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On a Dirichlet problem involving p-Laplacian

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Abstract

In this paper, the existence of at least three weak solutions for Dirichlet problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Omega \subset R^N (N \geq 1)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$ and $f : \Omega \times R \to R$ is a L^1-Caratheodory function, is established. The approach is based on variational methods and critical points.

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1 Introduction

We consider the boundary value problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Omega \subset R^N (N \geq 1)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$ and

$f : \Omega \times R \to R$ is a L^1-Caratheodory function.
In this paper, under novel assumptions, we are interested in ensuring the existence of at least three weak solutions for the problem (1). As usual, a weak solution of (1) is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \lambda \int_\Omega f(x,u(x))v(x)dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$.

Multiplicity results for problem (1) have been broadly investigated in recent years (see, for example, [1,2,4,5]); for instance, in [1], using variational methods, the authors ensure the existence of a sequence of arbitrarily small positive solutions for problem (1) when the function f has a suitable oscillating behaviour at zero.

Also, In [4], the author proves multiplicity results for the problem

$$\begin{cases} u'' + \lambda f(u) = 0, \\ u(0) = u(1) = 0, \end{cases} \tag{2}$$

which for each $\lambda \in [0, +\infty[$, admits at least three solutions in $W_0^{1,2}([0,1])$ when f is a continuous function.

In this paper, Theorem 2.3 which is our main result, under novel assumptions ensures the existence of an open interval $\Lambda \subseteq [0, \infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three weak solutions whose norms in $W_0^{1,p}(\Omega)$ are less than q.

Our approach is based on a three critical points Theorem proved in [7].

2 Main Results

First we recall its equivalent formulation [2, Theorem 1.1 and Remark 1.1] which equivalent with three critical points Theorem:

Theorem 2.1 *Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow R$ a continuously Gateâux differentiable and sequentially weakly lower semicontinuous functional whose Gateâux derivative admits a continuous inverse on X^*; $\Psi : X \longrightarrow R$ a continuously Gateâux differentiable functional whose Gateâux derivative is compact.*
Assume that

$$\lim_{||u|| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$$

for all $\lambda \in [0, +\infty[$, and that there exists $\rho \in R$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ *and a positive real number* q *such that , for each* $\lambda \in \Lambda$*, the equation*

$$\Phi'(u) + \lambda \Psi'(u) = 0$$

has at least three solutions in X *whose norms are less than* q*.*

Here and in the sequel, X will denote the Sobolev space $W_0^{1,p}(\Omega)$ with the norm

$$|| u ||= \left(\int_\Omega |\nabla u(x)|^p dx \right)^{1/p},$$

and put

$$g(x,t) = \int_0^t f(x,\xi) d\xi$$

for each $(x,t) \in \Omega \times R$.

Now, fix $x^0 \in \Omega$ and pick r_1, r_2 with $o < r_1 < r_2$ such that

$$S(x^0, r_1) \subset S(x^0, r_2) \subseteq \Omega.$$

Put

$$k_1 = \frac{1}{r_2 - r_1}((r_2^N - r_1^N)\frac{\pi^{N/2}}{\Gamma(1 + N/2)})^{1/p} cm(\Omega)^{\frac{1}{N} - \frac{1}{p}} \tag{3}$$

where Γ denotes the Gamma function, $c = c(N,p)$ is a positive constant and $m(\Omega)$ is the Lebesgue measure of the set Ω.

Our main results fully depend on the following Lemma:

Lemma 2.2 *Assume that there exist two positive constants* θ *and* τ *with* $k_1\tau > \theta$*, such that*

(i) $g(x,t) \geq 0$ *for each* $(x,t) \in \Omega \times [0, \tau]$,
(ii) $m(\Omega)(k_1\tau)^p \max_{(x,t) \in \overline{\Omega} \times [-\theta, \theta]} g(x,t) < \theta^p \int_{S(x^0, r_1)} g(x, \tau) dx$,

where k_1 *is given in (3).*
Then, there exist $r > 0$ *and* $w \in X$ *such that* $||w||^p > pr$ *and*

$$m(\Omega) \max g(x,t) < pr \frac{\int_\Omega g(x, w(x)) dx}{||w||^p}$$

where $(x,t) \in \overline{\Omega} \times [-c|\Omega|^{\frac{1}{N} - \frac{1}{p}} \sqrt[p]{pr}, c|\Omega|^{\frac{1}{N} - \frac{1}{p}} \sqrt[p]{pr}]$.
Proof: *We put*

$$w(x) = \begin{cases} 0 & , x \in \Omega \setminus S(x^0, r_2) \\ \frac{\tau}{r_2 - r_1}[r_2 - \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}\ ] & , x \in S(x^0, r_2) \setminus S(x^0, r_1) \\ \tau & , x \in S(x^0, r_1) \end{cases}$$

and $r = \frac{\theta^p}{pc^p m(\Omega)^{\frac{p}{N}-1}}$. *It is easy to see that* $w \in X$ *and, in particular, one has*

$$||w||^p = (r_2^N - r_1^N)\frac{\pi^{N/2}}{\Gamma(1+N/2)}(\frac{\tau}{r_2 - r_1})^p.$$

Hence, taking into account that $k_1\tau > \theta$, *one has*

$$pr < ||w||^p.$$

Since $0 \leq w(x) \leq \tau$ *for each* $x \in \Omega$, *condition (i) ensures that*

$$\int_{S(x^0,r_2)\setminus S(x^0,r_1)} g(x,w(x))dx + \int_{\Omega\setminus S(x^0,r_2)} g(x,w(x))dx \geq 0.$$

Moreover, owing to our assumptions, we have

$$m(\Omega)\max g(x,t) < (\frac{\theta}{k_1\tau})^p \int_{S(x^0,r_1)} g(x,\tau)dx \leq pr\frac{\int_\Omega g(x,w(x))dx}{||w||^p}$$

where $(x,t) \in \overline{\Omega} \times [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}]$.
So, the Proof is complete.

Now, we state our main result:

Theorem 2.3 *Assume that there exist three positive constants* θ, τ, γ *with* $k_1\tau > \theta$, $\gamma < p$ *and a positive function* $a \in L^1(\Omega)$ *such that*

(i) $g(x,t) \geq 0$ *for each* $(x,t) \in \Omega \times [0,\tau]$,
(ii) $m(\Omega)(k_1\tau)^p \max_{(x,t)\in\overline{\Omega}\times[-\theta,\theta]} g(x,t) < \theta^p \int_{S(x^0,r_1)} g(x,\tau)dx$,
(iii) $g(x,t) \leq a(x)(1+|t|^\gamma)$ *almost everywhere in* Ω *and for each* $t \in R$

where k_1 *is given in (3).*
Then, there exists an open interval $\Lambda \subseteq [0,+\infty[$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$, *problem* (1) *admits at least three solutions in* X *whose norms are less than* q.

Proof: *For each* $u \in X$, *we put*

$\Phi(u) = \frac{||u||^p}{p}$, $\quad \Psi(u) = -\int_\Omega g(x,u(x))dx \quad$ *and* $\quad J(u) = \Phi(u) + \lambda\Psi(u)$.

In particular, for each $u, v \in X$ *one has*

$$\Phi'(u)(v) = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx,$$

$$\Psi'(u)(v) = -\int_\Omega f(x,u(x))v(x)dx.$$

It is well known that the critical points of J are the weak solutions of (1), *our goal is to prove that Φ and Ψ satisfy the assumptions of Theorem 2.1. Clearly, Φ is a continuously Gateâux differentiable and sequentially weakly lower semi continuous functional whose Gateâux derivative admits a continuous inverse on X^* and Ψ is a continuously Gateâux differentiable functional whose Gateâux derivative is compact.*
Thanks to (ii), for each $\lambda > 0$ one has that

$$\lim_{||u||\to+\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty.$$

We claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < r\frac{-(\Psi(w))}{\Phi(w)}.$$

Now, taking into account that for every $u \in X$, one has

$$\sup_{x\in\Omega}|u(x)| \leq cm(\Omega)^{\frac{1}{N}-\frac{1}{p}}||u||$$

for each $u \in X$, it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) = \sup_{||u||^p\leq pr}\int_\Omega g(x,u(x))dx \leq m(\Omega)\max g(x,t)$$

where $(x,t) \in \overline{\Omega} \times [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}]$.
Now, thanks to Lemma 2.2, there exist $r > 0$ and $w \in X$ such that

$$m(\Omega)\max g(x,t) < pr\frac{\int_\Omega g(x,w(x))dx}{||w||^p}$$

where $(x,t) \in \overline{\Omega} \times [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}]$.
So

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < r\frac{-(\Psi(w))}{\Phi(w)}.$$

Fix ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < \rho < r\frac{-(\Psi(w))}{\Phi(w)},$$

from Proposition 3.1 of [6], we obtain

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Now, our conclusion follows from Theorem 2.1.

3 Two Corollary of Theorem 2.3

In this section, we put

$$k_2 = \frac{1}{r_2 - r_1}(\frac{r_2^N - r_1^N}{r_1^N})^{1/p} cm(\Omega)^{\frac{1}{N}-\frac{1}{p}}. \tag{4}$$

Then, with use Theorem 2.3, we have the following result:

Corollary 3.1 *Let $f : R \to R$ be a continuous function. Put $g(t) = \int_0^t f(\xi)d\xi$ for each $t \in R$ and assume that there exist four positive constants θ, τ, γ and σ with $k_1\tau > \theta$, $\gamma < p$ such that*

(j) $g(t) \geq 0$ for each $t \in [0, \tau]$,
(jj) $(k_2\tau)^p \max_{t\in[-\theta,\theta]} g(t) < \theta^p g(\tau)$,
(jjj) $g(t) \leq \sigma(1 + |t|^\gamma)$ for each $t \in R$,

where k_1 and k_2 are given in (3) and (4).
Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem

$$\begin{cases} \Delta_p u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

admits at least three solutions in X whose norms are less than q.

Proof: *From (jj) and since $\int_{S(x^0,r_1)} g(\tau)dx = r_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)} g(\tau)$, we have*

$$\max_{t\in[-\theta,\theta]} g(t) < \frac{\theta^p}{(\frac{\tau}{r_2-r_1})^p (r_2^N - r_1^N) c^p m(\Omega)^{\frac{p}{N}-1} \frac{\pi^{N/2}}{\Gamma(1+N/2)}} r_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)} g(\tau)$$
$$= (\frac{\theta}{k_1\tau})^p \int_{S(x^0,r_1)} g(\tau)dx.$$

Now, our conclusion follows from Theorem 2.3.

If $f : R \to R$ was a positive continuous function. Then we have:

Corollary 3.2 *Assume that there exist four positive constants θ, τ, γ and σ with $k_1\tau > \theta$, $\gamma < p$ such that*

(j′) $(k_2\tau)^p g(\theta) < \theta^p g(\tau)$,
(jj′) $g(t) \leq \sigma(1 + |t|^\gamma)$ for each $t \in R$,

where k_1 and k_2 are given in (3) and (4).
Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (5) admits at least three solutions in X whose norms are less than q

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