On the instability of positive solution of an elliptic equation

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Abstract

We study the stability of positive stationary solutions of

\[
\begin{align*}
-\Delta u(x) &= \lambda f(x, u), & x &\in \Omega, \\
Bu &= 0, & x &\in \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( Bu(x) = \alpha h(x)u + (1 - \alpha) \frac{\partial u}{\partial n} \) where \( \alpha \in [0, 1] \), \( h : \partial \Omega \rightarrow \mathbb{R}^+ \) with \( h = 1 \) when \( \alpha = 1 \), \( \lambda > 0 \) is a constant and \( f \) is a smooth function such that \( f_{uu}(x, u) > 0 \) for all fixed \( x \in \Omega \), \( u \in \mathbb{R}^+ \), \( f_x(0, 0) = 0 \), \( f(x, u) < 0 \) for \( u \in (0, \beta) \) and \( f(x, u) > 0 \) for \( u > \beta \) for some \( \beta > 0 \) (for all fixed \( x \in \Omega \)). We provide a simple proof to establish that every positive stationary solution is linearly unstable.

Keywords: Instability, linearized equation, positive solutions

Mathematics Subject Classification: 35B35, 35J65

1 Introduction

In this paper, we consider the stability of positive stationary solutions to the elliptic boundary value problem

\[
\begin{align*}
-\Delta u(x) &= \lambda f(x, u), & x &\in \Omega, \\
Bu &= 0, & x &\in \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( Bu(x) = \alpha h(x)u + (1 - \alpha) \frac{\partial u}{\partial n} \) where \( \alpha \in [0, 1] \), \( h : \partial \Omega \rightarrow \mathbb{R}^+ \) is a smooth function with \( h = 1 \) when \( \alpha = 1 \), i.e., the boundary condition may be of Dirichlet, Neumann or mixed type, \( \lambda > 0 \) is a constant and \( f \) is a smooth function satisfying
\[ f_{uu}(x, u) > 0 \text{ for all fixed } x \in \Omega \ (u \in R^+) \text{ and } f_x(0, 0) = 0, \quad (3) \]

and

\[ f(x, u) < 0 \text{ for } u \in (0, \beta) \text{ and } f(x, u) > 0 \text{ for } u > \beta \text{ for some } \beta > 0, \quad (4) \]

for all fixed \( x \in \Omega \), where \( f_u(x, u) \) denotes the partial derivative of \( f(x, u) \) with respect to \( u \).

In the case when \( f(x, u) \equiv f(u) \), was studied by several authors. Brown and his co-authors have altogether proved that if \( f'' > 0 \) and \( f(0) \leq 0 \), then every non-trivial nonnegative solution of the problem

\[
\begin{cases}
-\Delta u = \lambda f(u), & x \in \Omega, \\
 u = 0, & x \in \partial\Omega,
\end{cases}
\quad (5)
\]

is unstable. They first considered the monotone case, i.e., \( f' > 0 \) in [2]. The non-monotone case was first proved by Tertikas [4] using sub- and supersolution. For the case \( f(x, u) \equiv m(x) u(u - 1) \), Afrouzi and Rasouli [1] studied the instability of positive solutions. The purpose of this paper is to extend this study to problem (1). The main result of [2] is summarized in the following theorem.

**Theorem 1.1.** Let \( f : R \rightarrow R \) be a twice continuously differentiable function, then

(i) if \( f'' > 0 \) and \( f(0) \leq 0 \), then every nontrivial nonnegative solution of (5) is unstable. while

(ii) if \( f'' < 0 \) and \( f(0) \geq 0 \), then every nontrivial nonnegative solution of (5) is stable.

We recall that, if \( u \) be any nonnegative solution of

\[
\begin{cases}
-\Delta u = g(x, u), & x \in \Omega, \\
 u = 0, & x \in \partial\Omega,
\end{cases}
\quad (6)
\]

then the linearized equation about \( u \) is

\[
\begin{cases}
-\Delta \phi - g_u(x, u)\phi = \mu \phi, & x \in \Omega, \\
 \phi = 0, & x \in \partial\Omega.
\end{cases}
\quad (7)
\]

**Definition 1.3.** We call a solution \( u \) of (6) a linearly stable solution if all eigenvalues of (7) are strictly positive, which can be inferred if the principal eigenvalue \( \mu_1 > 0 \). Otherwise \( u \) is linearly unstable.
2 Main result

In this section we shall prove the instability of positive stationary solution $u$ by showing that the principal eigenvalue $\mu_1$, of the equation linearized about $u$ is negative; the stability of $u$ then follows from the well-known principle of linearized stability (see [3]). We overcome the difficulty of $f$ being nonmonotone, by re-writing $f$ as the sum of the monotone function and a linear function involving $f(0,0)$ and $f_u(0,0)$. By doing so we arrive at a much simpler proof clearly indicating the role of $f(0,0)$ in establishing the instability result. Our main result is the following theorem.

**Theorem 1.1** : Every positive stationary solution of (1)-(2) is linearly unstable.

**Proof.** Let $g(x,u) = f(x,u) - f(0,0) + |f_u(0,0)|u$. Then $g(x,0) = 0$ for all fixed $x \in \Omega$, $g_u(x,u) = f_u(x,u) + |f_u(0,0)|$, $g_{uu}(x,u) = f_{uu}(x,u) > 0$ for all fixed $x \in \Omega$ ($u \in R^+$) and, therefore $g_u(x,u) > 0$ for all fixed $x \in \Omega$ ($u \in R^+$) and $g(x,u) > 0$ for all fixed $x \in \Omega$ ($u \in R^+$). Now, (1) – (2) can be rewritten as

\[-\Delta u(x) = \lambda \{g(x,u) + f(0,0) - |f_u(0,0)|u\}, \quad x \in \Omega, \quad (8)\]

\[Bu(x) = 0, \quad x \in \partial\Omega. \quad (9)\]

Let $u_0$ be any positive stationary solution of (8) – (9). Then the linearized equation about $u_0$ is

\[-\Delta \phi(x) - \lambda \{g_u(x,u_0) - |f_u(0,0)|\} \phi(x) = \mu \phi(x), \quad x \in \Omega, \quad (10)\]

\[B\phi(x) = 0, \quad x \in \partial\Omega, \quad (11)\]

Let $\mu_1$ be the principal eigenvalue and let $\psi(x) \geq 0$ be a corresponding eigenfunction. Multiplying (8) by $g_u(x,u_0)\psi(x)$ and (10) by $g(x,u_0)$, then subtracting and integrating over $\Omega$, we obtain

\[\int_\Omega \{(-\Delta u_0) g_u(x,u_0) \psi(x) - (-\Delta \psi(x))g(x,u_0) - \lambda f(0,0)g_u(x,u_0)\psi(x) \]

\[+\lambda |f_u(0,0)|u_0\psi(x) - \lambda |f_u(0,0)|g(x,u_0)\psi(x)\}dx \]

\[= -\mu_1 \int_\Omega \psi(x)g(x,u_0)dx. \quad (12)\]
But by Green’s first identity, we have

\[
\int_{\Omega} (-\Delta u_0) g_u(x, u_0) \psi(x) dx = \int_{\Omega} \nabla (g_u(x, u_0) \psi(x)) \nabla u_0(x) dx
\]

\[- \int_{\partial \Omega} g_u(x, u_0) \psi(s)(\frac{\partial u_0}{\partial n}) ds = \int_{\Omega} g_{uu}(x, u_0) \psi(x) |\nabla u_0|^2 dx
\]

\[+ \int_{\Omega} g_u(x, u_0)(\nabla \psi \nabla u_0) dx - \int_{\partial \Omega} g_u(x, u_0) \psi(s) (\frac{\partial u_0}{\partial n}) ds,
\]  

(13)

and

\[
\int_{\Omega} (\Delta \psi(x)) g(x, u_0) dx = - \int_{\Omega} \nabla (g(x, u_0)) \nabla \psi(x) dx + \int_{\partial \Omega} g(x, u_0)(\frac{\partial \psi}{\partial n}) ds
\]

\[= - \int_{\Omega} g_u(x, u_0)(\nabla u_0 \nabla \psi) dx + \int_{\partial \Omega} g(x, u_0)(\frac{\partial \psi}{\partial n}) ds.
\]  

(14)

By using (13) - (14) in (12) we get

\[- \mu_1 \int_{\Omega} \psi(x) g(x, u_0) dx = \int_{\Omega} g_{uu}(x, u_0) \psi(x) |\nabla u_0|^2 dx
\]

\[ - \lambda f(0, 0) \int_{\Omega} g_u(x, u_0) \psi(x) dx + \lambda |f_u(0, 0)| \int_{\Omega} \{g_u(x, u_0) u_0 - g(x, u_0)\} \psi(x) dx
\]

\[+ \int_{\partial \Omega} \{g(x, u_0) (\frac{\partial \psi}{\partial n}) - g_{u}(x, u_0) \psi(s) (\frac{\partial \psi}{\partial n})\} ds.
\]  

(15)

We notice that when \( \alpha = 1 \) (then \( h = 1 \)) we have \( u_0 = 0 \) for \( s \in \partial \Omega \) and, therefore, \( g(x, u_0) = 0 \) for \( s \in \partial \Omega \) and also we have \( \psi = 0 \) for \( s \in \partial \Omega \). Hence,

\[
\int_{\partial \Omega} \{g(x, u_0) (\frac{\partial \psi}{\partial n}) - g_u(x, u_0) \psi(s) (\frac{\partial \psi}{\partial n})\} ds = 0,
\]  

(16)

and when \( \alpha \neq 1 \), we have

\[
\int_{\partial \Omega} \{g(x, u_0) (\frac{\partial \psi}{\partial n}) - g_u(x, u_0) \psi(s) (\frac{\partial \psi}{\partial n})\} ds
\]

\[= \int_{\partial \Omega} \{g(x, u_0) \frac{(-\alpha h(x) \psi(x))}{(1 - \alpha)} - g_u(x, u_0) \psi(s) \frac{(-\alpha h(x) u_0)}{(1 - \alpha)}\} ds
\]
On the instability of positive solution

\[ = \int_{\partial \Omega} \left\{ \frac{\alpha h(x) \psi(s)}{1 - \alpha} \right\} [u_0 g_u(x, u_0) - g(x, u_0)] ds. \]

But \( \alpha \geq 0, \ h > 0, \ \psi \geq 0 \) for \( s \in \partial \Omega \) and \( u_0 g_u(x, u_0) - g(x, u_0) > 0 \) for all fixed \( x \in \Omega \) \((u_0 \in \mathbb{R}^+)\). Therefore, if \( \alpha \neq 1 \)

\[ \int_{\partial \Omega} \left\{ \frac{\alpha h(x) \psi(s)}{1 - \alpha} \right\} [u_0 g_u(x, u_0) - g(x, u_0)] ds \geq 0. \tag{17} \]

Also, since \( g_{uu}(x, u_0) > 0 \) for all fixed \( x \in \Omega \) \((u_0 \in \mathbb{R}^+)\), we get

\[ \int_\Omega g_{uu}(x, u_0) \psi(x) |\nabla u_0|^2 dx > 0. \tag{18} \]

Thus, by (16) – (18) we have

\[ -\mu_1 \int_\Omega \psi(x) g(x, u_0) dx > -\lambda f(0, 0) \int_\Omega g_u(x, u_0) \psi(x) dx \]

\[ + \lambda |f_u(0, 0)| \int_\Omega \{g_u(x, u_0) u_0 - g(x, u_0)\} \psi(x) dx. \tag{19} \]

Now by using (3) – (4) and the fact that \( g_u(x, u_0) u_0 - g(x, u_0) > 0 \) for all fixed \( x \in \Omega \) \((u_0 \in \mathbb{R}^+)\) in (19) it is easy to see that

\[ -\mu_1 \int_\Omega \psi(x) g(x, u_0) dx > 0. \]

But \( \psi > 0 \) for \( x \in \Omega \) and \( g(x, u_0) > 0 \) for all fixed \( x \in \Omega \) \((u_0 \in \mathbb{R}^+)\) and hence, \( \mu_1 < 0 \) and the result follows (see [3]). \( \square \)

References


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