International Mathematical Forum, 2, 2007, no. 4, 187 - 193

Generalized Difference Sequence Spaces Defined by a Sequence of Orlicz Functions

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Abstract

The idea of difference sequence spaces was introduced by Kizmaz [5], and this concept was generalized by Et and Colak [2]. In this paper, we define the sequence spaces $c_0(\Delta_u^n, M_k, p, s), c\ (\Delta_u^n, M_k, p, s)$, and $l_\infty(\Delta_u^n, M_k, p, s)$, where $M = (M_k)$ is a sequence of Orlicz functions, and examine some inclusion relations and properties of these spaces , which will give as a special case the spaces $c_0(\Delta^n, M, p), c\ (\Delta^n, M, p)$, and $l_\infty(\Delta^n, M, p)$ of Gökhan, et al [3].

Mathematics Subject Classification: 40A05, 40C05, 46A45

Keywords: Difference sequence spaces, A sequence of Orlicz functions

1 Definitions and notations

Let w denote the set of all complex sequences $x = (x_k)$, and l_∞, c, and c_0 be the linear spaces of bounded, convergent, and null sequences with complex terms, respectively, normed by $\| x \| = \sup_k \mid x_k \mid$, where $k \in \mathbb{N}$, the set of positive integers.

A paranorm on a linear topological space X is a function $g : X \to \mathbb{R}$ which satisfies the following axioms :

for any $x, y, x_0 \in X$ and $\lambda, \lambda_0 \in \mathbb{C}$, the set of complex numbers,
(i) $g(\theta) = 0$, where $\theta = (0, 0, 0, \cdots)$, the zero sequence,
(ii) $g(x) = g(-x)$,
(iii) $g(x + y) \leq g(x) + g(y)$ (subadditivity),
and
(iv) the scalar multiplication is continuous, that is,

$$\lambda \to \lambda_0, x \to x_0 \text{ imply } \lambda x \to \lambda_0 x_0 \ ;$$

in other words,

$$\mid \lambda - \lambda_0 \mid \to 0, g(x - x_0) \to 0 \text{ imply } g(\lambda x - \lambda_0 x_0) \to 0.$$

A paranormed space is a linear space X with a paranorm g and is written (X, g), (see [8], p. 92).

Any function g which satisfies all the conditions (i)-(iv) together with the condition

(v) $g(x) = 0$ if and only if $x = \theta$, is called a total paranorm on X, and the pair (X, g) is called a total paranormed space, (see [8], p. 92).

For any sequence $x = (x_k)$, the difference sequence Δx is defined by $\Delta x = (\Delta x_k)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$.

Kizmaz [5] defined the sequence spaces

$$l_\infty(\Delta) = \{x \in w : \Delta x \in l_\infty\},$$

$$c\ (\Delta) = \{x \in w : \Delta x \in c\},$$

and

$$c_0(\Delta) = \{x \in w : \Delta x \in c_0\}.$$

Et and Colak [2] generalized the notion of difference sequence spaces and defined the spaces

$$l_\infty(\Delta^n) = \{x \in w : \Delta^n x \in l_\infty\},$$

$$c\ (\Delta^n) = \{x \in w : \Delta^n x \in c\},$$

and

$$c_0(\Delta^n) = \{x \in w : \Delta^n x \in c_0\},$$

where $\Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}$ and $\Delta^0 x_k = x_k$, for all $k \in \mathbb{N}$.

Taking $X = l_\infty(p), c\ (p)$, and $c_0(p)$, these sequence spaces has been generalized by Et and Basarir [1].

The generalized difference has the following binomial representation :

$$\Delta^n x_k = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x_{k+r}\ , \ \text{ for all } k \in \mathbb{N}.$$

The following inequality will be used throughout this paper : let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup_k p_k = G$, $C = \max(1, 2^{G-1})$. Then for $a_k, b_k \in \mathbb{C}$ and for all $k \in \mathbb{N}$, we have

$$\mid a_k + b_k \mid^{p_k} \leq C\ (\mid a_k \mid^{p_k} + \mid b_k \mid^{p_k}), \text{ (see [5]) (see [7])} \tag{1}$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of l, if there exist a constant $K > 0$ such that

$$M(2l) \leq KM(l)(l \geq 0);$$

equivalently,

$$M(hl) \leq KhM(l)$$

for every value of l and for $l > 1$.

Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to define what is called an Orlicz sequence space :

$$l_M := \{x \in w : \sum_{k=1}^{\infty} M(\frac{\mid x_k \mid}{\rho}) < \infty, \text{ for some } \rho\ > 0\}$$

which is a Banach space with the norm :

$$\parallel x \parallel_M = \inf\{\rho\ > 0 : \sum_{k=1}^{\infty} M(\frac{\mid x_k \mid}{\rho}) \leq 1\}.$$

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for every sequence (α_k) of scalars such that $\mid \alpha_k \mid \leq 1$, for all $k \in \mathbb{N}$.

A sequence space E is said to be monotone if E contains preimages of all its step spaces.

A sequence space E is said to be sequence algebra if $x \cdot y \in E$, whenever $x, y \in E$ (see [4]).

Lemma 1. A sequence space E is solid implies E is monotone.

Mursaleen, et al [9] defined and studied the sequence spaces $c_0(\Delta, M, p), c$ (Δ, M, p), and $l_\infty(\Delta, M, p)$. Recently, Gökhan, et al [3] generalized the spaces of Mursaleen, et al [9] to $c_0(\Delta^n, M, p), c\ (\Delta^n, M, p)$, and $l_\infty(\Delta^n, M, p)$, where n is apositive integer.

In this paper, we further generalize these spaces as follows :

Let $M = (M_k)$ be a sequence of Orlicz functions and $u = (u_k)$ be any sequence such that $u_k \neq 0$ for all k , then we define

$$c_0(\Delta_u^n, M_k, p, s) = \{x \in w : \lim_{k\to\infty} k^{-s}[M_k(\frac{|\Delta_u^n x_k|}{\rho})]^{p_k} = 0, \text{ for some } \rho > 0,\ s \geq 0\},$$

$$c\ (\Delta_u^n, M_k, p, s) = \{x \in w : \lim_{k\to\infty} k^{-s}[M_k(\frac{|\Delta_u^n x_k - l|}{\rho})]^{p_k} = 0,$$
$$\text{for some } \rho > 0,\ l \in \mathbb{C},\ s \geq 0\},$$

and

$$l_\infty(\Delta_u^n, M_k, p, s) = \{x \in w : \sup_k k^{-s}[M_k(\frac{|\Delta_u^n x_k|}{\rho})]^{p_k} < \infty, \text{ for some } \rho > 0, s \geq 0\},$$

where $\Delta_u^n x_k = (\Delta_u^{n-1} x_k - \Delta_u^{n-1} x_{k+1})$ such that $\Delta_u^n x_k = \sum_{r=0}^{n} (-1)^r \binom{n}{r} u_{k+r} x_{k+r}$, $\Delta_u^n x_k = (u_k x_k)$, $\Delta_u x_k = (u_k x_k - u_{k+1} x_{k+1})$.

If $(M_k) = M$ for all k, $s = 0$, and $u = e = (1, 1, 1, \cdots)$, then these spaces reduce to those of Gökhan, et al [3].

In the case p_k =constant for all k, we denote the above mentioned spaces as $c_0(\Delta_u^n, M_k, s), c\ (\Delta_u^n, M_k, s)$, and $l_\infty(\Delta_u^n, M_k, s)$, respectively.

2 Main results

We prove the following theorems :

Theorem 2.1. Let (p_k) be bounded, n be a positive integer, and (M_k) be a sequence of Orlicz functions. Then $l_\infty(\Delta_{u-1}^{n-1}, M_k, p, s) \subset l_\infty(\Delta_u^n, M_k, p, s)$ and the inclusion is strict.

Proof. Since M_k is nondecreasing and convex for each k, the result follows by (1.1). To show that the inclusion is strict, let $M_k(x) = x$, and $p_k = 1$ for all k, then the sequence $x = (k^m)$ belongs to $l_\infty(\Delta_u^n, M_k, p, s)$ but does not belong to $l_\infty(\Delta_{u-1}^{n-1}, M_k, p, s)$.

Remark. It is easy to show that these sequence spaces are paranormed spaces with

$$h(x) = \inf_m \{\rho^{p_m/H} : (\sup_k k^{-s}[M_k(\frac{|\Delta_u^n x_k|}{\rho})]^{p_k})^{1/H} \leq 1, m = 1, 2, \cdots\}, \qquad (1)$$

where $H = \max(1, \sup_k p_k)$.

Theorem 2.2. $l_\infty(\Delta_u^n, M_k, p, s)$ is a complete paranormed space with h defined in (2.1).

Proof. Using the same technique used in [9, Theorem 2.1], it is easy to prove the theorem.

Theorem 2.3. Let $0 < p_k \leq q_k < \infty$, for each k.
Then $c_0(\Delta_u^n, M_k, p, s) \subset c_0(\Delta_u^n, M_k, q, s)$.
Proof. The proof is easy, so we omit it.
Theorem 2.4.
(i) Let $0 < \inf p_k \leq p_k \leq 1$. Then $c_0(\Delta_u^n, M_k, p, s) \subset c_0(\Delta_u^n, M_k, s)$.
(ii) Let $1 \leq p_k \leq \sup p_k < \infty$. Then $c_0(\Delta_u^n, M_k, s) \subset c_0(\Delta_u^n, M_k, p, s)$.
Proof.
(i) Let $x \in c_0(\Delta_u^n, M_k, p, s)$, that is,

$$\lim_{k\to\infty} k^{-s}[M_k(\frac{|\Delta_u^n x_k|}{\rho})]^{p_k} = 0.$$

Since $0 < \inf p_k \leq p_k \leq 1$,

$$\lim_{k\to\infty} k^{-s}[M_k(\frac{|\Delta_u^n x_k|}{\rho})] \leq \lim_{k\to\infty} k^{-s}[M_k(\frac{|\Delta_u^n x_k|}{\rho})]^{p_k} = 0,$$

and hence $x \in c_0(\Delta_u^n, M_k, s)$.

(ii) Let $1 \leq p_k$, for each k, and $\sup p_k < \infty$. Let $x \in c_0(\Delta_u^n, M_k, s)$, then for each $\epsilon (0 < \epsilon < 1)$, there exists a positive integer N such that

$$k^{-s}[M_k(\frac{|\Delta_u^n x_k|}{\rho})] \leq \epsilon, \text{ for all } k \geq N.$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\lim_{k\to\infty} k^{-s}[M_k(\frac{|\Delta_u^n x_k|}{\rho})]^{p_k} \leq \lim_{k\to\infty} k^{-s}[M_k(\frac{|\Delta_u^n x_k|}{\rho})] \leq \epsilon < 1.$$

therefore $x \in c_0(\Delta_u^n, M_k, p, s)$.This completes the proof.

Theorem 2.5. Let (M_k) be a sequence of Orlicz functions such that M_k satisfies the Δ_2-condition for all k. Then $c_0(\Delta_u^n, M_k, p, s) \subset c\ (\Delta_u^n, M_k, p, s) \subset l_\infty(\Delta_u^n, M_k, p, s)$. and the inclusions are strict.

Proof. Let $x \in c\ (\Delta_u^n, M_k, p, s)$. Then we have

$$\begin{aligned} k^{-s}[M_k(\frac{\mid \Delta_u^n x_k \mid}{\rho})]^{p_k} &\leq C\ k^{-s}[M_k(\frac{\mid \Delta_u^n x_k - l \mid}{\rho})]^{p_k} + C\ k^{-s}[M_k(\frac{\mid l \mid}{\rho})]^{p_k} \\ &\leq C\ k^{-s}[M_k(\frac{\mid \Delta_u^n x_k - l \mid}{\rho})]^{p_k} + C\ k^{-s}[K\delta^{-1}(\frac{\mid l \mid}{\rho})M_k(2)]^H. \end{aligned}$$

Thus, we get $x \in l_\infty(\Delta_u^n, M_k, p, s)$. Then inclusion $c_0(\Delta_u^n, M_k, p, s) \subset c\ (\Delta_u^n, M_k, p, s)$ is obvious. To show that the inclusion is strict, consider the following example.

Example 1 : Let $M_k(x) = x$ and $p_k = 1$ for all k. Then the sequence $x = ((-1)^k)$ belongs to $l_\infty(\Delta_u^n, M_k, p, s)$, but does not belong to $c\ (\Delta_u^n, M_k, p, s)$.

Theorem 2.6. Let $n \geq 1$, then $c\ (\Delta_{u-1}^{n-1}, M_k, p, s) \subset c_0(\Delta_u^n, M_k, p, s)$.

Theorem 2.7. The spaces $c_0(M_k, p, s)$ and $l_\infty(M_k, p, s)$ are solid and therefore are monotone.

Proof. Let $x \in c_0(M_k, p, s)$. Then there exists $\rho > 0$ such that

$$\lim_{k\to\infty} k^{-s}[M_k(\frac{\mid \Delta_u x_k \mid}{\rho})] = 0.$$

Let (α_k) be a sequence of scalars such that $\mid \alpha_k \mid \leq 1$. Then we have

$$k^{-s}[M_k(\frac{\mid \alpha_k x_k \mid}{\rho})]^{p_k} \leq C\ k^{-s}[M_k(\frac{\mid x_k \mid}{\rho})]^{p_k}$$

From this inequality, it follows that $c_0(M_k, p, s)$ is also solid. The monotonicity of the spaces $c_0(M_k, p, s)$ and $l_\infty(M_k, p, s)$ follows from Lemma 1.

The spaces $c_0(\Delta_u^n, M_k, p, s)$, $c\ (\Delta_u^n, M_k, p, s)$, and $l_\infty(\Delta_u^n, M_k, p, s)$ are not solid in general.

To show that the above spaces are not solid, consider the following example.

Example 2 : Let $M_k(x) = x$ and $p_k = 1$ for all k. Then the sequence $x = (k^m)$ belongs to $l_\infty(\Delta_u^n, M_k, p, s)$. Let $(\alpha_k) = ((-1)^k)$, then $(\alpha_k x_k)$ does not belong to $l_\infty\ (\Delta_u^n, M_k, p, s)$.

Corollary. $c_0(\Delta_u^n, M_k, p, s)$ and $c\ (\Delta_u^n, M_k, p, s)$ are nowhere dense subsets of $l_\infty(\Delta_u^n, M_k, p, s)$.

Theorem 2.8. The spaces $c_0(\Delta_u^n, M_k, p, s)$, $c\ (\Delta_u^n, M_k, p, s)$ and $l_\infty(\Delta_u^n, M_k, p, s)$ are not sequence algebra, for $n \geq 2$.

Proof. Let $M_k(x) = x$ and $p_k = 1$ for all k. Consider the sequences $x = (k^m)$ and $y = (k)$ for all k, then $x, y \in l_\infty(\Delta_u^n, M_k, p, s)$ but $x \cdot y \notin l_\infty(\Delta_u^n, M_k, p, s)$.

Acknowledgement. This research is completed while the authors were awarded the Kuwait Junior Research Fellowship by the Kuwait Foundation for the Advancement of Sciences, for the six months period during February, 2006 to August, 2006, at the Department of Pure Mathematics and Mathematical Statistics, at the University of Cambridge, so the authors would like to express their sincere thanks and gratitude to the Kuwait Foundation for the Advancement of Sciences and to the Department of Pure Mathematics and Mathematical Statistics, at the University of Cambridge.

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Received: March 17, 2006