\( \Gamma - (\sigma, \tau) \) Derivation on Gamma Near Rings

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Abstract. In this paper \( \Gamma - (\sigma, \tau) \) Derivation of Gamma Near Rings is defined. Some results are given about this derivation.

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1. INTRODUCTION

All near-rings considered in this paper are left distributive. \( \Gamma - \) near ring is a triple \((M, +, \Gamma)\) where

(i) \((M, +)\) is a group (Not necessarily Abelian)

(ii) \(\Gamma\) is a non-empty set of binary operators \((M, +, \gamma)\) is near ring for each \(\gamma \in \Gamma\)

(iii) \((x \beta y) \gamma z = x \beta (y \gamma z), \forall x, y, z \in M, \beta, \gamma \in \Gamma\)

For a \(\Gamma - \)near ring \(M\) the set

\[ M_0 = \{ x \in M : 0 \gamma x = 0, \forall \gamma \in \Gamma \} \]

is called the zero-symmetric part of \(M\). If \(M = M_0\) then \(M\) is called zero symmetric. For \(\forall x \in M, u \in U, \gamma \in \Gamma\) if \(x \gamma u \in U\) then \(M\) is said to be left invariant. For \(\forall x \in M, u \in U, \gamma \in \Gamma\) if \(u \gamma x \in U\) then \(M\) is said to be right invariant. If \(U\) is both left and right invariant, we say that \(U\) is invariant.

For \(x \in M\) if \(x \Gamma M \Gamma y = \{0\} \) implies \(x = 0\) or \(y = 0\) then \(M\) is called prime \(\Gamma - \)near ring. If \(M\) and \(M'\) are \(\Gamma - \)near rings then a mapping \(d\), such that \(d : M \rightarrow M'\)

\[ d(x + y) = d(x) + d(y) \text{ and } d(x \gamma y) = d(x) \gamma d(y) \]

is called \(\Gamma - \)near ring homomorphism.
Lemma 3. Let \( \forall x, y \in M \) then for \( x + y - x - y \) we get \( x = y \) and \( y = x \). We can be commuted with \( x, y \) and \( y, x \) will be a zero-symmetric \( x, y \). The symbol \( C \) will denote the multiplicative center of \( M \).

\[
\begin{align*}
C_{(\sigma, \tau)} &= \{ x \in M : x\gamma\sigma(m) = \tau(m)\gamma x \quad \forall m \in M, \gamma \in \Gamma \} \\
\end{align*}
\]

\( \forall x, y \in M, [x, y]_{(\sigma, \tau)} = x\gamma\sigma(y) - \tau(y)\gamma x \) is defined as commutator. \( (x, y) = x + y - x - y \) will denote the additive-group commutator. Other commutators are like this.

\[
\begin{align*}
[x, y]_{(\sigma, \tau)} &= x\sigma(y) - \tau(y)x \\
[x, y]_{\gamma} &= x\gamma y - y\gamma x \\
[x\gamma y, z]_{(\sigma, \tau)} &= x\gamma [y, z]_{(\sigma, \tau)} + [x, \tau(z)]_{\gamma} \gamma y \\
\end{align*}
\]

2. Main Results

Lemma 1. For \( \forall x, y \in M, \gamma, \beta \in \Gamma \) if \( z \in C \) then \( [z\beta x, y]_{(\sigma, \tau)} = z\beta [x, y]_{(\sigma, \tau)} \)

Lemma 2. ([5], Lemma 2) Let \( M \) be a prime \( \Gamma \)-near ring.

(i) If \( z \in C/\{0\} \) then \( z \) is not a zero divisor.

(ii) Let \( z \in C/\{0\} \), be an element such that \( z + z \in C \) Then \( (M, +) \) is Abelian.

(iii) If \( z \in C/\{0\} \) and \( x \) is an element of \( M \) such that \( x\gamma z \in C \) or \( z\gamma x \in C \) then \( x \in C \)

Lemma 3. Let \( M \) be a \( \Gamma \)-near ring and \( d \) be a \( \Gamma - (\sigma, \tau) \) derivation on \( M \). Then for \( \forall x, y, z \in M, \gamma, \mu \in \Gamma \) the followings are satisfied.

(i) \( (x\gamma d(y) + d(x)\gamma y)\mu z = x\gamma d(y)\mu z + d(x)\gamma y\mu z \)

(ii) \( (d(x)\gamma y + x\gamma d(y))\mu z = d(x)\gamma y\mu z + x\gamma d(y)\mu z \)

Lemma 4. Let \( M \) be 2-torsion-free prime and \( d, \Gamma - (\sigma, \tau) \) derivation and \( d \), can be commuted with \( \sigma \) and \( \tau \). If \( d^2 = 0 \) then \( d = 0 \)

Proof. Let \( \forall x, y \in M, \gamma \in \Gamma \). From the hypothesis

\[
\begin{align*}
0 &= d^2(x\gamma y) = d(d(x)\gamma\sigma(y) + \tau(x)\gamma d(y)) \\
&= d^2(x)\gamma\sigma^2(y) + \tau(d(x))\gamma d(\sigma(y)) + d(\tau(x))\gamma\sigma(d(y)) + \tau^2(x)\gamma d^2(y) \\
\end{align*}
\]

We get

\[
\begin{align*}
0 &= \tau(d(x))\gamma d(\sigma(y)) + d(\tau(x))\gamma\sigma(d(y)) \\
&= 2\tau(d(x))\gamma\sigma(d(y)) \\
&= \tau(d(x))\gamma\sigma(d(y)) \\
\end{align*}
\]

Taking \( y\beta z \) instead of \( y \)
\[ 0 = \tau(d(x))\gamma \sigma(d(y\beta z)) = \tau(d(x))\gamma \sigma(d(y)\beta \sigma(z) + \tau(y)\beta d(z)) = \tau(d(x))\gamma \sigma(d(y)\beta \sigma(z)) + \tau(d(x))\gamma \sigma(\tau(y)\beta d(z)) = \tau(d(x))\gamma \sigma(\tau(y)\beta \sigma(d(z))) = \tau(d(x))\Gamma M \Gamma \sigma(d(z)) \]

Since \( M \) is prime then \( d(x) = 0 \) or \( d(z) = 0 \) and as a result we get \( d = 0 \). ▪

**Lemma 5.** Let \( d \) be a \( \Gamma - (\sigma, \tau) \) derivation on \( M \). Suppose that \( u \in M \) is not a left zero divisor. If

\[ [u, d(u)]_{(\sigma, \tau)}^\gamma = 0 \]

for all \( \gamma \in \Gamma \) then \( (x, u) \) is a constant on \( M \) for all \( x \in M \).

**Proof.** Let \( x \in M, \gamma \in \Gamma \). Then

\[ d(u\gamma(u + x)) = d(u)\gamma \sigma(u + x) + \tau(u)\gamma d(u + x) = d(u)\gamma \sigma(u) + d(u)\gamma \sigma(x) + \tau(u)\gamma d(u) + \tau(u)\gamma d(x) \]

and

\[ d(u\gamma u + u\gamma x) = d(u\gamma u) + d(u\gamma x) = d(u)\gamma \sigma(u) + \tau(u)\gamma d(u) + d(u)\gamma \sigma(x) + \tau(u)\gamma d(x) \]

Since the left sides are equal the right sides must be equal

\[ 0 = \tau(u)\gamma d(u) + \tau(u)\gamma d(x) - \tau(u)\gamma d(u) - \tau(u)\gamma d(x) \]

\[ = \tau(u)\gamma d((x, u)) \]

Since \( u \) is not a left zero divisor we have that \( d((x, u)) = 0 \) and \( (x, u) \) is constant. ▪

**Theorem 6.** Let \( d \) be a nontrivial \( \Gamma - (\sigma, \tau) \) derivation on \( M \) and let \( M \) have no nonzero divisor of zero. If for \( x \in M, \gamma \in \Gamma \)

\[ [x, d(x)]_{(\sigma, \tau)}^\gamma = 0 \]

then \( (M, +) \) is Abelian.

**Proof.** For \( a, b \in M \) Let \( c = (a, b) \) be any additive commutator. Then \( c \) is a constant from lemma 5. That is \( d(c) = 0 \) Since \( w\gamma c = (w\gamma a, w\gamma c) \) where \( w \in M \) and \( \gamma \in \Gamma \) then \( w\gamma c \) is constant. Then we get

\[ 0 = d(w\gamma c) = d(w)\gamma \sigma(c) + \tau(w)\gamma d(c) = d(w)\gamma \sigma(c) \]

Since \( d(w) \neq 0 \) and \( M \) have no zero divisor we get \( c = 0 \). So \( (M, +) \) is Abelian. ▪
**Theorem 7.** Let $M$ be prime and let $d$ be a nontrivial $\Gamma - (\sigma, \tau)$ derivation on $M$. If $d(x) \in C$ then $(M, +)$ is Abelian. Furthermore if $M$ is 2-torsion-free then $M$ is commutative.

**Proof.** Let $c$ be an arbitrary constant and $x$ be a nonconstant. Then we get
\[
d(x\gamma c) = d(x)\gamma \sigma(c) + \tau(x)\gamma d(c) = d(x)\gamma \sigma(c) \in C
\]
Since $d(x) \in C/\{0\}$ and from lemma 2(iii) we get $c \in C/\{0\}$. Since $d(c+c) = 0$ from lemma 2(ii) we get $(M, +)$ is Abelian. Let 0 be the only constant. Let $u$ be not a zero divisor. For $x \in M, \gamma \in \Gamma$ we get
\[
d(u\gamma(u + x)) = d(u)\gamma \sigma(u + x) + \tau(u)\gamma d(u + x) = d(u)\gamma \sigma(u) + d(u)\gamma \sigma(x) + \tau(u)\gamma d(u) + \tau(u)\gamma d(x)
\]
and then
\[
d(u\gamma u + u\gamma x) = d(u\gamma u) + d(u\gamma x) = d(u)\gamma \sigma(u) + \tau(u)\gamma d(u) + d(u)\gamma \sigma(x) + \tau(u)\gamma d(x)
\]
From the hypothesis and $d(x) \in C$ we obtain
\[
0 = \tau(u)\gamma(d(x) + d(u) - d(x) - d(u)) = u\gamma(d(x + u - x - u) = u\gamma d((x, u))
\]
Since $u$ is not a zero divisor we get $d((x, u)) = 0$. Then $(x, u) = 0$. That is if $u \in C(M)$ then it is on the center of $(M, +)$.

Now let $x$ be a nonzero element. Since $d(M) \subseteq C$ and from lemma 2(i) $d(x)$ is not a zero divisor. From this $d(x) \in C(M)$. Let $0 \neq y \in M$ then
\[
0 = d(x) + d(y) - d(x) - d(y) = d((x, y))
\]
Since $(x, y) = 0$ then $(M, +)$ is Abelian. Taking $x\beta y$ instead of $x$ in the hypothesis where $\beta \in \Gamma$ we obtain
\[
0 = [d(x\beta y), z]_{(\sigma, \tau)} = [d(x)\beta \sigma(y) + \tau(x)\beta d(y), z]_{(\sigma, \tau)} = [d(x)\beta \sigma(y), z]_{(\sigma, \tau)} + [\tau(x)\beta d(y), z]_{(\sigma, \tau)}
\]
Using commutator properties from lemma 1(i) for $\forall x, y, z \in M, \beta, \gamma \in \Gamma$ we get
\[
d(x)\beta [y, z]_{(\sigma, \tau)} = d(y)\beta [z, x]_{(\sigma, \tau)}
\]
Taking $d(x)$ instead of $x$ in this equality and using the hypothesis we obtain
\[
0 = d^2(x)\beta [y, z]_{(\sigma, \tau)}
\]
Let us suppose that $M$ is not commutative. Choosing $y, z \in M$ such that $[y, z]_{(\sigma, \tau)} \neq 0$. Since a central element $d^2(x)$ can not be a nonzero divisor of...
zero we obtain that \( d^2(x) = 0 \) for all \( x \in M \). Then from lemma 4 \( d = 0 \) is obtained. But this is a contradiction. That is \( M \) is commutative. ■

**Lemma 8.** Let \( M \) be prime and \( x, y \in M \). If \( x \in C \) and \( x\Gamma y = \{0\} \) then \( x = 0 \) or \( y = 0 \).

**Proof.** Obvious. ■

**Theorem 9.** Let \( M \) be prime and let \( d \) be a nontrivial \( \Gamma - (\sigma, \tau) \) derivation on \( M \). If \( [d(x), d(y)]_{(\sigma, \tau)} = 0 \) then \( (M, +) \) is Abelian.

**Proof.** Since \( [d(x), d(y)]_{(\sigma, \tau)} = 0 \) for all \( x, y \in M \) and \( \gamma \in \Gamma \). If both \( w \) and \( w + w \) commute elementwise with \( d(x) \)

\[
0 = [w, d(x + y)]_{(\sigma, \tau)} = w\gamma(d(x, y))
\]

is obtained. Taking \( d(z) \) instead of \( w \) in the last equation we have

\[
0 = d(z)\gamma d(x + y)
\]

for \( \forall x, y, z \in M, \gamma \in \Gamma \).

Substituting \( z\beta v \) for \( z \) in last equation and using lemma 3(i) we obtain

\[
0 = [d(z\beta v), d((x, y))]_{(\sigma, \tau)} = d(z)\beta\sigma(v)\gamma d((x, y)) + \tau(x)\beta d(v)\gamma d(x, y))
\]

By primeness of \( M \) we get \( d((x, y)) = 0 \).

Since \( z\gamma(x, y) \) is also an additive commutator for any \( z \in M \) and \( \gamma \in \Gamma \) we get

\[
0 = d(z\gamma(x, y))
\]

\[
= d(z)\gamma\sigma(x, y) + \tau(z)\gamma d((x, y))
\]

\[
= d(z)\gamma\sigma(x, y)
\]

By primeness of \( M \) we get \( (x, y) = 0 \). This means \( (M, +) \) is Abelian. ■

**Lemma 10.** Let \( M \) be prime and \( U \neq \{0\} \) be a left (resp. right) invariant subset of \( M \). If \( U \subseteq C \) then \( M \) is commutative.

**Proof.** Since a central left invariant subset of \( M \) is a right invariant subset of \( M \), we may assume that \( U \neq \{0\} \) is a right invariant subset of \( M \). Let \( x \in M, u \in U, \gamma \in \Gamma \). From the hypothesis we get that \( [u, x]_{(\sigma, \tau)} = 0 \). Replacing \( u \) by \( u\beta y \) in the previous equation, we obtain

\[
0 = [u\beta y, x]_{(\sigma, \tau)} = u\beta [y, x]_{(\sigma, \tau)}
\]

Thus since \( U \neq \{0\} \) we have \( [y, x]_{(\sigma, \tau)} = 0 \) from lemma 6. This completes the proof. ■

**Theorem 11.** Let \( M \) be prime and \( 2 \)-torsion-free and \( U \neq \{0\} \) be left (resp. right) invariant subset of \( M \) and \( d \) be a \( \Gamma - (\sigma, \tau) \) derivation on \( M \). If \( d(U) \subseteq C \setminus \{0\} \) then \( M \) is commutative.
Proof. Let $x \in M, u \in U, \gamma \in \Gamma$. From the hypothesis $[d(u), x]_{(\sigma, \tau)}^\gamma = 0$. Taking $u\beta u$ instead of $u$ in the previous equation, we have

$$2 [d(u)\beta u, x]_{(\sigma, \tau)}^\gamma$$

Since $M$ is prime 2- torsion free , for $x \in M, u \in U, \gamma \in \Gamma$ we obtain

$$[d(u)\beta u, x]_{(\sigma, \tau)}^\gamma = 0$$

From lemma 2(i) we get

$$d(u)\beta [u, x]_{(\sigma, \tau)}^\gamma = 0$$

From hypothesis, last equation and lemma 6 for $x \in M, u \in U, \gamma \in \Gamma$ we get

$$[u, x]_{(\sigma, \tau)}^\gamma = 0$$

As a result from lemma 7 $M$ is commutative.

References


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