The Commutativity in Prime Gamma Rings with Left Derivation

Mustafa AŞCI

Gazi University, Science and Art Faculty
Department of Mathematics, Ankara, Turkey
masci@gazi.edu.tr

Şahin CERAN

Pamukkale University, Science and Art Faculty
Department of Mathematics, Denizli, Turkey
sceran@pamukkale.edu.tr

Abstract. Let M be a Γ -ring, U an ideal of M , $\text{char} M \neq 2, 3$ and $0 \neq d : M \rightarrow M$ be a left derivation. In this paper we have proved the following results.

- (i) If $d(U) \subset U$ and $d^2(U) \subset Z$ then M is commutative where $Z = \{c \in M : c\gamma m = m\gamma c : m \in M, \gamma \in \Gamma\}$ is the center of M .
- (ii) Let d_1, d_2 be non zero left and right derivations of M and $d_2(U) \subset U$. If $d_1 d_2(U) \subset Z$ then M is commutative.

Mathematics Subject Classification: Primary 16Y30, 16W25; Secondary 16U80

Keywords: Left derivation, commutativity, prime rings

1. INTRODUCTION

The gamma ring is defined in [1] as follows. Let M and Γ be additive Abelian groups. If for $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ the following conditions are satisfied.

(G₁) $a\beta b \in M$,

(G₂) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$

(G₃) $(a\alpha b)\beta c = a\alpha(b\beta c)$

Then M is called a Γ -ring (in the sense of Barnes). Every ring is a Γ -ring. A right (left) ideal of a Γ -ring M is an additive subgroup U of M such that $U\Gamma M \subset U$ ($M\Gamma U \subset U$). If U is both a right and left ideal, then we say that U is an ideal of M . If the following conditions hold for a gamma ring M then M

is called a prime gamma ring [2]. For $a, b \in M$

$$a\Gamma M\Gamma b = 0 \Rightarrow a = 0 \text{ or } b = 0$$

In this paper we will prove some results in gamma rings which are known in ring theory. Throughout this paper M will be a prime Γ -ring, Z will be the center of M .

Let M be a Γ -ring and $D : M \rightarrow M$ be an additive map. D is called a left derivation if for all $x, y \in M, \alpha \in \Gamma$

$$D(x\alpha y) = x\alpha D(y) + y\alpha D(x)$$

A right derivation is defined similarly. A Γ -ring M is called commutative if $x\alpha y = y\alpha x$ for any $x, y \in M$ and $\alpha \in \Gamma$. An element $m \in M$ is called a right nonzero divisor if $x\alpha m = 0$ implies $x = 0$ for any $\alpha \in \Gamma$. A left nonzero divisor is defined similarly.

2. MAIN RESULTS

Lemma 1. *Let U be a non-zero right ideal of M and $d : M \rightarrow M$ left derivation (right derivation).*

(i) *If $d(U) = 0$ then $d = 0$*

(ii) *Let U be a non-zero ideal of M and $d : M \rightarrow M$ right derivation, $d(M)$ left non-zero divisor. If $a\Gamma d(U) = 0$ then $a = 0$*

(iii) *Let U be a non-zero ideal of M and $d : M \rightarrow M$ left derivation, $D(M)$ right non-zero divisor. If $d(U)\Gamma a = 0$ then $a = 0$*

(iv) *Let U be a non-zero ideal of M and $\text{char} M \neq 2$, $d : M \rightarrow M$ right derivation. If $d^2(U) = 0$ then $d = 0$*

(v) *Let U be a non-zero ideal of M , $\text{char} M = 2$, $d_1 : M \rightarrow M$ left derivation and $d_2 : M \rightarrow M$ right derivation. If $d_2(U) \subset U$ and $d_1 d_2(U) = 0$ then $d_1 = 0$ or $d_2 = 0$*

Proof. (i) For any $u \in U, \alpha \in \Gamma$ and $m \in M$ we have

$$\begin{aligned} 0 &= d(u\alpha m) = u\alpha d(m) + m\alpha d(u) \\ &= u\alpha d(m) \end{aligned}$$

That is, $U\Gamma d(M) = 0$. This implies that $d(M) = 0$, by [[4] Lemma 2 (ii)] and so $d = 0$

(ii) Let $\alpha, \beta \in \Gamma, u \in U$ and $m \in M$. Then

$$\begin{aligned} 0 &= a\gamma d(u\beta m) \\ &= a\gamma d(u)\beta m + a\gamma d(m)\beta u \\ &= a\gamma d(m)\beta u \end{aligned}$$

is obtained. That is, $a\Gamma d(M)\Gamma U = 0$ Hence $a\Gamma d(M) = 0$ by [[4] Lemma 2 (ii)] and

$$\begin{aligned} 0 &= a\Gamma d(m\gamma)d^{-1}(m) \\ &= a\Gamma d(m)\gamma d^{-1}(m) + a\Gamma m\Gamma m \\ &= a\Gamma M\Gamma M \end{aligned}$$

[[4] Lemma 2 (ii)] $a = 0$

(iii) Let $\alpha, \beta \in \Gamma$, $u \in U$ and $m \in M$. Then

$$\begin{aligned} 0 &= d(u\beta m)\gamma a \\ &= u\beta d(m)\gamma a + m\beta d(u)\gamma a \\ &= u\beta d(m)\gamma a \end{aligned}$$

is obtained. That is, $U\Gamma d(M)\Gamma a = 0$ hence $d(M)\Gamma a = 0$, [[4] by Lemma 2 (ii)] and

$$d(m\gamma d^{-1}(m))\Gamma a = m\gamma m\Gamma a + d^{-1}(m)\gamma d(m)\Gamma a \Rightarrow M\Gamma M\Gamma a = 0$$

[[4] Lemma 2 (ii)], $a = 0$

(iv) Let $\text{char} M \neq 2$ and $d^2(U) = 0$. For any $u \in U$ and $\alpha \in \Gamma$ we have $0 = d^2(u\alpha u) = d(u)\alpha d(u)$. Therefore, $d(U)\Gamma d(U) = 0$ and so $d(U) = 0$ by (ii). Hence $d = 0$ by (i)

(v) If $d_1 d_2(U) = 0$ and $d_2(U) \subset U$ then for all $u, v \in U$ and $\alpha \in \Gamma$

$$\begin{aligned} 0 &= d_1 d_2(u\alpha v) = d_1(d_2(u)\alpha v + d_2(v)\alpha u) \\ &= d_2(u)\alpha d_1(v) + v\alpha d_1 d_2(v) + d_2(v)\alpha d_1(u) + u\alpha d_1 d_2(v) \\ &= d_2(u)\alpha d_1(v) + d_2(v)\alpha d_1(u) \end{aligned}$$

If we replace u by $d_2(u)$ in the last equation we get $d_2^2(U)\Gamma d_1(U) = 0$ By (ii) and (iii), we obtain $d_1 = 0$ or $d_2 = 0$ ■

In the rest of this paper U will be a nonzero ideal of M .

Lemma 2. *Let d be a non-zero right derivation of M and $\text{char} M \neq 2$. If $d(U) \subset Z$ then M is commutative.*

Proof. For all $u \in U, y \in M, z \in Z$ and $\gamma \in \Gamma$

$$\begin{aligned} 0 &= [d(u\gamma z), y]_\gamma \\ &= [d(u)\gamma z + d(z)\gamma u, y]_\gamma \\ &= d(z)\gamma [u, y]_\gamma \end{aligned}$$

This implies that

$$(2.1) \quad [U, M]_\gamma = 0 \quad \text{or} \quad d(Z) = 0$$

by [[4] Lemma 1(vi)] for all $\gamma \in \Gamma$. If $d(Z) = 0$ then $d^2(U) = 0$, which implies that $d = 0$ by [[4] Lemma 1 (iv)]. This is a contradiction. Thus $U \subset Z$ is obtained and so M is commutative by [[4] Lemma 2 (i)]. ■

Theorem 3. *Let d be a non-zero right derivation of M and $\text{char}M \neq 2, 3$. If $d^2(U) \subset Z$ and $d(U) \subset U$ then M is commutative.*

Proof. For any $u \in U, \gamma, \beta \in \Gamma$ and $y \in M$ we have,

$$\begin{aligned} 0 &= [d^2(d(u)\gamma d(u)), y]_\beta \\ &= 2[d(d^2(u)\gamma d(u)), y]_\beta \\ &= 2[d^3(u)\gamma d(u) + d^2(u)\gamma d^2(u), y]_\beta \\ &= 2d^3(u)\gamma[d(u), y]_\beta \end{aligned}$$

and so by [[4]Lemma 1(v)]

$$(2.2) \quad d^3(u) = 0 \quad \text{or} \quad d(u) \in Z$$

is obtained. If $d^3(U) = 0$ then

$$\begin{aligned} 0 &= [d^2(u\beta d(u)), y]_\gamma \\ &= [d(d(u)\beta d(u) + d^2(u)\beta u), y]_\gamma \\ &= [2d^2(u)\beta d(u) + d^3(u)\beta u + d(u)\beta d^2(u), y]_\gamma \\ &= 2d^2(u)\beta[d(u), y]_\gamma + [d(u), y]_\gamma\beta d^2(u) \\ &= 3d^2(u)\beta[d(u), y]_\gamma \end{aligned}$$

Considering $\text{char}M \neq 3$ and using [[4] Lemma 1 (v)] we get for all $u \in U$

$$d^2(u) = 0 \quad \text{or} \quad d(u) \in Z$$

the sets $K = \{u \in U : d^2(u) = 0\}$ and $L = \{u \in U : d(u) \in Z\}$ are additive subgroups of U and $U = K \cup L$. If $U = K$ then $d = 0$ by Lemma 1 (iv). This is a contradiction. Hence we have $U = L$ by Brauer trick and so M is commutative by Lemma 2. ■

Lemma 4. *Let $0 \neq d_1 : M \rightarrow M$ left derivation and $0 \neq d_2 : M \rightarrow M$ right derivation such that $d_1d_2(U) \subset Z$ and $d_2(U) \subset U$, $\text{char}M \neq 2, 3$. If $d_1d_2^2(U) = 0$ then M is commutative.*

Proof. For any $u \in U$ and $\gamma \in \Gamma$ we have for all $y \in M$.

$$\begin{aligned} 0 &= [d_1(d_2(d_2(u)\beta d_2(u))), y]_\gamma \\ &= 2[d_1(d_2^2(u)\beta d_2(u)), y]_\gamma \\ &= 2[d_2^2(u)\beta d_1d_2(u) + d_2(u)\beta d_1d_2^2(u), y]_\gamma \\ &= 2[d_2^2(u)\beta d_1d_2(u), y]_\gamma \end{aligned}$$

Since $\text{char}M \neq 2$, if we use [lemma 1(iv)] we get

$$(2.3) \quad d_1d_2(u) = 0 \quad \text{or} \quad d_2^2(u) \in Z$$

Let $K = \{u \in U : d_1d_2(u) = 0\}$ and $L = \{u \in U : d_2^2(u) \in Z\}$. In this case K and L are subgroups of U and $U = K \cup L$. If $U = K$ then $d_1 = 0$ or $d_2 = 0$ by Lemma 1 (v). This is a contradiction. Therefore $U = L$ by Brauer trick, which implies that M is commutative by Theorem 1. ■

Lemma 5. *Let $0 \neq d_1 : M \rightarrow M$ left derivation and $0 \neq d_2 : M \rightarrow M$ right derivation such that $d_1 d_2(U) \subset Z$, $d_2(U) \subset U$, $\text{char} M \neq 2$ and $a \in M$. If $[d_1(U), a]_\gamma = 0$ then $[d_2(U), a]_\gamma = 0$ for all $\gamma \in \Gamma$.*

Proof. For any $x \in U$ and $\gamma \in \Gamma$ we have

$$\begin{aligned} 0 &= [d_1 d_2(x) \gamma d_2(x), a]_\gamma \\ &= 2[d_2(x) \gamma d_1 d_2(x), a]_\gamma \\ &= 2[d_2(x), a]_\gamma \gamma d_1 d_2(x) \end{aligned}$$

By [[4] Lemma 1 (iv)] for all $x \in U, \gamma \in \Gamma$

$$(2.4) \quad d_1 d_2(x) = 0 \quad \text{or} \quad [d_2(x), a]_\gamma = 0$$

is obtained. Let $K = \{x \in U : d_1 d_2(x) = 0\}$ and $L = \{x \in U : [d_2(x), a]_\gamma = 0\}$. Then K and L are subgroups of U and $U = K \cup L$. Since $d_1 \neq 0$ and $d_2 \neq 0$, U can not equal to K by Lemma 1 (v). Hence $U = L$ by Brauer trick. That is $[d_2(U), a]_\gamma = 0$ for all $\gamma \in \Gamma$, is obtained. ■

Lemma 6. *Let d_1 be non-zero left derivation of M and d_2 be non-zero right derivation of M such that $d_1 d_2(U) \subset Z$, $d_2(U) \subset U$ and $a \in M$. If $[d_1(U), a]_\gamma = 0$ for all $\gamma \in \Gamma$ then $a \in Z$.*

Proof. By Lemma 4 we have $[d_2(U), a]_\gamma = 0$ for all $\gamma \in \Gamma$. Hence for all $x \in U$ and $\gamma \in \Gamma$

$$\begin{aligned} 0 &= [d_1 d_2(x \gamma x), a]_\gamma \\ &= 2[d_1(d_2(x) \gamma x), a]_\gamma \\ &= 2[d_2(x) \gamma d_1(x) + x \gamma d_1 d_2(x), a]_\gamma \\ &= 2[x \gamma d_1 d_2(x), a]_\gamma \end{aligned}$$

And so

$$(2.5) \quad d_1 d_2(x) = 0 \quad \text{or} \quad [x, a]_\gamma = 0$$

[[4] Lemma 1 (iv)]. If we apply to (2.5) Brauer trick and use Lemma 1 (v) we have $a \in Z$. ■

Theorem 7. *Let d_1 be non-zero left derivation of M and $d_1 d_2(U) \subset Z$ and $d_2(U) \subset U$ and $a \in M$, $\text{char} M \neq 2, 3$. If $[d_1(U), a]_\gamma \in Z$ for all $\gamma \in \Gamma$ then $a \in Z$.*

Proof. If $Z = 0$ $d_1 d_2(U) = 0$ and so $d_1 = 0$ or $d_2 = 0$ by Lemma 1. It is a contradiction. Hence we can assume that $Z \neq 0$. For any $x \in U, z \in Z$ and $\gamma, \beta \in \Gamma$ we have

$$\begin{aligned} Z &\ni [d_1(x \beta z), a]_\gamma = [x \beta d_1(z) + z \beta d_1(x), a]_\gamma \\ &= [x, a]_\gamma \beta d_1(z) + z \beta [d_1(x), a]_\gamma \end{aligned}$$

That is, for all $\gamma, \beta \in \Gamma$

$$(2.6) \quad [U, a]_\gamma \beta d_1(z) \subset Z$$

By [[4] Lemma 1 (v)] and (2.6) we obtain for all $\gamma \in \Gamma$

$$(2.7) \quad [U, a]_\gamma \subset Z \quad \text{or} \quad d_1(z) = 0$$

If $d_1(z) \neq 0$ then $[U, a]_\gamma \subset Z$, for all $\gamma \in \Gamma$. Now, for any $m \in M, \beta \in \Gamma$, we have $[[U, a]_\gamma, m]_\beta = 0$. Let us take $u\gamma u$ instead of u in the last equation. Then we get,

$$\begin{aligned} 0 &= [[u\gamma u, a]_\gamma, m]_\beta \\ &= [u\gamma[u, a]_\gamma + [u, a]_\gamma\gamma u, m]_\beta \\ &= 2[u\gamma[u, a]_\gamma, m]_\beta \end{aligned}$$

Since M is a prime gamma ring, then for all $m \in M, \gamma, \beta \in \Gamma, u \in U$ we have

$$(2.8) \quad [u, m]_\beta\gamma[u, a]_\gamma = 0$$

This implies that, [[4] by Lemma 1 (iv)] for all $m \in M, \beta, \gamma \in \Gamma$

$$[u, m]_\beta = 0 \quad \text{or} \quad [u, a]_\gamma = 0$$

That is, for all $\gamma \in \Gamma$

$$(2.9) \quad [U, a]_\gamma = 0$$

If we consider Lemma 5 and (2.9) we get $a \in Z$.

If $d_1(z) = 0$ then $d_1d_1d_2(U) = 0$. Thus for any $x \in U$ and $\gamma \in \Gamma$ we have

$$\begin{aligned} 0 &= d_1d_1d_2(d_2(x)\gamma d_2(x)) \\ &= d_1d_1(2d_2^2(x)\gamma d_2(x)) \\ &= 2d_1(d_2^2(x)\gamma d_1d_2(x) + d_2(x)\gamma d_1d_2^2(x)) \\ &= 4d_1d_2(x)\gamma d_1d_2^2(x) \end{aligned}$$

and so for all $x \in U$

$$(2.10) \quad d_1d_2(x) = 0 \quad \text{or} \quad d_1d_2^2(x) = 0$$

If we say that $K = \{x \in U : d_1d_2(x) = 0\}$ and $L = \{x \in U : d_1d_2^2(x) = 0\}$ then K and L are subgroups of U and $U = K \cup L$. Since $d_1 \neq 0$ and $d_2 \neq 0$, we have $U \neq K$ by Lemma 1. Thus if we consider Brauer trick we get $U = L$. Thus it $d_1d_2^2(U) = 0$. Now, $d_1d_2^2(U) = 0$, $d_1d_2(U) \subset Z$, $d_2(U) \subset U$ and $\text{char} M \neq 2, 3$ implies that M is commutative by Lemma 3 and $a \in Z$. ■

REFERENCES

- [1] Barnes W. E., On the Γ -rings of Nobusawa, Pacific J. of Math. Vol. 18, No 3, (1966).
- [2] Kyum S., On prime gamma rings, Pacific J. of Math. Vol. 25, No 1, (1978).
- [3] Feng Jie Jing, On derivations of Γ -rings; Qu fu shifon Daxue Xuebeo Ziran Kexue Ban 13 No 4, 159 - 161 (1987)
- [4] Soyrk M., The commutativity in prime gamma rings with derivation, Tr. J.Of Math. Vol. 18. 149 - 155, (1994).

Received: May 30, 2006