On concepts of PDE-Software: The cellwise oriented approach in DEAL.

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Abstract

In this note, we review and refine our concepts of a cellwise oriented approach to PDE-software described in the dissertations of G. Kanschat and F.T. Suttmeier. The problem of non-matching cells in finite element discretisations on locally refined meshes is addressed. General concepts of the treatment of so-called hanging nodes are illustrated at the standard bilinear and -quadratic FE-spaces. These results carry over to nonconforming elements.

Furthermore, two features we focus on are on one hand the significant reduction of memory requirements employing local scaling techniques for representing operators. The second aspect is data locality, which is the basis for vectorisation/parallelisation and for employing efficient cache-optimised hardware-techniques.

We illustrate our ideas and provide numerical results by treating a standard benchmark problem (*disc with a hole*) from structural mechanics.

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1 Introduction

DEAL (Differential Equations Analysis Library) is available via [2]. This C++ class library for finite elements was designed by Guido Kanschat and Franz-Theo Suttmeier in 1992. The implementation was continuously improved between 1993 and 1998 by G. Kanschat, F.T.Suttmeier and R. Becker.

With DEAL as the heart in research on adaptive FEM at the University of Heidelberg, pioneering work on the so called Dual Weighted Residual (DWR)

approach was done by R. Rannacher and his co-workers in R. Becker [1], G. Kanschat [4] and F.T.Suttmeier [8].

The following text is partly taken from [4]: Scientific computing software to solve “real life” problems tends to become more and more complex. The necessary validation of program code becomes a hard task by this development. A proof of correctness is possible only for simple data structures and small programs. The classic approach to more reliable software used in numerical computations is *modular programming*. Applying this paradigm, a complex algorithm is decomposed into small simple parts which can be tested independently. But, as mentioned above, not only methods are complex, but data structures too. *Object oriented programming* now allows the modularization of algorithms and data by combining both of these aspects into the same structure. This leads to the notion of *classes* and *objects*.

The tight coupling of data and methods operating on them in object oriented programming, forces design to become an important step in software development. In previous numerical codes, this has been regarded only with the aim of reducing memory usage and computation time.

In a complex piece of software, design of classes has to optimize a combination of four development aims:

1. computing speed,
2. memory requirements,
3. verifiability of code and
4. flexibility.

While the first two points have been investigated very thoroughly in the last 30 years — see e. g. BLAS routines — points three and four are real white patches on the map. Additionally to the known trade-off between speed and memory consumption points three and four introduce a much more complicated balance. From the economic point of view we have to supplement run-time efficiency by development efficiency. In particular, software designed to develop new algorithms must obey this point, since implementation time usually exceeds run-time by orders of magnitude.

The main idea to improve flexibility and correctness of a program lies in the restriction of data access. In usual FORTRAN or C code a great deal of data is handled by common blocks and global variables, respectively. These allow unrestricted read and write access to their members, so the programmer can not be sure where these structures are changed or corrupted. This verification problem is augmented by the fixed structure of implementation: a change in the data structure necessitates a change in all functions using it.

Considering application to partial differential equations, there are several parts of the algorithms which may be separated to a high degree. There is a base level of classes describing the adaptive grid generation and handling of multi-grid structures. This part of software is invariant for a huge class of finite element problems. A second level on top of the first defines basic numerical operations like application of an operator to a discrete function. Here, the dependence on the structure of the physical problem is very strong. A third part of code provides standard numerical solvers as cg and Bi-cgstab for linear systems and Newton's method for nonlinear problems. This level should be implemented in an abstract way to allow usage for different applications.

```
class Triangulation
{
    void      read(File);
    void      refine(int levels);
    void      adaptive_refine(double tolerance);
    Cell*     first_cell();
    Cell*     next_cell();
    Vertex*   first_vertex();
    Vertex*   next_vertex();
};
```

Figure 1: The Triangulation class

2 Grid handling

The following text is partly taken from [4]: Let us first regard the abstract concept of a triangulation of a domain Ω as a hierarchy of grid cells with a certain topology. Its basic functionality is shown in Figure 1, where the triangulation is reduced to a set of cells and vertices with some additional mesh generation functions. This mesh should be able to consist of triangles and quadrangles as well as tetrahedra, prisms and hexahedra in two and three dimensions respectively. This means that we have to extract basic information from all these geometric objects to identify them as a *cell*. The cell information necessary for adaptive refinement was first collected in [7] in a very abstract way. This led to the definition of a cell essentially following Figure 2.

The function *refine* of *Triangulation* works by traversing all cells and forcing each cell to refine itself. Since a correct triangulation is characterized by consistent values of vertices, neighbors and father/child information, *refine* and *coarse* function of *Cell* ensure conservation of these data. Certainly, a cell can hardly supply any of these functions, since topology information is

specified only for more concrete objects. Therefore, they are declared as an abstract interface and are implemented in derived classes for e. g. triangles and quadrilateral cells, where we know the number of neighbors is three or four respectively.

```
class Cell
{
    int      number_of_vertices();
    int      number_of_neighbors();

    Vertex*  vertex(int nr);
    Cell*    neighbor(int nr);
    Cell*    father();
    Cell*    child(int nr);

    int      level();
    int      index();

    void      refine();
    void      coarse();
    double    refinement_criterion();
};
```

Figure 2: The Cell class (topology and refinement)

The functional interface to these values enables high flexibility regarding implementation. Considering e. g. the children there are two possibilities of obtaining the corresponding pointers. They may be stored in a cell producing memory consumption, but making fast access possible. Alternatively, they can be reconstructed from the list of cells stored in the triangulation. According to the algorithms used as well as preference for memory or speed optimization, these techniques should be chosen appropriately. The interface of Figure 2 allows the necessary internal change of representation on a purely local basis, since implementation details are hidden for other functions. The correctness of working code using this function *child* is not influenced by internal changes, if only the result of *child* is correct.

We ensure for each function, that all objects involved in its operation — not only explicitly modified objects — are in an admissible state after the function returns control to the calling context.

Take e. g. cell refinement as shown in Figure 3. We start with a valid locally refined triangulation (a). The cells are linked by the mutual neighborhood relation $\circlearrowleft$ and the “is child of” relation $\Downarrow$. We display two consecutive layers of refinement on top of each other. After splitting the middle cell (*) into four children, the topology is corrupted (b). Setting topology information

for the children (c), the cell we operated on is valid, but the operation has destroyed the triangulation structure: there is only an unidirectional neighborhood relation $\curvearrowright$ from the children of $(*)$ to their neighbors. Finally, the topology information of neighboring cells is updated too and the topology is valid again (d). Compared to other strategies, this process not only avoids expensive postprocessing, but enables us to stop refinement after processing an arbitrary cell (which is actually done in adaptive algorithms). While the states (b) and (c) are “virtual” grids never to be seen in an application, (a) and (d) represent states, where further operations may be inserted.

Figure 3: Refinement of a cell

3 Hanging nodes

The problem of non-matching cells in finite element discretisations on locally refined meshes is addressed. General concepts of the treatment of so-called hanging nodes are illustrated at the standard bilinear and -quadratic FE-spaces. Furthermore these results carry over to nonconforming elements.

The Laplace problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1)$$

on the domain $\Omega \subset \mathbb{R}^2$, is taken as a model problem to present our ideas.

The variational formulation

$$(\nabla u, \nabla \varphi) = (f, \varphi) \quad \forall \varphi \in V := H_0^1(\Omega), \quad (2)$$

serves as a basis for applying FE methods to (1). Here and below, $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$ and $\|\cdot\|$ the corresponding norm, while $(\nabla \cdot, \nabla \cdot)$ is the natural energy inner product of problem (2). Further, $H^m(\Omega)$, for $m \in \mathbb{N}$, denotes the usual m -th order Sobolev space with norm $\|\cdot\|_m$, and $H_0^1(\Omega)$ the subset of $H^1(\Omega)$ of functions vanishing on $\partial\Omega$. These are all spaces of either scalar or vector-valued functions and no distinction will be made in the notation of the corresponding inner products and norms. Now, we apply the finite element method on decompositions $\mathbb{T}_h = \{T\}$ of Ω consisting of quadrilaterals T , satisfying the usual condition of shape regularity. The width of the mesh $\mathbb{T}_h$ is characterised in terms of a piecewise constant mesh size function $h = h(x) > 0$, where $h_T := h|_T = \text{diam}(T)$. Using the notation $V_h \approx V$ for the corresponding finite element spaces approximating V , the discrete solution $u_h \in V_h$ is determined by the equation

$$(\nabla u_h, \nabla \varphi) = (f, \varphi) \quad \forall \varphi \in V_h. \quad (3)$$

Figure 4: Domain and sample mesh with hanging nodes

Figure 5: Sketch of the appearance of a hanging node $1/2$ depending on the nodes 0 and 1 for bilinear finite elements.

For ease of mesh refinement and coarsening hanging nodes are allowed in our implementation, as shown in Figure 4. The treatment of non-matching cells in the case of bilinear shape functions is sketched in Figure 5.

Let $\langle \varphi_i \rangle, i = 1..N$ denote the standard nodal basis for V_h with dimension N . We define the vectors of nodal values $U, F \in \mathbb{R}^N$ with

$$U_i = (\nabla u_h, \nabla \varphi_i), \quad F_i = (f, \varphi_i),$$

and the corresponding matrix $A \in \mathbb{R}^{N \times N}$ with $A_{ij} = (\nabla \varphi_j, \nabla \varphi_i)$.

Since continuity of the solution is required, u_h has to be linear along $\overline{01}$, i.e. assuming u_h determined by U_0 and U_1 in nodes 0 and 1, the value for u_h in node 1/2 has to be subjected to the constraint $U_{1/2} = 0.5(U_0 + U_1)$.

Then introducing the averaging operator M the constraints for ensuring continuity of u_h can be written in the form $U = MU$. The operator M acts as on identity on regular points 0 and 1 and performs an averaging on hanging nodes 1/2 giving the structure

$$M = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & \\ & & \ddots & & \\ & & & & 1 \end{pmatrix}.$$

Consequently one has to solve $AU = F$ with the additional constraint $U = MU$. Remarks on possibilities of incorporating the constraint into FE-implementations are discussed separately in a section below.

3.1 Elements of higher order and nonconforming discretisations

The techniques described above for the case of bilinear shape functions can be extended to other situations. We discuss here the biquadratic and the nonconforming Crouzeix-Raviart element.

3.1.1 Biquadratic

For the biquadratic shape functions, let the vector on the edge with the hanging nodes be arranged in the form

$$U^T = (U_0 \quad U_{1/4} \quad U_{1/2} \quad U_{3/4} \quad U_1)$$

(see Figure 6, right). In order to preserve the continuity, u_h has to be quadratic along $\overline{01}$, u_h determined by U_0 , $U_{1/2}$ and U_1 . Hence

$$\begin{aligned} U_{1/4} &= \frac{3}{8}U_0 + \frac{3}{4}U_{1/2} - \frac{1}{8}U_1 \\ U_{3/4} &= -\frac{1}{8}U_0 + \frac{3}{4}U_{1/2} + \frac{3}{8}U_1 \end{aligned}$$

and the average operator is given by

$$M = \begin{pmatrix} 1 & & & & \\ 3/8 & 0 & 3/4 & 0 & -1/8 \\ & & 1 & & \\ -1/8 & 0 & 3/4 & 0 & 3/8 \\ & & & & 1 \end{pmatrix}.$$

Figure 6: Hanging nodes for rotated bilinear (left) and biquadratic (right) finite elements. Bars or points indicate degrees of freedom.

3.1.2 Rotated bilinear

Eventually, we discuss the extension of the Crouzeix-Raviart element on triangles to quadrilaterals, i.e., the rotated bilinear element $\tilde{Q}_1$ analyzed by Rannacher and Turek in [6]. Analogously to the triangular elements, we take the mean value along each edge as node values. The required restrictions to the degrees of freedom in situations sketched in Figure 6 are imposed as follows. Remarking, that the node values of the rotated bilinear element are related to the element edges and are in fact the mean value along these lines, conformity with respect to the chosen discrete space is ensured by

$$\begin{aligned} U_{01} &= U_0 + \frac{1}{4}U_1 - \frac{1}{4}U_2 \\ U_{02} &= U_0 - \frac{1}{4}U_1 + \frac{1}{4}U_2. \end{aligned}$$

4 Filtering techniques

In this section we present a possibility how the proposed treatment of hanging nodes can be incorporated into a FE-implementation. For illustration we choose a pseudo C++ description.

Representation of an operator

For simplicity, we assume that discretisation by the finite element method leads to a symmetric and positive definite operator A . The cg-method is one of the most frequently used solvers to treat such a linear system $Ax = b$. In Figure 7 we sketch parts of the template implementation of the cg-method used in DEAL [2]. This solver requires only the abstract ability of the operator to do a matrix vector multiplication `vmult`.

```
template<class MT, class VT, class MEM>
cg_simple(MT& A, VT& x, VT& b, ...)
{
    double res, alpha, beta;
    VT&    d, g, Ad;
    initialisation();
    for (int it=0;!reached;it++)
    {
        A.vmult(Ad,d);           // operator property

        alpha = Ad*d;    alpha = beta/alpha;
        x.add(alpha,d);  g.add(alpha,Ad);
        res = g*g;
        d.sadd(res/beta,-1.,g);
        beta = res;
        check_residual();
    }
}
```

Figure 7: Pseudo code for template implementation of the cg-method

Filtering techniques and hanging nodes

In traditional approaches the treatment of Dirichlet data can be done by eliminating rows and columns of the matrix. We propose to use filtering techniques so that the representation of the operator is independent of the desired boundary values. We assume that the operator is implemented in form of a class `Matrix` with a default matrix-vector multiplication `vmult`.

The used cg-like methods are based on an update step for the solution x of the form $x_{k+1} = x_k + \alpha_k d_k$, where d_k is a certain correction vector. So after initialising the iteration vector with the correct Dirichlet values, we have to ensure that they are not changed during the iteration. This is simply done by modifying the matrix vector multiplication. With terms introduced above, all entries in the correction vector d_k , representing Dirichlet points, are set to

zero after the `vmult` of the operator has taken place.

```
void FilterMatrix::vmult(Vector dst, Vector src)
{
    A.vmult(dst,src);          // default vmult
    set_zero_boundary(dst);
}
```

Carrying over this idea to the treatment of hanging nodes, we rewrite the discrete Poisson equation as a minimum problem

$$\begin{aligned} U^T A U - 2U^T F &= \min, \\ U &= M U \end{aligned}$$

or in compact form

$$U^T (M^T A M) U - 2U^T M^T F = \min.$$

We see that the appropriate modification of the matrix vector multiplication consists simply of the two processes M and M^T referred to as **average** and **distribute** respectively. So in our pseudo programming language the modified `vmult`-version looks like

```
void FilterMatrix::vmult(Vector dst, Vector src)
{
    average(src);
    A.vmult(dst,src);          // default vmult
    distribute(dst);
    set_zero_boundary(dst);
}
```

4.1 Numerical tests

The technique presented above has been proved to be succesful for various applications, see e.g. [1], [4], [8]. As a further example, we show some results of a comparative study of the convergence behaviour of bilinear and biquadratic shape functions on locally refined meshes. In Figure 8 (left), we show the structure of the locally refined grids produced on the basis of an *a posteriori* energy estimate. In Figure 8 (right) the errors measured in the energy norm on globally and locally refined grids are depicted for both discretisations. We observe the expected behaviour indicating the averaging technique to treat the hanging nodes in the correct way.

Figure 8: Adaptively refined grid for the Poisson Problem (left) and comparison of convergence behaviour of bilinear and biquadratic shape functions (right).

5 Matrix implementation

The main work for a finite element scheme is performed locally on the cells. Afterwards, information is transferred to the triangulation, where all degrees of freedom (DOF) are treated globally. Those processes can be described by introducing the function I_T , identifying the local numbering of l DOF on a cell T to n DOF (numbered globally) on the triangulation.

$$I_T : \{1, \dots, l\} \rightarrow \{1, \dots, n\}.$$

Now, employing I_T , one can describe how local information in form of vectors and matrices becomes global. These operators are named p_T and P_T respectively and defined as follows.

$$\begin{aligned} p_T : \mathbb{R}^l &\rightarrow \mathbb{R}^n \\ u_n(I_T(i)) &= u_l(i), \quad i = 1, \dots, l \end{aligned}$$

and

$$\begin{aligned} P_T : \mathbb{R}^{l \times l} &\rightarrow \mathbb{R}^{n \times n} \\ A_n(I_T(i), I_T(j)) &= A_l(i, j), \quad i, j = 1, \dots, l. \end{aligned}$$

The traditional approach to represent the operator is the use of a classical matrix. All local information A_T on a cell T is saved in a global matrix by

$$A = \sum_T P_T A_T. \quad (4)$$

But, within an iterative processes the result Ax of a matrix-vector-multiplication is required only, i.e., based on

$$Ax = \sum_T p_T A_T p_T^{-1} x,$$

Ax may be interpreted as the sum of results of local matrix-vector-multiplications on cells T .

5.1 Evaluation

Here, we discuss, how the concept of employing local matrices performs.

More operations: First, one observes that according to the ratio

$$\frac{\text{entries per line in } A_T \times \text{cells per node}}{\text{entries per line in } A} \quad (5)$$

which is $16/9$ and $64/27$ for quads and hexs on tensorproduct meshes, the number of iterations per matrix-vector-multiplication increases.

Higher hardware performance: On the other hand, a higher hardware performance is achieved. This is illustrated below by evaluating MFLOP/s rates for a standard 3D configuration in linear elasticity, yielding an overall speed-up.

	sparse matrix		local matrix		
DOF	MV-time	MFLOP	MV-time	MFLOP	Speed-up
375	0.11	28.04	0.03	205.64	3.67
2187	0.83	26.43	0.22	199.43	3.77
14739	6.39	25.85	1.72	192.07	3.72
107811	93.11	13.76	17.70	144.79	5.26
823875	778.14	12.97	145.97	138.29	5.33

Table 1: MFLOP/s rates: local matrices versus global sparse matrix, SUN Fire

Memory reduction: Most parts of the hierarchical structured triangulation consist of triangles or parallelograms. Consequently, the domain transformations are nearly all linear. Computing the relevant element matrices on the lowest level, the matrices required on the working level are computed simply by scaling the available knowledge from the lowest level according to the following lemma.(see [4])

	sparse matrix		local matrix		
DOF	MV-time	MFLOP	MV-time	MFLOP	Speed-up
375	0.03	102.82	0.02	308.46	1.50
2187	0.18	121.88	0.10	438.75	1.80
14739	1.61	102.60	1.36	242.91	1.18
107811	13.05	98.19	8.16	314.07	1.60
823875	101.41	99.53	64.92	310.95	1.56

Table 2: MFLOP/s rates: local matrices versus global sparse matrix, PC, Pentium 4

Lemma 5.1. *Let T_0 be a parallelogram of dimension d and the matrix A_0 the discretisation of an arbitrary differential operator of order ω on T_0 .*

For a cell T obtained by ν -fold regular refinement of T_0 into similar cells, the according matrix is obtained by

$$A_T = (2^{\omega-d})^\nu A_0 \quad (6)$$

Multithreading/Parallelisation: Eventually, employing cell matrices, parallelisation of a matrix-vector-multiplication becomes straight forward indicated by the pseudo-code below.

```
vmult(Vector d, Vector s) {
    ...
    for i=1 to N/P do {
        for p=1 to P do {
            Cell c = cells(i*P+p);
            thread(p,c.vmult(d,s));
        }
    }
    ...
}
```

Summary: Collecting in Table 3 the considerations above, we recommend to use this concept of data locality.

Cell matrices	
-	more operations
+	higher hardware performance
+	drastical memory reduction
+	multithreading/parallelisation

Table 3: Evaluation of the performance of employing local matrices.

5.2 Implementation

In order to describe the behaviour of an operator within a iterative solution process in an adequate way, the abstract ability to perform a matrix vector multiplication `vmult` is required only. This is realised by the class `MVTriangulation` in Figure 9.

Classical matrix

The traditional approach to represent the operator is the use of a classical matrix. All information is saved in a global matrix. For example one can use a `SparseMatrix` where only non-zero entries are stored. So our classical approach leads to the class `MatrixTriangulation` in Figure 9.

The required matrix vector multiplication uses the already available properties of `SparseMatrix`. The method `A.vmult` is called indirectly by the `vmult` of the triangulation to have the possibility to incorporate for example filter routines described above. Essentially we have the structure (see Figure 9)

```
void MatrixTriangulation::vmult(Vector dst, Vector src)
{
    A.vmult(dst,src);
    ...
}
```

Cell matrices

As indicated above, the global operator `MVTriangulation::vmult` may alternatively be implemented as a loop over all cells, where a matrix-vector-multiplication is performed locally.

This is implemented by the class `LocalTriangulation` in Figure 9, requiring the cells to provide an abstract `vmult`, which is realised by the class `MVCell` in Figure 10.

Next, we want to describe a concept of storing the assemble information locally. Therefore we introduce the class `MatrixCell` which contains the required memory in form of `Matrix`. Furthermore this cell is provided with a function `vmult`, which does the local multiplication. (see Figure 10)

Remark: During an adaptive process only parts of the whole domain are refined. Computation time is saved because the local assembling is done only on new cells and the element matrices are stored locally.

Scaled matrices

Most parts of the hierarchical structured triangulation consist of triangles or parallelograms. Consequently, the domain transformations are nearly all linear. Computing the relevant element matrices on the lowest level, the matrices required on the working level are computed simply by scaling the available knowledge from the lowest level as described above. This is implemented by the class `ScaleCell` in Figure 10.

Remark: Even for nonlinear problems one can use the scaling technique. For example in a primal finite element formulation for plasticity the nonlinearity is given in terms of ∇u_h . Treating this problem with a stationary iteration and using P_1 -elements, the nonlinearity is constant over a cell. So the nonlinear term is simply a further scaling factor in addition to the domain scaling described above.

Summary

We can summarise, that already reliable and tested modules, for example the `SparseMatrix`, are used for the implementation. All work to implement the different operator classes is done locally and therefore global code like the the cg-method remains unchanged. So the point *verifiability of code* is fulfilled. If there are errors in the own program, we only have to examine the very few new code lines we did ourselves.

Due to the high degree of flexibility, there is the possibility to combine the points *computing speed* and *memory requirements* by employing BLAS-routines to accelerate the local matrix-vector-multiplications. On the other hand we use the scaling technique to solve problems which can only be treated reducing the memory requirements drastically.

Figure 9: Concept for MVTriangulation

5.3 Multigrid

Since the hierarchical structure of the triangulation is available, it suggests itself to use a multigrid algorithm in the context of solving algebraic equations resulting from the finite element method. The implementation was mainly done by Roland Becker and Guido Kanschat and this concept is explained in detail in [1]. Up to now the implementation requires the ability of cells to do `vmult` locally. So the used cells are based on the structures of `MVCell` described above. The user defined operator classes are derived from

```
class MGTriangulation : Triangulation
{
    hierarchical structure;
};
```

```
class Cell {
    ...
};
```

```
class MVCell {
    void vmult(Vector d, Vector s);
};
```

```
class MatrixCell {
    Matrix A;
};

void MatrixCell
::vmult(Vector d, Vector s) {

    s -> loc_s;
    A.vmult(loc_d,loc_s);
    d <- loc_d;
}
```

```
class ScaleCell {
    Matrix get_scaled_matrix();
};

void ScaleCell
::vmult(Vector d, Vector s) {
    loc_A = get_scaled_matrix();
    s -> loc_s;
    loc_A.vmult(loc_d,loc_s);
    d <- loc_d;
}
```

Figure 10: Concept of MVCell

Our experience is, that local multigrid is very successful if a fixed number of iterations (say one or two) is used as a preconditioner in the cg-method. Using the cell-wise approach sketched above one loses the possibility of matrix oriented smoothers like for example ILU-decomposition. But this is not so important since only spectral equivalence is needed between the operator A

and the preconditioner. So for our purpose it is convenient to choose Jacobi-like steps as a smoother. Alternatively, fixed numbers of cg-steps may be employed to construct a local smoother on each level of the triangulation. This concept only requires the operator property too. Furthermore, we want to remark that during an adaptive process in each step we have good initial guesses for the solution, coming from computations on grids of the previous adaptive step.

6 Elasto-plastic benchmark problem

In order to demonstrate our adaptive FE-concepts to work within a time-stepping scheme on locally varying meshes in space including hanging nodes, we show some results concerning this subject.

The fundamental problem in the flow theory of linear-elastic perfectly plastic material in classical notation reads (see, e.g., [3])

$$\begin{aligned} -\operatorname{div} \sigma &= f, \quad \varepsilon(\dot{u}) = A\dot{\sigma} + \lambda \quad \text{in } \Omega, \\ \lambda(\tau - \sigma) &\leq 0 \quad \forall \tau \text{ with } \mathcal{F}(\tau) \leq 0, \quad \lambda\dot{\sigma} = 0 \quad \text{in } \Omega, \\ \dot{u} &= 0 \quad \text{on } \Gamma_D, \quad \sigma \cdot n = g \quad \text{on } \Gamma_N, \end{aligned} \tag{7}$$

where σ and u are the stress tensor and displacement vector, respectively. We assume a stress-free initial state $\sigma(0) = 0$ and $u(0) = 0$. This idealized model describes the deformation of an elasto-plastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) under the action of a body force f and a surface traction g along Γ_N . Along the remaining part of the boundary, $\Gamma_D = \partial\Omega \setminus \Gamma_N$, the body is fixed. The plastic growth is denoted by λ , and $\mathcal{F}(\cdot)$ is the (convex) von Mises yield function. We assume a quasi-static process, i.e., acceleration effects are neglected. Accordingly, the loading is prescribed in terms of functions $f(x, t) = a(t)f_0(x)$ and $g(x, t) = b(t)g_0(x)$ depending on a time-like parameter $t \in I := [0, T]$, where the functions $a(t)$ and $b(t)$ vary only slowly (relative to the elasto-plastic changes in the model). Further, the deformation is supposed to be small, so that the strain tensor can be written as $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. The material tensor A is assumed to be symmetric and positive definite.

We apply our adaptive approach to a standard benchmark problem. A geometrically two-dimensional square disc with a hole is subjected to a constant boundary traction acting upon two opposite sides. No body force is applied, i.e., $f \equiv 0$. In the 2-dimensional case we use the plane-strain approximation and assume perfectly plastic material behavior. In virtue of symmetry the consideration can be restricted to a quarter of the domain as shown in Figure 11. The height and width of the quarter corresponding to lines $\overline{45}$ and $\overline{15}$ are 100, and the radius of the hole is 10. The ratio of the lines $\overline{57}$ and $\overline{67}$ is 2 : 1.

Figure 11: Geometry of the benchmark problem.

Figure 12: Plot of the reference solution (500.000 DOF) and the one obtained by our adaptive scheme (12.000 DOF).

The boundary traction is given in the form

$$g(t) = tg_0, \quad g_0 = 100, \quad t \in [0, 4.5]. \quad (8)$$

We consider the case of a linear-elastic isotropic material law expressed in the form

$$\sigma = C\varepsilon := 2\mu\varepsilon^D(u) + \kappa \operatorname{div} u I,$$

with material dependent constants $\mu > 0$ and $\kappa > 0$. Here, $C = A^{-1}$, and $\tau^D := \tau - \frac{1}{3}\text{tr}(\tau)I$ is the deviatoric part of a tensor τ . The von Mises flow function has the form

$$\mathcal{F}(\sigma) = |\sigma^D|^2 - \sigma_0^2 \leq 0.$$

In our test calculations, we have chosen the material parameters as commonly used for aluminium: $\kappa = 164,206 \text{ N/mm}^2$, $\mu = 80,193.80 \text{ N/mm}^2$, $\sigma_0 = \sqrt{2/3} 450$.

Figure 13: Sequence of the locally refined meshes including hanging nodes. Benchmark problem with zoom to one quarter of the configuration.

We evaluate values of the stresses in point P_7 under cycling loading. We compare our discretisation, which uses an *a posteriori* error estimate controlling the errors in time and space (c.f. [5]) to a *reference solution*, produced on the basis of a primal formulation approximating the displacements by the continuous, standard (bi)-linear shape functions. The pressure is treated by

an inner variable (see [9]). About 500.000 degrees of freedom (DOF) were employed to guarantee an error below one percent. In Figure 12, there is shown the plot of the reference solution and the one obtained by our adaptive scheme using only about 12.000 DOF to reach the prescribed accuracy.

Further, Figure 13 shows a sequence of zooms into adaptively refined meshes (including hanging nodes) for computing $\sigma_{11}(P_7)$ over the loading path $3.6 \leq t \leq 4.2$.

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