Factorizable Rpp Monoids

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Abstract

It is well known that the rpp semigroups are generalized regular semigroups. In order to further investigate the structure of rpp semigroups, we introduce the type-\((I, \mathcal{L}^*)\) factorizable monoids. In this paper, we study such factorizable rpp monoids admitting a left univocal type-\((I, \mathcal{L}^*)\) factorization. In particular, we prove that a semigroup \(S\) is a factorizable rpp monoid admitting a left univocal type-\((I, \mathcal{L}^*)\) factorization if and only if \(S\) is isomorphic to a semidirect product of a band and a left cancellative monoid. Some results of Catino and Tolo on factorizable semigroups are extended and amplified.

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1 Introduction

A semigroup \( S \) is called right principal projective, in short, rpp if for all \( a \in S \), \( aS^1 \) regarded as an \( S^1 \)-system is projective. We can also dually define the left principal projective semigroups (in short, lpp) semigroups. It was shown by Fountain [3] that a semigroup \( S \) is rpp if and only if every \( \mathcal{L}^* \)-class of \( S \) contains at least one idempotent. According to Fountain [3] and [4], a semigroup \( S \) is called abundant if every \( \mathcal{L}^* \)-class and every \( \mathcal{R}^* \)-class of \( S \) contains an idempotent. Equivalently, \( S \) is an abundant semigroup if and only if \( S \) is both an rpp and lpp semigroup. An abundant semigroup \( S \) is called quasi-adequate if \( E(S) \), the set of idempotents of \( S \), forms a subsemigroup of \( S \). Regular semigroups are abundant semigroups and orthodox semigroups quasi-adequate semigroups.

For an abundant semigroup \( S \), Lawson [9] defined a partial order “\( \leq \) ” on \( S \) by \( a \leq b \) if and only if for some \( e, f \in E(S) \), \( a = eb = bf \). Later on, Guo [5] called an abundant semigroup \( S \) F-abundant if for every \( a \in S \), the congruence class of the minimal cancellative congruence \( \sigma \) containing \( a \) has a greatest element \( m_a \) with respect to the partial order “\( \leq \) ”. It is clear that F-abundant semigroups are generalized F-regular semigroups (for F-regular semigroups, see [2], [10] etc.) Recently, Zhang-Chen [13] considered a special subclass of F-abundant semigroups, namely, the u-IC abundant semigroups. On the other hand, Ni-Chen-Guo [11] have provided a method of construction for the general F-abundant semigroups. Along this direction, Guo-Li-Shum [6] have further generalized F-abundant semigroups to F-rpp semigroups.

According to Catino[1], a semigroup \( S \) is called a factorizable semigroup if there exist some proper subsemigroups \( U \) and \( V \) of \( S \) such that \( S = UV \), where \( UV \) is the set \( \{uv \mid u \in U, v \in V \} \). In this case, we call the pair \( (U, V) \) a factorization pair of the factorizable semigroup \( S \). A factorization pair \( (U, V) \) of a factorizable semigroup \( S \) is called \( \text{left; right} \) univocal if

\[
(\forall u_1, u_2 \in U, v_1, v_2 \in V) \quad u_1 v_1 = u_2 v_2 \Rightarrow [u_1 = u_2; \; v_1 = v_2] u_1 = u_2 \text{ and } v_1 = v_2.
\]

We shall call a semigroup \( S \) a \( \text{left; right} \) univocal factorizable semigroup if no confusion arises. For researches in this topic, the readers are referred to Catino[1] and Tolo [12].

In this paper, we study a class of left univocal factorizable rpp semigroups, namely, the left univocal factorizable rpp monoids of type-(\( \mathcal{I}, \mathcal{L}^* \)). This class of left univocal factorizable rpp semigroups is an F-rpp semigroup. Moreover, the u-IC quasi-adequate semigroups are special cases of factorable rpp monoids of type \( (\mathcal{I}, \mathcal{L}^*) \). We shall prove that a monoid is a left univocal factorizable rpp monoid of type-(\( \mathcal{I}, \mathcal{L}^* \)) if and only if it is a semidirect product of a band and a left cancellative monoid.
2 Preliminaries

Throughout this paper, we adopt the terminologies and notions given in [1], [4] and [7]. Now we cite some known results in the literature which will be used in the sequel.

To begin with, we first state some facts concerning the Green \( \mathcal{L}^* \) and \( \mathcal{R}^* \) on the semigroup \( S \).

Lemma 2.1 [4] Let \( S \) be an semigroup and \( a, b \in S \). Then the following statements are equivalent:

1. \( a \mathcal{L}^* [\mathcal{R}^*] b \).
2. For all \( x, y \in S^1 \), \( ax = ay \Leftrightarrow bx = by \) \( \Rightarrow [xa = ya \Leftrightarrow xb = yb] \).

Lemma 2.2 [4] Let \( S \) be an semigroup and \( a, e^2 = e \in S \). Then the following statements are equivalent:

1. \( a \mathcal{L}^* [\mathcal{R}^*] e \).
2. \( a = ae [a = ea] \) and \( ax = ay \Rightarrow ex = ey \) \( [xa = ya \Rightarrow xe = ye] \) for all \( x, y \in S^1 \).

Evidently, the Green \( \ast \)-relation \( \mathcal{L}^* \) is a right congruence on \( S \) while the Green \( \ast \)-relation \( \mathcal{R}^* \) is a left congruence on \( S \). In general, \( \mathcal{L} \subseteq \mathcal{L}^* \) and \( \mathcal{R} \subseteq \mathcal{R}^* \). And, for the regular elements \( a \) and \( b \) of \( S \), \( a \mathcal{L}^* b [a \mathcal{R}^* b] \) if and only if \( a \mathcal{L} b [a \mathcal{R} b] \) (for more details, see [4] ). For the sake of brevity, we denote by \( a^\dagger [a^\ast] \) the typical idempotents related to \( a \) by \( \mathcal{L}^* [\mathcal{L}^*] \) and use \( E(S) \) to denote the set of all idempotents of the semigroup \( S \). If \( \mathcal{K} \) is one of the Green \( \ast \)-relations \( \mathcal{L}^*, \mathcal{R}^* \) and \( \mathcal{H}^* \) on \( S \), then we denote the \( \mathcal{K} \)-class of \( S \) containing \( a \) by \( K_a \).

Now suppose that \( S \) is an rpp semigroup. For \( x, y \in S \), we define

\[ x \leq_{\ell} y \text{ if and only if } L^*_x \leq L^*_y, \text{ that is, } L^*(x) \subseteq L^*(y), \text{ and there} \]

\[ \text{exists } f \in E(S) \text{ such that } x = yf. \]

It was pointed out by Lawson [9] that \( x \leq_{\ell} y \) if and only if there exist \( y^* \) and an uniquely determined \( f \in \omega(y^*) \) such that \( x = yf \), where \( \omega(y^*) = \{ e \in E(S) \mid e \omega y^* \} \) (here \( \omega \) is defined on \( E(S) \) by \( e \omega f \Leftrightarrow e = ef = fe \)). In this case, for a fixed \( y^* \), the idempotent \( f \) is unique. In fact, if \( g \in \omega(y^*) \) and \( x = yg \), then \( yf = yg \) and \( y^*f = y^*g \). Thus \( f = g \). In addition, it is noteworthy that any idempotent related to \( y \) by \( \mathcal{L}^* \) acts as the same as \( y^* \).

A congruence \( \rho \) on a semigroup \( S \) is called a left cancellative monoid congruence on \( S \) if \( S/\rho \) is a left cancellative monoid. It is easy to check that the
meet of left cancellative monoid congruences on the semigroup $S$ is also a left cancellative monoid congruence on $S$. Note that the universal relation is of course a left cancellative monoid congruence. Thus there exists a smallest left cancellative monoid congruence on an rpp semigroup $S$. We shall denote by $\sigma_S$ the smallest left cancellative monoid congruence on $S$ if exists. Also, we write $\sigma_S$ as $\sigma$ if there is no ambiguity.

An rpp semigroup $S$ is called $F$-rpp if each $\sigma$-class of $S$ has a greatest element with respect to the partial $\leq_\ell$ (see [6], for the detail).

Now we assume that $S$ is an $F$-rpp semigroup. We denote by $m_s$ the greatest element in the $\sigma$-class of $S$ containing $s$ with respect to $\leq_\ell$, and $M_S = \{m_s \mid s \in S\}$. In general, with respect to the multiplication of $S$, $M_S$ need not be a semigroup, however, it is easy to see that under the following multiplication “$*$”,

$$m_s * m_t = m_{st},$$

$M_S$ forms a left cancellative monoid which is isomorphic to $S/\sigma$.

We now have the following result:

**Lemma 2.3** [6] Let $S$ be an F-rpp semigroup. Then the following statements hold:

(1) $E(S)$ is a band with identity.

(2) For all $m \in M_S$ and $m^*$, $m^*E(S) \subseteq E(S)m^*$ and $E(S)m \subseteq mE(S)$.

(3) For all $s \in S$ and for each $m_s^*$, there exists a unique $f_s \in \omega(m_s^*)$ such that $s = m_s f_s$.

Let $S$ be a monoid with identity 1. Denote by $H_1^*$ the $H^*$-class of $S$ containing 1. Then $H_1^*$ is a cancellative monoid with identity 1. And for $a \in S$, we write $A^*(a) = \{b \in S \mid a = ebf, eR^*aL^*f, e, f \in E(S)\}$ and $U^*(a) = A^*(a) \cap H_1^*$. Now, we call an IC quasi-adequate monoid $S$ a $u$-IC quasi-adequate semigroup if $|U^*(a)| = 1$, for all $a \in S$.

The following result was obtained by Zhang and Chen in [13].

**Lemma 2.4** [13] Let $S$ be an F-abundant semigroup. Then $S$ is a $u$-IC quasi-adequate semigroup if and only if $M_S = H_1^*$.

### 3 Definitions and characterizations

**Definition 3.1** We call a factorization $(A, B)$ of a factorizable monoid $S$ type-$(I, L^*)$ if the following conditions are satisfied:
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(1) A is contained in the $\mathcal{L}^*$-class of $S$ containing 1, where 1 is the identity of $S$.

(2) $B = E(S)$.

(3) For each $a \in A$, $Ba \subseteq aB$.

We also call $S$ a factorizable monoid of type-(\(\mathcal{I}, \mathcal{L}^*\)).

By Definition 3.1, the following corollary is immediate.

**Corollary 3.2** Let $(A, B)$ be a factorization of a factorizable monoid $S$. If $(A, B)$ is type-(\(\mathcal{I}, \mathcal{L}^*\)), then $E(S)$ is a band.

**Proposition 3.3** Let $S$ be a factorizable rpp monoid admitting a left univocal type-(\(\mathcal{I}, \mathcal{L}^*\)) factorization $(A, B)$. Then the factorization $(A, B)$ is univocal.

*Proof.* Let $a_1, a_2 \in A, b_1, b_2 \in B$ and $a_1b_1 = a_2b_2$. Then by our hypothesis, $a_1b_1 = a_1b_2$ and $1b_1 = 1b_2$, since $a_1\mathcal{L}^*1$. That is, $b_1 = b_2$. Thus $(A, B)$ is univocal. \(\square\)

In the following proposition, we describe the relationship between the factorizable rpp monoid admitting a left univocal type-(\(\mathcal{I}, \mathcal{L}^*\)) factorization and the F-rpp semigroups.

**Proposition 3.4** Let $S$ be a factorizable monoid with factorization $(A, B)$ and with identity 1. Then $S$ is a factorizable rpp monoid admitting a left univocal type-(\(\mathcal{I}, \mathcal{L}^*\)) factorization $(A, B)$ if and only if $S$ is an F-rpp semigroup with $A = M_S$ and $B = E(S)$.

*Proof.* Suppose that $S$ is an F-rpp semigroup with $A = M_S$ and $B = E(S)$. Then by Lemma 2.3 (3), $(A, B)$ is a left univocal factorization of $S$ since $m^* = 1$ and $\omega(m^*) = E(S)$, for all $m \in A$. Again by Lemma 2.3 (2), $(A, B)$ is type-(\(\mathcal{I}, \mathcal{L}^*\)). This proves the sufficient part.

Conversely, let $S$ be a factorizable rpp monoid admitting a left univocal type-(\(\mathcal{I}, \mathcal{L}^*\)) factorization $(A, B)$. We now prove that

$$
\rho = \{(a, b) \in S \times S \mid a = me, b = mf \text{ for some } e, f \in E(S) \text{ and } m \in A\}
$$

is a left cancellative monoid congruence on $S$. Obviously, $\rho$ is both reflexive and symmetric. If $apb$ and $bpc$, then there exist elements $m, n \in A$ and $e, f, g, h \in E(S)$ such that $a = me, b = mf = ng$ and $c = nh$. The second equality implies that $m = n$ and $g = f$ since $(A, B)$ is a univocal factorization of $S$. 
Thus $\rho$ is transitive so that $\rho$ is an equivalence relation on $S$. Note that for $e, f, g \in E(S)$ and $m, n \in A$, we have

$$mnhf = (me)(nf) = (me)(ng) = mnhg$$

with $h \in E(S)$ such that $en = nh$ (by Definition 3.1 (3)). Thus $\rho$ is a left congruence on $S$. Similarly, we can prove that $\rho$ is a right congruence on $S$. Thus, $\rho$ is indeed a congruence on $S$.

Now let $a, b, c \in S$ and $(ab)\rho = (ac)\rho$. Then, by our assumptions, there exist elements $m, n, p \in A$ and $e, f, g \in E(S)$ such that $a = me, b = nf$ and $c = pg$. Hence $ab = menf = mnhf$ with $h \in E(S)$, and similarly $ac = mpkg$ where $k \in E(S)$. On the other hand, since $(ab)\rho = (ac)\rho$, we have $q \in A$ and $u, v \in E(S)$ such that $ab = qu$ and $ac = qv$. Observe that $(A, B)$ is a univocal factorization of $S$. Since $mL^*1, mn = q = mp$ and $1n = 1p$ and consequently, $n = p$. This shows that $b\rho c$, and thereby $\rho$ is a left cancellative monoid congruence on $S$.

Consider that $\sigma_S$ is the smallest left cancellative monoid congruence on $S$. We have $\sigma_S \subseteq \rho$. On the other hand, it is clear that $\rho \subseteq \sigma_S$ since $e\sigma_S 1$ for all $e \in E(S)$. Hence, $\sigma_S = \rho$.

Note that each element in $A$ is related to 1 by $L^*$ and that for all $e \in E(S)$, $e\omega 1$. Thus $me \leq_\ell m$. On the other hand, by the above proof, every $\sigma$-class of $S$ consists of the elements of the form $me$ with $m \in A$ and $e \in E(S)$, and hence every element of $A$ is the greatest element of some $\sigma$-class of $S$ with respect to “$\leq_\ell$”.

Now, it is not difficult to see that $S$ is an F-rpp semigroup with $M_S = A$.

It was observed by Guo [6] that an F-abundant semigroup is an IC abundant F-rpp semigroup and vice versa. Thus, from the above observation and Lemma 2.4, the following proposition is immediate.

**Proposition 3.5** A semigroup is a u-IC quasi-adequate semigroup if and only if it is a factorizable rpp semigroup one of whose factorizations is left univocal of type-$(\mathcal{I}, \mathcal{L}^*)$, and which is an IC abundant semigroup.

**Proof.** $(\Rightarrow)$ Assume that $S$ is a u-IC quasi-adequate semigroup. Note that in an IC abundant semigroup “$\leq = \leq_\ell$” (for detail, see [9]). Thus by Lemma 2.4, $S$ is an F-rpp semigroup and $M_S = H_1^* (\subseteq L_1^*)$. This implies that $S$ is a factorizable monoid admitting a left univocal type-$(\mathcal{I}, \mathcal{L}^*)$, by Proposition 3.4.

$(\Leftarrow)$ Suppose that $S$ is a factorizable rpp semigroup. Assume that one of its factorizations is left univocal and is also of type-$(\mathcal{I}, \mathcal{L}^*)$, say $(A, B)$ which is IC abundant. Then, by Proposition 3.4, $S$ is an F-rpp semigroup with $A = M_S$.
and $B = E(S)$. On the other hand, by the using the same arguments as in Proposition 3.5, we see that $S$ is an $F$-abundant semigroup. Note that $S$ is IC abundant so that on $S$, we have “$≤_e ≤_ℓ$”. Thus, $M_S = H^*_I$, that is, $A = H^*_I$. Now, by Lemma 2.4, $S$ is a u-IC quasi-adequate semigroup. □

4 A structure theorem

In this section we shall establish a semidirect product theorem for the factorizable rpp monoids admitting a left univocal type-$(I, L^*)$ factorizations.

Let $S$ be a semigroup and $T$ a monoid. Denote by $End(S)$ the semigroup of endomorphisms of $S$ into itself. Let

$\varphi : T \rightarrow End(S)$ defined by $x \mapsto \varphi_x$

be a monoid homomorphism from $T$ into $End(S)$. Then, we define a multiplication “$\circ$” on the set $S \times T$ such that for $(a, x), (b, y) \in S \times T$,

$$(a, x) \circ (b, y) = ((a \varphi_y)b, xy).$$

It is easy to check that $(S \times T, \circ)$ is a semigroup. We call this semigroup the semidirect product of $S$ by $T$ with the structure mapping $\varphi$, and we denote this semigroup by $S \times_\varphi T$. If, in addition, the component $S$ is a monoid, then the semidirect product $S \times_\varphi T$ is also a monoid with identity $(i, 1)$, where $i$ and 1 are the identities of $S$ and $T$, respectively. (for more details, see [8, Definitions 1.8]).

**Proposition 4.1** Let $B \times_\varphi M$ be the semidirect product of a band $B$ with identity $i$ by a left cancellative monoid $M$ with identity 1 for the structure mapping $\varphi$. Then

1. $E(B \times_\varphi M) = \{(e, 1) \in B \times_\varphi M\}$. Moreover, $E(B \times_\varphi M)$ is isomorphic to $B$.

2. For all $(e, m), (f, n) \in B \times_\varphi M$, $(e, m)L^*(f, n)$ if and only if $eLf$. Moreover, $L^*_{(i, 1)} = \{(i, m) \mid m \in M\}$.

3. For all $(e, m), (f, n) \in B \times_\varphi M$, $(e, m) ≤_ℓ (f, n)$ if and only if $eωf$ and $m = n$.

4. $B \times_\varphi M$ is a factorizable rpp monoid with left univocal type $(I, L^*)$ factorization $(\{i\} \times M, E(B \times_\varphi M))$. 
Proof. (1) If \((e, x) \in E(B \times_\varphi M)\), then \((e, x) = (e, x)^2 = ((e\varphi_x)e, x^2)\) and \(x^2 = x\). Thus \(x = 1\) because \(M\) is a left cancellative monoid. This leads to \(E(B \times_\varphi M) \subseteq \{(e, 1) \in B \times_\varphi M \text{ and } e \in B\}\). By the multiplication of semidirect product, the reverse inclusion is obvious. Thus,
\[
E(B \times_\varphi M) = \{(e, 1) \in B \times_\varphi M \text{ and } e \in B\}.
\]
The remaining proof is clear.

(2) Let \((e, x) \in B \times_\varphi M\). Suppose that \((f, y), (g, z) \in (B \times_\varphi M)^1\) and \((e, x)(f, y) = (e, x)(g, z)\), that is, \(((e\varphi_y)f, xy) = ((e\varphi_z)g, xz)\). Then \((e\varphi_y)f = (e\varphi_z)g\) and \(xy = xz\). The latter equality implies that \(y = z\) since \(M\) is a left cancellative monoid. Thus
\[
(e, 1)(f, y) = ((e\varphi_y)f, y) = ((e\varphi_z)g, z) = (e, 1)(g, z).
\]
On the other hand, we have \((e, x)(e, 1) = (e, x)\). Consequently, by Lemma 2.2, \((e, x)\mathcal{L}^*(e, 1)\), and \(B \times_\varphi M\) is an rpp semigroup. Note that any idempotent related to the identity by \(\mathcal{L}^*\) must be the identity. Now, it is clear that \(L^*_{(e, 1)} = \{(m, m) \mid m \in M\}\).

By using the same arguments as above, we have
\[
(e, m)\mathcal{L}^*(f, n) \iff (e, 1)\mathcal{L}^*(f, 1) \iff (e, 1)\mathcal{L}(f, 1) \iff e\mathcal{L}f.
\]

(3) If \((e, x) \leq_\ell (f, y)\), then there exists \((e, 1) \in \omega((f, 1))\) such that \((e, x) = (f, y)(e, 1)\). This shows that \((e, x) = ((f\varphi_1)e, 1y) = (fe, y)\), and hence \(x = y\) and \(e = fe\). On the other hand, by the multiplication of semidirect product, we have \((e, 1) \in \omega((f, 1))\) if and only if \(e = ef = fe\), that is, \(e \in \omega(f)\).

Conversely, if \(x = y\) and \(\omega f\), then, by the above proof, \((e, 1) \in \omega((f, 1))\). By assumption, we can obtain that \((e, x) = (f, y)(e, 1)\). Thus \((e, x) \leq_\ell (f, y)\).

(4) Define a relation \(\rho\) on \(B \times_\varphi M\) by: for all \((e, m), (f, n) \in B \times_\varphi M\)
\[
(e, m)\rho(f, n) \text{ if and only if } m = n.
\]
Then, it is easy to check that \(\rho\) is a congruence on \(B \times_\varphi M\). On the other hand, it is easy to see that the mapping
\[
\theta : B \times_\varphi M/\rho \to M \text{ defined by } (e, m)\rho \mapsto m
\]
is a semigroup isomorphism. This shows that \(\rho\) is a left cancellative monoid congruence on \(B \times_\varphi M\). It is also clear that \(\rho_{(e, m)} = B \times \{m\}\). Furthermore, \((\ell, m)\) is the greatest element of \(\rho_{(e, m)}\) with respect to the Lawson order “\(\leq_\ell\)”. Thus, \(B \times_\varphi M\) is an F-rpp semigroup and \(M_{B \times_\varphi M} = L^*_{(\ell, 1)}\). By Proposition 3.4, \(B \times_\varphi M\) is a factorizable rpp monoid with left univocal type-\((\mathcal{I}, \mathcal{L}^*)\) factorization \((\{1\} \times M, E(B \times_\varphi M))\).

We now establish the following main theorem.
Theorem 4.2 (Main Theorem) Let $B$ be a band with an identity, and $M$ a left cancellative monoid. Let $\varphi$ be a monoid homomorphism of $M$ into $\text{End}(B)$. Then the semidirect product of $B$ by $M$ for the structure mapping $\varphi$ is a left univocal factorizable rpp monoid of type-$(I, L^*)$. Conversely, any left univocal factorizable rpp monoid of type-$(I, L^*)$ can be constructed by the above manner.

Proof. By Proposition 4.1, we only need to prove the converse part. Now assume that $S$ is a factorizable rpp monoid with the identity 1 which admits a left univocal type-$(I, L^*)$ factorization $(M, E)$. Then, by Proposition 3.5, $S$ is an F-rpp semigroup with $M = M_S$ and $E = E(S)$. For $m \in M$ and $e \in E$, by Lemma 2.3(3), there exists a unique $(em)^\circ \in E$ such that $em = m(em)^\circ$, since $m^* = 1$ and $\omega(m^*) = E(S)$. Now, we consider the following mapping $\varphi_m$:

$$\varphi_m : E \rightarrow E \text{ defined by } e \mapsto (em)^\circ.$$ 

Let $f \in E$. Then $m(efm)^\circ = efm = em(fm)^\circ = m(em)^\circ(fm)^\circ$, and by the uniqueness of $(em)^\circ$, $(efm)^\circ = (em)^\circ(fm)^\circ$. Thus, $(ef)\varphi_m = (e\varphi_m)(f\varphi_m)$. This shows that $\varphi_m$ is a homomorphism of $E$ into itself.

Let $\varphi$ be a mapping which maps from $M$ into $\text{End}(E)$:

$$\varphi : M \rightarrow \text{End}(E) \text{ defined by } m \mapsto \varphi_m.$$ 

Then, it is clear that $\varphi$ is a semigroup homomorphism. Now, let $m, n \in M$ and $e \in E$. Then we deduce that

$$mn(emn)^\circ = emn = m(em)^\circ n = mn((em)^\circ n)^\circ.$$ 

By the uniqueness of $(em)^\circ$, we have $((em)^\circ n)^\circ = (emn)^\circ$ and

$$e\varphi_m\varphi_n = (em)^\circ \varphi_n = ((em)^\circ n)^\circ = (emn)^\circ = e\varphi_{mn}.$$ 

Hence, $\varphi_{mn} = \varphi_m\varphi_n$, and thereby, $\varphi$ is a semigroup homomorphism. Denote by 1 the identity of $S$. Since $(e1)^\circ = e$, we see that $\varphi_1$ is the identity of $E$. Thus $\varphi$ is a monoid homomorphism.

Let $s \in S$. Note that $s \leq t m_s$. Then, by Lemma 2.3(3), we have a unique $s^\circ \in E$ such that $s = m_s s^\circ$ because $m^* = 1$ and $\omega(m^*) = E(S)$. Since $L^*$ is a right congruence and $m_s L^* 1$, $s = m_s s^\circ L^* 1 s^\circ = s^\circ$. Now, we define

$$\theta : S \rightarrow E \times_M M \text{ by } s \mapsto (s^\circ, m_s).$$ 

Obviously, $\theta$ is well defined. It remains to verify that $\theta$ is a semigroup isomorphism. If $(s^\circ, m_s) = (t^\circ, m_t)$, then $s^\circ = t^\circ$ and $m_s = m_t$. Hence $s = m_s s^\circ = m_t t^\circ = t$ and this shows that $\theta$ is injective. By Lemma 2.3, $(me)^\circ = e$ and
Proof. Since $m_e = m$, for any $e \in E$ and $m \in M$. Thus, $(me)\theta = (e, m)$ and hence $\theta$ is surjective.

Finally, if $s, t \in S$, then $m_{st} = m_s m_t$. By routine computation, we have

$$m_{st}(st)^{\circ} = st = m_s s^\circ m_t t^\circ = m_s m_t (s^\circ m_t)^{\circ} = m_{st} (s^\circ \varphi_{m_t}) t^\circ.$$  

Since $m_{st} \mathcal{L}^* 1$, by Lemma 2.2, we deduce that

$$(s^\circ \varphi_{m_t}) t^\circ = 1 (s^\circ \varphi_{m_t}) t^\circ = 1 (s t)^{\circ} = (s t)^{\circ}$$

and

$$(s \theta)(t \theta) = (s^\circ, m_s) (t^\circ, m_t) = ((s^\circ \varphi_{m_t}) t^\circ, m_s m_t) = ((s^\circ m_t)^{\circ} t^\circ, m_{st}) = ((st)^{\circ}, m_{st}) = (s t)^{\circ} \theta.$$  

This shows that $\theta$ is a semigroup homomorphism. Consequently, $\theta$ is an isomorphism. \qed

Remark 4.3 By Lemma 2.4 and Proposition 3.5, we can easily see that a u-IC quasi-adequate semigroup $S$ is always a factorable rpp monoid admitting a left univocal type $(I, \mathcal{L}^*)$ factorization $(H^*, E(S))$. Thus, by Theorem 4.2, we can see immediately that a u-IC quasi-adequate semigroup is a semidirect product of a band and a cancellative monoid, and vice versa. Thus, in this way, we can re-obtain the main result recently obtained by Zhang-Chen for IC quasi-adequate semigroups in [13].

References


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