\mu)}y - \frac{\mu-4}{d(\mu)}\right\}^2. \tag{3.2b}$$

Secondly, with **B** and **M**-coefficients being defined respectively by

$$B(\lambda,\mu) = \frac{(\mu-2)d^2(\lambda)}{c(\lambda,\mu)} \tag{3.3a}$$

$$M(\lambda,\mu) = \frac{(\lambda-\mu)(\lambda\mu-2\lambda-2\mu)}{c(\lambda,\mu)} \tag{3.3b}$$

we have evidently $\gcd\Big\{B(\lambda,\mu), B(\mu,\lambda)\Big\} = 1$ in view of (3.1b). Under the linear transformation

$$\mathcal{S}: \begin{cases} X = \dfrac{2\lambda-4}{d(\lambda)}x - \dfrac{\lambda-4}{d(\lambda)} \\ Y = \dfrac{2\mu-4}{d(\mu)}y - \dfrac{\mu-4}{d(\mu)}. \end{cases} \tag{3.4}$$

the Diophantine equation (1.4) is equivalent to the following

$$B(\lambda,\mu)\,X^2 - B(\mu,\lambda)\,Y^2 = M(\lambda,\mu). \tag{3.5}$$

Observe that $B(\lambda,\mu)$ is an integer on account (3.1a) and (3.1b). For the same reason, we can check that $M(\lambda,\mu)$ is an integer either, just specifying (3.2a-3.2b) by $x = y = 1$, which is the "universal solution" of (1.4).

Lastly, applying the square-free factorization

$$B(\lambda,\mu) = p(\lambda,\mu)q^2(\lambda,\mu) \tag{3.6}$$

and introducing **D** and **N**-coefficients:

$$D(\lambda,\mu) = p(\lambda,\mu)\, p(\mu,\lambda) \tag{3.7a}$$

$$N(\lambda,\mu) = p(\lambda,\mu)\, M(\lambda,\mu) \tag{3.7b}$$

we can restate (3.5) as the generalized Pell equation:

$$U^2 - D(\lambda,\mu)\, V^2 = N(\lambda,\mu) \tag{3.8}$$

where we have further introduced the linear transformation

$$\mathcal{T}: \begin{cases} U = p(\lambda,\mu)q(\lambda,\mu)X \\ V = q(\mu,\lambda)Y. \end{cases} \tag{3.9}$$

Summing up, the Diophantine equation (1.4) on regular (λ,μ)-polygonal numbers has been reduced to the generalized Pell equation (3.8) under the composite transformation

$$\Omega[\lambda,\mu] = \mathcal{T} \circ \mathcal{S}:\ (x,y) \longrightarrow \{U(x), V(y)\}$$

explicitly given by:

$$\Omega[\lambda,\mu]: \begin{cases} U(x) = \left\{\dfrac{2\lambda-4}{d(\lambda)}x - \dfrac{\lambda-4}{d(\lambda)}\right\}p(\lambda,\mu)q(\lambda,\mu) \\ V(y) = \left\{\dfrac{2\mu-4}{d(\mu)}y - \dfrac{\mu-4}{d(\mu)}\right\}q(\mu,\lambda) \end{cases} \tag{3.10}$$

whose inverse transformation reads as

$$\Omega'[\lambda,\mu]: \begin{cases} x = \dfrac{d(\lambda)U + (\lambda-4)\, p(\lambda,\mu)q(\lambda,\mu)}{(2\lambda-4)p(\lambda,\mu)q(\lambda,\mu)} \\ y = \dfrac{d(\mu)V + (\mu-4)q(\mu,\lambda)}{(2\mu-4)q(\mu,\lambda)}. \end{cases} \tag{3.11}$$

The antisymmetry $M(\lambda,\mu) = -M(\mu,\lambda)$ allows us to choose always the non-negative parameter $N(\mu,\lambda) = p(\mu,\lambda)\, M(\mu,\lambda)$ by interchanging the positions of λ and μ.

Remark: Recalling that $(x_0, y_0) = (0,0)$ and $(x_1, y_1) = (1,1)$ are always solutions of (1.4), we affirm that $\{U(0), V(0)\}$ and $\{U(1), V(1)\}$ deduced from the composite transformation $\Omega[\lambda,\mu]$ are always solutions of the generalized Pell equation (3.8).

4. Regular (λ,μ)-Polygonal Numbers

When the D-coefficient defined by (3.7a) is equal to one, we say that the corresponding Diophantine equation (1.4) is reducible. In that case, for $(\lambda,\mu) \neq (3,6)$ or $(6,3)$, equation (1.4) would have finitely many solutions (see §4.1 and §4.2). Otherwise, the Diophantine equation (1.4) corresponding to $D \neq 1$ is said to be irreducible and the generalized Pell equation (3.8) has infinitely many solutions (see §4.3 below). Therefore, the original Diophantine equation (1.4) on regular (λ,μ)-polygonal numbers has infinitely many solutions too. These solutions can be divided into equivalent classes and the solutions in each class can be determined by recurrence relations and accordingly generating functions.

4.1. **Triangular and hexagonal numbers.** Consider the extreme case of (3.8) when $N(\lambda,\mu)=0$, or $M(\lambda,\mu)=0$ equivalently, iff

$$\lambda\mu-2\lambda-2\mu \quad = \quad 0$$

which may be reformulated as

$$(\lambda-2)(\mu-2) \quad = \quad 4.$$

Under the restrictions $\lambda\neq\mu\geq 3$, there is a unique solution $(\lambda,\mu)=(3,6)$. In this case, we have $D(3,6)=1$ and the equations (3.8) and (3.10) reduce to

$$U^2-V^2 \;=\; 0: \qquad \Omega[3,6]\begin{cases} U &= 2x+1\\ V &= 4y-1\end{cases}$$

which can be written explicitly as

$$(U+V)\times(U-V) \;=\; 0 \quad \rightleftharpoons \quad (x+2y)\times(1+x-2y) \;=\; 0.$$

On account of solutions in natural numbers, we can write explicitly the solutions as:

$$\left.\begin{array}{l} x_n=2n-1\\ y_n=n\end{array}\right\} \quad (n\in\mathbb{N})$$

n	x_n	y_n	z_n
1	1	1	1
2	3	2	6
3	5	3	15
4	7	4	28
5	9	5	45

This means that *all the hexagon numbers are also triangle numbers.* This is the only reducible case where the equation (1.4) admits infinitely many solutions.

4.2. **Reducible cases: Finite solutions.** When $D(\lambda,\mu)=p(\lambda,\mu)p(\mu,\lambda)$ is a perfect square number, then $D(\lambda,\mu)=1$ thanks to the square free factorization. In this case, equation (3.8) reduces to

$$(U+V)\times(U-V) \;=\; N(\lambda,\mu).$$

This equation can be resolved by factorizing $N(\lambda,\mu)$ into two integers and therefore has only finitely many solutions for $N(\lambda,\mu)\neq 0$, which is equivalent to $(\lambda,\mu)\neq(3,6)$ and $(6,3)$.

In view of (3.3a), (3.6) and (3.7a), we have

$$D(\lambda,\mu) \;=\; \frac{B(\lambda,\mu)B(\mu,\lambda)}{q^2(\lambda,\mu)q^2(\mu,\lambda)} \;=\; \frac{(\lambda-2)(\mu-2)d^2(\lambda)d^2(\mu)}{c^2(\lambda,\mu)q^2(\lambda,\mu)q^2(\mu,\lambda)}. \tag{4.1}$$

Therefore $(\lambda-2)(\mu-2)$ must be a perfect square number. In this case, the Diophantine equation (1.4) has only finitely many solutions with $(\lambda,\mu)\neq(3,6)$ and $(6,3)$.

Here we present a couple of examples to show the reducible cases.

When $(\lambda,\mu)=(11,6)$, both $\lambda-2=9$ and $\mu-2=4$ are perfect square numbers. The Diophantine equation (1.4) reads as

$$x(9x-7)=2y(2y-1)$$

and the reduced Pell equation (3.8) becomes

$$u^2-v^2=40: \quad \Omega[11,6]\begin{cases} u=18x-7\\ v=12y-3\end{cases}$$

which has only the solutions $(\pm7,\pm3)$ and $(\pm11,\pm9)$. Among these solutions, only $(11,9)$ gives natural numbers $(x,y)=(1,1)$ under the corresponding inverse transformation $\Omega'[11,6]$ defined by (3.11). Therefore we have the only trivial regular $(11,6)$-polygonal number $(1,1)$.

For $(\lambda,\mu)=(29,5)$, neither $\lambda-2=27$ nor $\mu-2=3$ is perfect square number. But their product 81 is a perfect square number. The corresponding Diophantine equation (1.4) becomes

$$x(27x-25)=y(3y-1)$$

and the reduced Pell equation (3.8) reads as

$$u^2-v^2=616: \quad \Omega[29,5]\begin{cases} u=54x-25\\ v=18y-3.\end{cases}$$

It has only the solutions $(\pm25,\pm3)$, $(\pm29,\pm15)$, $(\pm79,\pm75)$ and $(\pm155,\pm153)$. Among these solutions, only $(29,15)$ gives natural numbers $(x,y)=(1,1)$ under the corresponding inverse transformation $\Omega'[29,5]$ defined by (3.11). Therefore we have again in this case the only trivial regular $(29,5)$-polygonal number $(1,1)$.

4.3. **Irreducible cases: Infinite solutions.** When $(\lambda-2)(\mu-2)$ is not a perfect square number, the generalized Pell equation (3.8) is not reducible. Recalling that (3.8) has two universal solutions $\{U(0),V(0)\}$ and $\{U(1),V(1)\}$, we deduce that there exists at least one equivalent class of solutions for (3.8). Hence there are infinitely many solutions for the irreducible generalized Pell equation (3.8). These solutions can be classified, according to any fixed solution (u,v) of the associated Pell equation, into equivalent classes, each of which satisfies recurrences:

$$\left.\begin{aligned} U_{1+n}&=uU_n+vV_nD\\ V_{1+n}&=uV_n+vU_n\end{aligned}\right\} \quad\text{and}\quad \left\{\begin{aligned} U_{1+n}&=2uU_n-U_{n-1}\\ V_{1+n}&=2uV_n-V_{n-1}.\end{aligned}\right.$$

In what follows, we will determine the recurrence relations satisfied by the solutions of the Diophantine equation (1.4) on the regular (λ,μ)-polygonal numbers and the corresponding generating functions.

4.4. **Recurrence relations.** All the solutions of the Diophantine equation (1.4) turn, under Ω-transform, into solutions of the generalized Pell equation

$$U^2 - D(\lambda,\mu)\, V^2 \quad = \quad N(\lambda,\mu).$$

Instead, each equivalent class of solutions of the generalized Pell equation obey crossing recursions (2.4a-2.4b), which are converted, under the inverse transform Ω', into *rational* solutions of the original Diophantine equation (1.4). It is not difficult to check that the latter obey again crossing recursions, which can be figured out, by substituting the transformation (3.10) into the crossing recurrence relations (2.4a-2.4b), as follows:

$$x_{1+n} \;=\; u\, x_n + vH(\lambda,\mu)\, y_n + E(\lambda,\mu|u,v) \qquad (4.2a)$$
$$y_{1+n} \;=\; u\, y_n + vH(\mu,\lambda)\, x_n + E(\mu,\lambda|u,v) \qquad (4.2b)$$

where

$$H(\lambda,\mu) \;=\; \sqrt{\tfrac{\mu-2}{\lambda-2}D(\lambda,\mu)} \;=\; \frac{d(\mu)p(\lambda,\mu)q(\lambda,\mu)}{d(\lambda)q(\mu,\lambda)} \qquad (4.3a)$$
$$E(\lambda,\mu|u,v) \;=\; \frac{4-\lambda}{4-2\lambda}(1-u) - H(\lambda,\mu)\frac{4-\mu}{4-2\mu}v. \qquad (4.3b)$$

In order to determine the crossing recurrence relations satisfied by the *integer* solutions of the Diophantine equation (1.4) on the regular (λ,μ)-polygonal numbers, it is enough to find out the minimum (u,v) among the solutions of the associated Pell-equation (2.1) such that (4.2a-4.2b) have integer coefficients.

Since $(1,1)$ is always a solution of the Diophantine equation (1.4), we assert that any class of equivalent solutions must have the initial values (x_0,y_0) with x_0 being from 1 to the sum of coefficients of (4.2a). Therefore **the number of equivalent classes** of the solutions corresponding to the solution (u,v) of the associated Pell equation is **less than** $u + vH(\lambda,\mu) + E(\lambda,\mu)$.

Similar to the derivation from (2.4a-2.4b) to (2.5a-2.5b), we can establish from (4.2a-4.2b) the following simplified independent recurrence relations

$$x_{1+n} \;=\; 2u\, x_n - x_{n-1} + \frac{4-\lambda}{2-\lambda}(1-u) \qquad (4.4a)$$
$$y_{1+n} \;=\; 2u\, y_n - y_{n-1} + \frac{4-\mu}{2-\mu}(1-u). \qquad (4.4b)$$

4.5. **Generating functions.** For an equivalent class of solutions satisfying the crossing recurrence relations (4.2a-4.2b) with initial solutions (x_0,y_0), let us denote the generating functions (cf. [9, §1.3-1.4]) by

$$f(x) \;:=\; \sum_{n=0}^{\infty} x^n x_n \quad \text{and} \quad g(y) \;:=\; \sum_{n=0}^{\infty} y^n y_n.$$

Multiplying (4.2a-4.2b) by z^n and then performing the summation with respect to n with $1 \le n < \infty$, we get the following simplified functional equations

$$\begin{aligned}(1-uz)f(z)-H(\lambda,\mu)vzg(z) &= \frac{x_0+\big\{E(\lambda,\mu)-x_0\big\}z}{1-z}\\ (1-uz)g(z)-H(\mu,\lambda)vzg(z) &= \frac{y_0+\big\{E(\mu,\lambda)-y_0\big\}z}{1-z}\end{aligned}$$

Resolving this system of equations, we establish two generating functions

$$f(x) = \frac{(1-ux)\big\{x_0-xx_0+xE(\lambda,\mu)\big\}+vxH(\lambda,\mu)\big\{y_0-xy_0+xE(\mu,\lambda)\big\}}{(1-x)(1-2ux+x^2)} \tag{4.5a}$$

$$g(y) = \frac{(1-uy)\big\{y_0-yy_0+yE(\mu,\lambda)\big\}+vyH(\mu,\lambda)\big\{x_0-yx_0+yE(\lambda,\mu)\big\}}{(1-y)(1-2uy+y^2)}. \tag{4.5b}$$

5. Computer Algebra and Examples

For two natural numbers (λ,μ) with $\lambda \ne \mu \ge 3$ and $(\lambda-2)(\mu-2)$ being a non perfect square number, the following procedure will be carried out in order to determine the recurrence relations, compute the generating functions and therefore to resolve the problem concerning the regular (λ,μ)-polygonal numbers.

A. Write down the Diophantine equation (1.4) on the regular (λ,μ)-polygonal numbers.
B. Figure out the generalized Pell equation $U^2 - D(\lambda,\mu)V^2 = N(\lambda,\mu)$ and the linear transformation $\Omega[\lambda,\mu]$.
C. Resolve the associated Pell equation $u^2-D(\lambda,\mu)v^2 = 1$ by PQ-algorithm (cf. [4, §8.1] and [8, §7.9].
D. Find out the minimum (u,v) such that the crossing recurrence relations (4.2a-4.2b) have integer coefficients.
E. Determine the number of classes of equivalent solutions and initial values for each class through Brute-force search ([4, §8.3] and [6, §5.3]) and LMM-algorithm [5].
F. Exhibit crossing recurrence relations (4.2a-4.2b) of the first degree and independent recursions of the second degree (4.4a-4.4b).
G. Display explicitly generating functions (4.5a-4.5b) for each equivalent class of solutions.

In order to realize this procedure, a **Mathematica** package has been developed based on the theoretical preparation of the previous sections. It can provide us the necessary information about the regular (λ,μ)-polygonal numbers. For the limit of space, here we present only a collection of the first twenty examples for $3 \le \lambda \ne \mu \le 9$ with $(\lambda-2)(\mu-2)$ being non perfect number, i.e. $(\lambda,\mu) \ne (3,6)$.

Example 1. *Triangular and square numbers:*

- *Diophantine equation:* $x(1+x) = 2y^2$.
- *Pell equation:* $u^2 - 2v^2 = 1$ *where* $u = 1 + 2x$ & $v = 2y$.
- *Recurrences with initial condition* $(1,1)$:

$$\left.\begin{array}{l} x_{1+m} = 3x_m + 4y_m + 1 \\ y_{1+m} = 3y_m + 2x_m + 1 \end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{l} x_{1+n} = 6x_n - x_{n-1} + 2 \\ y_{1+n} = 6y_n - y_{n-1} - 0. \end{array}\right.$$

- *Generating functions:*

$$f(x) = \frac{1+x}{(1-x)\big(1-6x+x^2\big)} \quad \textit{and} \quad g(y) = \frac{1}{1-6y+y^2}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	8	6	36
2	49	35	1225
3	288	204	41616
4	1681	1189	1413721
5	9800	6930	48024900

Example 2. *Triangular and pentagonal numbers:*

- *Diophantine equation:* $x(1+x) = y(3y-1)$.
- *Pell equation:* $u^2 - 3v^2 = 6$ *where* $u = 3 + 6x$ & $v = 6y - 1$.
- *Recurrences with initial condition* $(1,1)$:

$$\left.\begin{array}{l} x_{1+m} = 7x_m + 12y_m + 1 \\ y_{1+m} = 7y_m + 4x_m + 1 \end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{l} x_{1+n} = 14x_n - x_{n-1} + 6 \\ y_{1+n} = 14y_n - y_{n-1} - 2. \end{array}\right.$$

- *Generating functions:*

$$f(x) = \frac{1+5x}{(1-x)\big(1-14x+x^2\big)} \quad \textit{and} \quad g(y) = \frac{1-3y}{(1-y)\big(1-14y+y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	20	12	210
2	285	165	40755
3	3976	2296	7906276
4	55385	31977	1533776805
5	771420	445380	297544793910

Example 3. *Pentagonal and square numbers:*

- *Diophantine equation:* $x(3x-1) = 2y^2$.
- *Pell equation:* $u^2 - 6v^2 = 1$ *where* $u = 6x - 1$ & $v = 2y$.
- *Recurrences with initial condition* $(1,1)$:

$$\left.\begin{array}{l} x_{1+m} = 49x_m + 40y_m - 8 \\ y_{1+m} = 49y_m + 60x_m - 10 \end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{l} x_{1+n} = 98x_n - x_{n-1} - 16 \\ y_{1+n} = 98y_n - y_{n-1} - 0. \end{array}\right.$$

- *Generating functions:*

$$f(x)=\frac{1-18x+x^2}{(1-x)\big(1-98x+x^2\big)} \quad \textit{and} \quad g(y)=\frac{1-y^2}{(1-y)\big(1-98y+y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	81	99	9801
2	7921	9701	94109401
3	776161	950599	903638458801
4	76055841	93149001	8676736387298001
5	7452696241	9127651499	83314021887196947001

Example 4. *Hexagonal and square numbers:*

- *Diophantine equation:* $2x(2x-1)=2y^2$.
- *Pell equation:* $u^2-2v^2=1$ *where* $u=4x-1$ & $v=2y$.
- *Recurrences with initial condition* $(1,1)$*:*

$$\left.\begin{aligned} x_{1+m}&=17x_m+12y_m-4\\ y_{1+m}&=17y_m+24x_m-6\end{aligned}\right\} \quad \textit{and} \quad \left\{\begin{aligned} x_{1+n}&=34x_n-x_{n-1}-8\\ y_{1+n}&=34y_n-y_{n-1}-0.\end{aligned}\right.$$

- *Generating functions:*

$$f(x)=\frac{1-10x+x^2}{(1-x)\big(1-34x+x^2\big)} \quad \textit{and} \quad g(y)=\frac{1-y^2}{(1-y)\big(1-34y+y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	25	35	1225
2	841	1189	1413721
3	28561	40391	1631432881
4	970225	1372105	1882672131025
5	32959081	46611179	2172602007770041

Example 5. *Regular* $(6,5)$*-polygonal numbers:*

- *Diophantine equation:* $2x(2x-1)=y(3y-1)$.
- *Pell equation:* $u^2-3v^2=6$ *where* $u=3(4x-1)$ & $v=6y-1$.
- *Recurrences with initial condition* $(1,1)$*:*

$$\left.\begin{aligned} x_{1+m}&=97x_m+84y_m-38\\ y_{1+m}&=97y_m+112x_m-44\end{aligned}\right\} \quad \textit{and} \quad \left\{\begin{aligned} x_{1+n}&=194x_n-x_{n-1}-48\\ y_{1+n}&=194y_n-y_{n-1}-32.\end{aligned}\right.$$

- *Generating functions:*

$$f(x)=\frac{1-52x+3x^2}{(1-x)\big(1-194x+x^2\big)} \quad \textit{and} \quad g(y)=\frac{1-30y-3y^2}{(1-y)\big(1-194y+y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	143	165	40755
2	27693	31977	1533776805
3	5372251	6203341	57722156241751
4	1042188953	1203416145	2172315626468283465
5	202179284583	233456528757	81752926228785223683195

Example 6. *Regular* $(7,3)$*-polygonal numbers:*

- *Diophantine equation:* $x(5x-3)=y(1+y)$.
- *Pell equation:* $u^2-5v^2=4$ *where* $u=10x-3$ & $v=1+2y$.
- *Recurrences with initial conditions* $(1,1)$ *and* $(5,10)$*:*

$$\left.\begin{aligned} x_{1+m} &= 161x_m+72y_m-12 \\ y_{1+m} &= 161y_m+360x_m-28 \end{aligned}\right\} \quad \textit{and} \quad \left\{\begin{aligned} x_{1+n} &= 322x_n-x_{n-1}-96 \\ y_{1+n} &= 322y_n-y_{n-1}+160. \end{aligned}\right.$$

- *Generating functions:*

$$f(x)=\frac{3}{10(1-x)}+\frac{1-x}{2(1-18x+x^2)}+\frac{1-x}{5(1+18x+x^2)}$$
$$g(y)=\frac{-1}{2(1-y)}+\frac{1+y}{1-18y+y^2}+\frac{1+y}{2(1+18y+y^2)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	5	10	55
2	221	493	121771
3	1513	3382	5720653
4	71065	158905	12625478965
5	487085	1089154	593128762435

Example 7. *Regular* $(7,4)$*-polygonal numbers:*

- *Diophantine equation:* $x(5x-3)=2y^2$.
- *Pell equation:* $u^2-10v^2=9$ *where* $u=10x-3$ & $v=2y$.
- *Recurrences with initial conditions* $(1,1)$, $(6,9)$ *and* $(49,77)$*:*

$$\left.\begin{aligned} x_{1+m} &= 721x_m+456y_m-216 \\ y_{1+m} &= 721y_m+1140x_m-342 \end{aligned}\right\} \quad \textit{and} \quad \left\{\begin{aligned} x_{1+n} &= 1442x_n-x_{n-1}-432 \\ y_{1+n} &= 1442y_n-y_{n-1}+0. \end{aligned}\right.$$

- *Generating functions:*

$$f(x)=\frac{3}{10(1-x)}+\frac{(1-x)\big(7+64x+551x^2+64x^3+7x^4\big)}{10(1-1442x^3+x^6)}$$
$$g(y)=\frac{(1+y)\big(1+8y+69y^2+8y^3+y^4\big)}{1-1442y^3+y^6}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	6	9	81
2	49	77	5929
3	961	1519	2307361
4	8214	12987	168662169
5	70225	111035	12328771225

Example 8. *Regular* $(7,5)$*-polygonal numbers:*

- *Diophantine equation:* $x(5x-3) = y(3y-1)$.
- *Pell equation:* $u^2 - 15v^2 = 66$ *where* $u = 3(10x-3)$ & $v = 6y-1$.
- *Recurrences with initial condition* $(1,1)$*:*

$$\left.\begin{aligned} x_{1+m} &= 31x_m + 24y_m - 13 \\ y_{1+m} &= 31y_m + 40x_m - 17 \end{aligned}\right\} \quad \textit{and} \quad \left\{\begin{aligned} x_{1+n} &= 62x_n - x_{n-1} - 18 \\ y_{1+n} &= 62y_n - y_{n-1} - 10. \end{aligned}\right.$$

- *Generating functions:*

$$f(x) = \frac{1 - 21x + 2x^2}{(1-x)\big(1 - 62x + x^2\big)} \quad \textit{and} \quad g(y) = \frac{1 - 9y - 2y^2}{(1-y)\big(1 - 62y + y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	42	54	4347
2	2585	3337	16701685
3	160210	206830	64167869935
4	9930417	12820113	246532939589097
5	615525626	794640166	947179489733441251

Example 9. *Regular* $(7,6)$*-polygonal numbers:*

- *Diophantine equation:* $x(5x-3) = 2y(2y-1)$.
- *Pell equation:* $u^2 - 5v^2 = 4$ *where* $u = 10x-3$ & $v = 4y-1$.
- *Recurrences with initial condition* $(1,1)$*:*

$$\left.\begin{aligned} x_{1+m} &= 161x_m + 144y_m - 84 \\ y_{1+m} &= 161y_m + 180x_m - 94 \end{aligned}\right\} \quad \textit{and} \quad \left\{\begin{aligned} x_{1+n} &= 322x_n - x_{n-1} - 96 \\ y_{1+n} &= 322y_n - y_{n-1} - 80. \end{aligned}\right.$$

- *Generating functions:*

$$f(x) = \frac{1 - 102x + 5x^2}{(1-x)\big(1 - 322x + x^2\big)} \quad \textit{and} \quad g(y) = \frac{1 - 76y - 5y^2}{(1-y)\big(1 - 322y + y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	221	247	121771
2	71065	79453	12625478965
3	22882613	25583539	1309034909945503
4	7368130225	8237820025	135723357520344181255
5	2372515049741	2652552464431	14072069153115290487843091

Example 10. *Regular* $(8,3)$*-polygonal numbers:*

- *Diophantine equation:* $2x(3x-2) = y(1+y)$.
- *Pell equation:* $u^2 - 6v^2 = 10$ *where* $u = 4(3x-1)$ & $v = 1+2y$.
- *Recurrences with initial conditions* $(1,1)$ *and* $(3,6)$*:*

$$\left.\begin{aligned} x_{1+m} &= 49x_m + 20y_m - 6 \\ y_{1+m} &= 49y_m + 120x_m - 16 \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} x_{1+n} &= 98x_n - x_{n-1} - 32 \\ y_{1+n} &= 98y_n - y_{n-1} + 48. \end{aligned}\right.$$

- *Generating functions:*

$$f(x) = \frac{1}{3(1-x)} + \frac{1-x}{2(1-10x+x^2)} + \frac{1-x}{6(1+10x+x^2)}$$
$$g(y) = \frac{-1}{2(1-y)} + \frac{1+y}{1-10y+y^2} + \frac{1+y}{2(1+10y+y^2)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	3	6	21
2	63	153	11781
3	261	638	203841
4	6141	15041	113123361
5	25543	62566	1957283461

Example 11. *Regular* $(8,4)$*-polygonal numbers:*

- *Diophantine equation:* $2x(3x-2) = 2y^2$.
- *Pell equation:* $u^2 - 3v^2 = 1$ *where* $u = 3x-1$ & $v = y$.
- *Recurrences with initial condition* $(1,1)$*:*

$$\left.\begin{aligned} x_{1+m} &= 7x_m + 4y_m - 2 \\ y_{1+m} &= 7y_m + 12x_m - 4 \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} x_{1+n} &= 14x_n - x_{n-1} - 4 \\ y_{1+n} &= 14y_n - y_{n-1} - 0. \end{aligned}\right.$$

- *Generating functions:*

$$f(x) = \frac{1-6x+x^2}{(1-x)\big(1-14x+x^2\big)} \quad \text{and} \quad g(y) = \frac{1-y^2}{(1-y)\big(1-14y+y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	9	15	225
2	121	209	43681
3	1681	2911	8473921
4	23409	40545	1643897025
5	326041	564719	318907548961

Example 12. *Regular* $(8,5)$*-polygonal numbers:*

- *Diophantine equation:* $2x(3x-2)=y(3y-1)$.
- *Pell equation:* $u^2-2v^2=14$ *where* $u=4(3x-1)$ & $v=6y-1$.
- *Recurrences with initial conditions* $(1,1)$ *and* $(8,11)$*:*

$$\left.\begin{aligned} x_{1+m}&=577x_m+408y_m-260\\ y_{1+m}&=577y_m+816x_m-368\end{aligned}\right\}\ \textit{and}\ \left\{\begin{aligned} x_{1+n}&=1154x_n-x_{n-1}-384\\ y_{1+n}&=1154y_n-y_{n-1}-192.\end{aligned}\right.$$

- *Generating functions:*

$$f(x)=\frac{1}{3(1-x)}+\frac{1-5x}{2(1-34x+x^2)}+\frac{1-7x}{6(1+34x+x^2)}$$
$$g(y)=\frac{1}{6(1-y)}+\frac{1+7y}{2(1-34y+y^2)}+\frac{1+5y}{3(1+34y+y^2)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	8	11	176
2	725	1025	1575425
3	8844	12507	234631320
4	836265	1182657	2098015778145
5	10205584	14432875	312461813932000

Example 13. *Regular* $(8,6)$*-polygonal numbers:*

- *Diophantine equation:* $2x(3x-2)=2y(2y-1)$.
- *Pell equation:* $u^2-6v^2=10$ *where* $u=4(3x-1)$ & $v=4y-1$.
- *Recurrences with initial condition* $(1,1)$*:*

$$\left.\begin{aligned} x_{1+m}&=49x_m+40y_m-26\\ y_{1+m}&=49y_m+60x_m-32\end{aligned}\right\}\quad \textit{and}\quad \left\{\begin{aligned} x_{1+n}&=98x_n-x_{n-1}-32\\ y_{1+n}&=98y_n-y_{n-1}-24.\end{aligned}\right.$$

- *Generating functions:*

$$f(x)=\frac{1-36x+3x^2}{(1-x)\big(1-98x+x^2\big)}\quad \textit{and}\quad g(y)=\frac{1-22y-3y^2}{(1-y)\big(1-98y+y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	63	77	11781
2	6141	7521	113123361
3	601723	736957	1086210502741
4	58962681	72214241	10429793134197921
5	5777740983	7076258637	100146872588357936901

Example 14. *Regular* $(8,7)$*-polygonal numbers:*

- *Diophantine equation:* $2x(3x-2) = y(5y-3)$.
- *Pell equation:* $u^2 - 30v^2 = 130$ *where* $u = 20(3x-1)$ & $v = 10y-3$.
- *Recurrences with initial condition* $(1,1)$*:*

$$\left.\begin{aligned} x_{1+m} &= 241x_m + 220y_m - 146 \\ y_{1+m} &= 241y_m + 264x_m - 160 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} x_{1+n} &= 482x_n - x_{n-1} - 160 \\ y_{1+n} &= 482y_n - y_{n-1} - 144. \end{aligned}\right.$$

- *Generating functions:*

$$f(x) = \frac{1 - 168x + 7x^2}{(1-x)\big(1-482x+x^2\big)} \quad \text{and} \quad g(y) = \frac{1-138y-7y^2}{(1-y)\big(1-482y+y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	315	345	297045
2	151669	166145	69010153345
3	73103983	80081401	16032576845184901
4	35235967977	38599068993	3724720317758036481633
5	16983663460771	18604671173081	865334473646149974640821781

Example 15. *Regular* $(9,3)$*-polygonal numbers:*

- *Diophantine equation:* $x(7x-5) = y(1+y)$.
- *Pell equation:* $u^2 - 7v^2 = 18$ *where* $u = 14x-5$ & $v = 1+2y$.
- *Recurrences with initial condition* $(1,1)$*:*

$$\left.\begin{aligned} x_{1+m} &= 8x_m + 3y_m - 1 \\ y_{1+m} &= 8y_m + 21x_m - 4 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} x_{1+n} &= 16x_n - x_{n-1} - 5 \\ y_{1+n} &= 16y_n - y_{n-1} + 7. \end{aligned}\right.$$

- *Generating functions:*

$$f(x) = \frac{1-7x+x^2}{(1-x)\big(1-16x+x^2\big)} \quad \text{and} \quad g(y) = \frac{1+8y-2y^2}{(1-y)\big(1-16y+y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	10	25	325
2	154	406	82621
3	2449	6478	20985481
4	39025	103249	5330229625
5	621946	1645513	1353857339341

Example 16. *Regular* $(9, 4)$*-polygonal numbers:*

- *Diophantine equation:* $x(7x-5) = 2y^2$.
- *Pell equation:* $u^2 - 14v^2 = 25$ *where* $u = 14x - 5$ & $v = 2y$.
- *Recurrences with initial conditions* $(1, 1)$ *and* $(2, 3)$*:*

$$\left.\begin{array}{l} x_{1+m} = 15x_m + 8y_m - 5 \\ y_{1+m} = 15y_m + 28x_m - 10 \end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{l} x_{1+n} = 30x_n - x_{n-1} - 10 \\ y_{1+n} = 30y_n - y_{n-1} + 0. \end{array}\right.$$

- *Generating functions:*

$$f(x) = \frac{5}{14(1-x)} + \frac{(1-x)\big(9 + 32x + 9x^2\big)}{14(1 - 30x^2 + x^4)}$$

$$g(y) = \frac{(1+y)^3}{1 - 30y^2 + y^4}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	2	3	9
2	18	33	1089
3	49	91	8281
4	529	989	978121
5	1458	2727	7436529

Example 17. *Regular* $(9, 5)$*-polygonal numbers:*

- *Diophantine equation:* $x(7x-5) = y(3y-1)$.
- *Pell equation:* $u^2 - 21v^2 = 204$ *where* $u = 3(14x-5)$ & $v = 6y - 1$.
- *Recurrences with initial conditions* $(1, 1)$ *and* $(14, 21)$*:*

$$\left.\begin{array}{l} x_{1+m} = 6049x_m + 3960y_m - 2820 \\ y_{1+m} = 6049y_m + 9240x_m - 4308 \end{array}\right\} \textit{and} \left\{\begin{array}{l} x_{1+n} = 12098x_n - x_{n-1} - 4320 \\ y_{1+n} = 12098y_n - y_{n-1} - 2016. \end{array}\right.$$

- *Generating functions:*

$$f(x) = \frac{5}{14(1-x)} + \frac{1 - 31x}{2(1 - 110x + x^2)} + \frac{1 - 71x}{7(1 + 110x + x^2)}$$

$$g(y) = \frac{1}{6(1-y)} + \frac{1 + 71y}{3(1 - 110y + y^2)} + \frac{1 + 31y}{2(1 + 110y + y^2)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	14	21	651
2	7189	10981	180868051
3	165026	252081	95317119801
4	86968201	132846121	26472137730696901
5	1996480214	3049673901	13950766352135999751

Example 18. *Regular* $(9,6)$*-polygonal numbers:*

- *Diophantine equation:* $x(7x-5) = 2y(2y-1)$.
- *Pell equation:* $u^2 - 7v^2 = 18$ *where* $u = 14x - 5$ & $v = 4y - 1$.
- *Recurrences with initial conditions* $(1,1)$ *and* $(10,13)$*:*

$$\left.\begin{aligned} x_{1+m} &= 32257x_m + 24384y_m - 17616 \\ y_{1+m} &= 32257y_m + 42672x_m - 23304 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} x_{1+n} &= 64514x_n - x_{n-1} - 23040 \\ y_{1+n} &= 64514y_n - y_{n-1} - 16128. \end{aligned}\right.$$

- *Generating functions:*

$$f(x) = \frac{5}{14(1-x)} + \frac{9(1-127x)}{14(1-254x+x^2)} - \frac{72x}{1+254x+x^2}$$

$$g(y) = \frac{1}{4(1-y)} + \frac{108y}{1-254y+y^2} + \frac{3(1+127y)}{4(1+254y+y^2)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	10	13	325
2	39025	51625	5330229625
3	621946	822757	1353857339341
4	2517635809	3330519121	22184715227362706161
5	40124201194	53079328957	5634830324997758086741

Example 19. *Regular* $(9,7)$*-polygonal numbers:*

- *Diophantine equation:* $x(7x-5) = y(5y-3)$.
- *Pell equation:* $u^2 - 35v^2 = 310$ *where* $u = 5(14x-5)$ & $v = 10y - 3$.
- *Recurrences with initial condition* $(1,1)$*:*

$$\left.\begin{aligned} x_{1+m} &= 71x_m + 60y_m - 43 \\ y_{1+m} &= 71y_m + 84x_m - 51 \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} x_{1+n} &= 142x_n - x_{n-1} - 50 \\ y_{1+n} &= 142y_n - y_{n-1} - 42. \end{aligned}\right.$$

- *Generating functions:*

$$f(x) = \frac{1-55x+4x^2}{(1-x)\big(1-142x+x^2\big)} \quad \text{and} \quad g(y) = \frac{1-39y-4y^2}{(1-y)\big(1-142y+y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	88	104	26884
2	12445	14725	542041975
3	1767052	2090804	10928650279834
4	250908889	296879401	220343446399977901
5	35627295136	42154784096	4442564555387704166896

Example 20. *Regular* $(9, 8)$*-polygonal numbers:*

- *Diophantine equation:* $x(7x - 5) = 2y(3y - 2)$.
- *Pell equation:* $u^2 - 42v^2 = 57$ *where* $u = 3(14x - 5)$ & $v = 2(3y - 1)$.
- *Recurrences with initial condition* $(1, 1)$*:*

$$\left.\begin{aligned} x_{1+m} &= 337x_m + 312y_m - 224 \\ y_{1+m} &= 337y_m + 364x_m - 242 \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} x_{1+n} &= 674x_n - x_{n-1} - 240 \\ y_{1+n} &= 674y_n - y_{n-1} - 224. \end{aligned}\right.$$

- *Generating functions:*

$$f(x) = \frac{1 - 250x + 9x^2}{(1-x)\big(1 - 674x + x^2\big)} \quad \text{and} \quad g(y) = \frac{1 - 216y - 9y^2}{(1-y)\big(1 - 674y + y^2\big)}.$$

- *The first five numbers:*

n	x_n	y_n	z_n
1	425	459	631125
2	286209	309141	286703855361
3	192904201	208360351	130242107189808901
4	130017145025	140434567209	59165603001256545014625
5	87631362842409	94652689938291	26877395137662573622784125461

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