On the instability of positive solution of an elliptic equation

G. A. Afrouzi and S. H. Rasouli

Department of Mathematics, Faculty of Basic Sciences
Mazandaran University, Babolsar, Iran
afrouzi@umz.ac.ir

Abstract

We study the stability of positive stationary solutions of

$$\begin{cases} -\Delta u(x) = \lambda f(x, u), & x \in \Omega, \\ Bu = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in R^n with smooth boundary $Bu(x) = \alpha h(x)u + (1 - \alpha)\frac{\partial u}{\partial n}$ where $\alpha \in [0, 1]$, $h : \partial\Omega \rightarrow R^+$ with $h = 1$ when $\alpha = 1$, $\lambda > 0$, f is a smooth function such that $f_{uu}(x, u) > 0$ for all fixed $x \in \Omega$ ($u \in R^+$), $f_x(0, 0) = 0$, $f(x, u) < 0$ for $u \in (0, \beta)$ and $f(x, u) > 0$ for $u > \beta$ for some $\beta > 0$ (for all fixed $x \in \Omega$). We provide a simple proof to establish that every positive stationary solution is linearly unstable.

Keywords: Instability, linearized equation, positive solutions

Mathematics Subject Classification: 35B35, 35J65

1 Introduction

In this paper, we consider the stability of positive stationary solutions to the elliptic boundary value problem

$$-\Delta u(x) = \lambda f(x, u), \quad x \in \Omega, \quad (1)$$

$$Bu = 0, \quad x \in \partial\Omega, \quad (2)$$

where Ω is a bounded domain in R^n with smooth boundary $Bu(x) = \alpha h(x)u + (1 - \alpha)\frac{\partial u}{\partial n}$ where $\alpha \in [0, 1]$, $h : \partial\Omega \rightarrow R^+$ is a smooth function with $h = 1$ when $\alpha = 1$, i.e., the boundary condition may be of Dirichlet, Neumann or mixed type, $\lambda > 0$ is a constant and f is a smooth function satisfying

$$f_{uu}(x, u) > 0 \text{ for all fixed } x \in \Omega \text{ (} u \in R^+ \text{) and } f_x(0, 0) = 0, \quad (3)$$

and

$$f(x, u) < 0 \text{ for } u \in (0, \beta) \text{ and } f(x, u) > 0 \text{ for } u > \beta \text{ for some } \beta > 0, \quad (4)$$

for all fixed $x \in \Omega$, where $f_u(x, u)$ denotes the partial derivative of $f(x, u)$ with respect to u .

In the case when $f(x, u) \equiv f(u)$, was studied by several authors. Brown and his co-authors have altogether proved that if $f'' > 0$ and $f(0) \leq 0$, then every non-trivial nonnegative solution of the problem

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (5)$$

is unstable. They first considered the monotone case, i.e., $f' > 0$ in [2]. The non-monotone case was first proved by Tertikas [4] using sub- and supersolution. For the case $f(x, u) \equiv m(x)u(u-1)$, Afrouzi and Rasouli [1] studied the instability of positive solutions. The purpose of this paper is to extend this study to problem (1). The main result of [2] is summarized in the following theorem.

Theorem 1.1. Let $f : R \rightarrow R$ be a twice continuously differentiable function, then

- (i) if $f'' > 0$ and $f(0) \leq 0$, then every nontrivial nonnegative solution of (5) is unstable. while
- (ii) if $f'' < 0$ and $f(0) \geq 0$, then every nontrivial nonnegative solution of (5) is stable.

We recall that, if u be any nonnegative solution of

$$\begin{cases} -\Delta u = g(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (6)$$

then the linearized equation about u is

$$\begin{cases} -\Delta \phi - g_u(x, u)\phi = \mu\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \quad (7)$$

Definition 1.3. We call a solution u of (6) a linearly stable solution if all eigenvalues of (7) are strictly positive, which can be inferred if the principal eigenvalue $\mu_1 > 0$. Otherwise u is linearly unstable.

2 Main result

In this section we shall prove the instability of positive stationary solution u by showing that the principal eigenvalue μ_1 , of the equation linearized about u is negative; the stability of u then follows from the well-known principle of linearized stability (see [3]). We overcome the difficulty of f being nonmonotone, by re-writing f as the sum of the monotone function and a linear function involving $f(0, 0)$ and $f_u(0, 0)$. By doing so we arrive at a much simpler proof clearly indicating the role of $f(0, 0)$ in establishing the instability result. Our main result is the following theorem.

Theorem 1.1 : Every positive stationary solution of (1)-(2) is linearly unstable.

Proof. Let $g(x, u) = f(x, u) - f(0, 0) + |f_u(0, 0)|u$. Then $g(x, 0) = 0$ for all fixed $x \in \Omega$, $g_u(x, u) = f_u(x, u) + |f_u(0, 0)|$, $g_{uu}(x, u) = f_{uu}(x, u) > 0$ for all fixed $x \in \Omega$ ($u \in R^+$) and, therefore $g_u(x, u) > 0$ for all fixed $x \in \Omega$ ($u \in R^+$) and $g(x, u) > 0$ for all fixed $x \in \Omega$ ($u \in R^+$). Now, (1) – (2) can be rewritten as

$$-\Delta u(x) = \lambda\{g(x, u) + f(0, 0) - |f_u(0, 0)|u\}, \quad x \in \Omega, \quad (8)$$

$$Bu(x) = 0, \quad x \in \partial\Omega. \quad (9)$$

Let u_0 be any positive stationary solution of (8) – (9). Then the linearized equation about u_0 is

$$-\Delta\phi(x) - \lambda\{g_u(x, u_0) - |f_u(0, 0)|\}\phi(x) = \mu\phi(x), \quad x \in \Omega, \quad (10)$$

$$B\phi(x) = 0, \quad x \in \partial\Omega, \quad (11)$$

Let μ_1 be the principal eigenvalue and let $\psi(x) (\geq 0)$ be a corresponding eigenfunction. Multiplying (8) by $g_u(x, u_0)\psi(x)$ and (10) by $g(x, u_0)$, then subtracting and integrating over Ω , we obtain

$$\begin{aligned} & \int_{\Omega} \{(-\Delta u_0) g_u(x, u_0) \psi(x) - (-\Delta \psi(x)) g(x, u_0) - \lambda f(0, 0) g_u(x, u_0) \psi(x) \\ & \quad + \lambda |f_u(0, 0)| u_0 \psi(x) - \lambda |f_u(0, 0)| g(x, u_0) \psi(x)\} dx \\ & = -\mu_1 \int_{\Omega} \psi(x) g(x, u_0) dx. \end{aligned} \quad (12)$$

But by Green's first identity, we have

$$\begin{aligned}
\int_{\Omega} (-\Delta u_0) g_u(x, u_0) \psi(x) dx &= \int_{\Omega} \nabla(g_u(x, u_0) \psi(x)) \nabla u_0(x) dx \\
&- \int_{\partial\Omega} g_u(x, u_0) \psi(s) \left(\frac{\partial u_0}{\partial n}\right) ds = \int_{\Omega} g_{uu}(x, u_0) \psi(x) |\nabla u_0|^2 dx \\
&+ \int_{\Omega} g_u(x, u_0) (\nabla \psi \nabla u_0) dx - \int_{\partial\Omega} g_u(x, u_0) \psi(s) \left(\frac{\partial u_0}{\partial n}\right) ds, \tag{13}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega} (\Delta \psi(x)) g(x, u_0) dx &= - \int_{\Omega} \nabla(g(x, u_0)) \nabla \psi(x) dx + \int_{\partial\Omega} g(x, u_0) \left(\frac{\partial \psi}{\partial n}\right) ds \\
&= - \int_{\Omega} g_u(x, u_0) (\nabla u_0 \nabla \psi) dx + \int_{\partial\Omega} g(x, u_0) \left(\frac{\partial \psi}{\partial n}\right) ds. \tag{14}
\end{aligned}$$

By using (13) – (14) in (12) we get

$$\begin{aligned}
-\mu_1 \int_{\Omega} \psi(x) g(x, u_0) dx &= \int_{\Omega} g_{uu}(x, u_0) \psi(x) |\nabla u_0|^2 dx \\
-\lambda f(0, 0) \int_{\Omega} g_u(x, u_0) \psi(x) dx &+ \lambda |f_u(0, 0)| \int_{\Omega} \{g_u(x, u_0) u_0 - g(x, u_0)\} \psi(x) dx \\
&+ \int_{\partial\Omega} \{g(x, u_0) \left(\frac{\partial \psi}{\partial n}\right) - g_u(x, u_0) \psi(s) \left(\frac{\partial \psi}{\partial n}\right)\} ds. \tag{15}
\end{aligned}$$

We notice that when $\alpha = 1$ (then $h = 1$) we have $u_0 = 0$ for $s \in \partial\Omega$ and, therefore, $g(x, u_0) = 0$ for $s \in \partial\Omega$ and also we have $\psi = 0$ for $s \in \partial\Omega$. Hence,

$$\int_{\partial\Omega} \{g(x, u_0) \left(\frac{\partial \psi}{\partial n}\right) - g_u(x, u_0) \psi(s) \left(\frac{\partial \psi}{\partial n}\right)\} ds = 0, \tag{16}$$

and when $\alpha \neq 1$, we have

$$\begin{aligned}
&\int_{\partial\Omega} \{g(x, u_0) \left(\frac{\partial \psi}{\partial n}\right) - g_u(x, u_0) \psi(s) \left(\frac{\partial \psi}{\partial n}\right)\} ds \\
&= \int_{\partial\Omega} \{g(x, u_0) \left(\frac{(-\alpha h(x) \psi(x))}{(1 - \alpha)}\right) - g_u(x, u_0) \psi(s) \left(\frac{(-\alpha h(x) u_0)}{(1 - \alpha)}\right)\} ds
\end{aligned}$$

$$= \int_{\partial\Omega} \left\{ \frac{\alpha h(x)\psi(s)}{(1-\alpha)} \right\} [u_0 g_u(x, u_0) - g(x, u_0)] ds.$$

But $\alpha \geq 0$, $h > 0$, $\psi \geq 0$ for $s \in \partial\Omega$ and $u_0 g_u(x, u_0) - g(x, u_0) > 0$ for all fixed $x \in \Omega$ ($u_0 \in R^+$). Therefore, if $\alpha \neq 1$

$$\int_{\partial\Omega} \left\{ \frac{\alpha h(x)\psi(s)}{(1-\alpha)} \right\} [u_0 g_u(x, u_0) - g(x, u_0)] ds \geq 0. \quad (17)$$

Also, since $g_{uu}(x, u_0) > 0$ for all fixed $x \in \Omega$ ($u_0 \in R^+$), we get

$$\int_{\Omega} g_{uu}(x, u_0) \psi(x) |\nabla u_0|^2 dx > 0. \quad (18)$$

Thus, by (16) – (18) we have

$$\begin{aligned} -\mu_1 \int_{\Omega} \psi(x) g(x, u_0) dx &> -\lambda f(0, 0) \int_{\Omega} g_u(x, u_0) \psi(x) dx \\ &+ \lambda |f_u(0, 0)| \int_{\Omega} \{g_u(x, u_0) u_0 - g(x, u_0)\} \psi(x) dx. \end{aligned} \quad (19)$$

Now by using (3) – (4) and the fact that $g_u(x, u_0) u_0 - g(x, u_0) > 0$ for all fixed $x \in \Omega$ ($u_0 \in R^+$) in (19) it is easy to see that

$$-\mu_1 \int_{\Omega} \psi(x) g(x, u_0) dx > 0.$$

But $\psi > 0$ for $x \in \Omega$ and $g(x, u_0) > 0$ for all fixed $x \in \Omega$ ($u_0 \in R^+$) and hence, $\mu_1 < 0$ and the result follows (see [3]). $\square$

References

- [1] G.A. Afrouzi, S.H. Rasouli. Instability of nonnegative solutions for a boundary value problem with indefinite weight function. Global. J. Pure. Appl. Math. Vol 1, no. 1, (2005), pp. 9-12.
- [2] K.J. Brown, R. Shvaji. Instability of nonnegative solution for a class of semipositone problems. Proc. Amer. Math. Soc., 112 (1) (1991), pp. 121-124.

- [3] D.H. Sattinger. Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J., 21 (11)(1972), pp. 979-1000.
- [4] A. Tertikas. Stability and instability of positive solutions of semi-positone problems, Proc. AMS 114(4)(1992), pp. 1035-1040.

Received: March 1, 2006