Adomian decomposition method for nonlinear reaction diffusion system of Lotka-Volterra type

M. Alabdullatif and H. A. Abdusalam, E.S. Fahmy

1Mathematics Department, Faculty of Science
Cairo University, Giza, Egypt
hosny@operamail.com

2King Saud University
Women Students Medical Studies & Sciences Sections
Mathematics Department, P. O. Box 22452, Riyadh 11495
Saudi Arabia
esfahmy@operamail.com

3King Saud University
College of Science, Mathematics Department
P.O. Box 2455, Riyadh 11451
Saudi Arabia
mosaadal@ksu.edu.sa

Abstract

In this paper the Adomian decomposition method is used to find an analytic approximate solution for nonlinear reaction diffusion system of Lotka-Volterra type. The results obtained indicate that the method is efficient and accurate.

Mathematics Subject Classification: 35D05, 35J70, 35R35, 49J40
Keywords: Reaction diffusion system, decomposition method, Numerical solution, Lotka-Volterra system

1 Introduction

Recently, some new methods such as extended tanh function method [5,12] extended Jacobi elliptic function expansion method [7], and the simplest equation method [8] have been used in literature to find exact solutions for both partial differential equations and system of partial differential equations. By
using the Lie symmetry analysis [4], the exact solution of the nonlinear reaction differential system of the form:

\[ u_t = (D(u)u_x)_x + f(u, v), \quad v_t = (D(v)v_x)_x + g(u, v) \]  
(1)

is obtained for particular choices of the functions \( f(u, v) \) and \( g(u, v) \). It is known that this system generalizes a great number of the well-known nonlinear second order evolution systems describing various processes in Physics, Chemistry, and Biology [10,9,11,6]. In this paper we concentrate our work to the following particular case of system (1) which is of Lotka-Volterra type [10] that takes the form:

\[ u_t = (uu_x)_x + u(a_1 + b_1 u) + h_1 + c_1 v, \]
\[ v_t = (vv_x)_x + v(a_2 + b_2 v) + h_2 + c_2 u, \]  
(2)

\( a_1, a_2, b_1, b_2, c_1, c_2, h_1, h_2 \) are arbitrary constants such that \( b_1 b_2 \neq 0 \) and \( c_1 c_2 \neq 0 \), i.e. system (2) contains quadratic nonlinearity in reaction terms and the two equations are coupled.

A periodic exact solution of system (2) was obtained in [4] and is given by:

\[ u = \varphi_0(t) \pm (\varphi_0(t) + \frac{2a_1}{b}) \cos \left( \sqrt{\frac{2}{\beta}} \frac{c}{c_1} \left( \frac{a}{b} \right) \sin \left( \sqrt{\frac{2}{\beta}} x \mp \left| c_1 \right| t - \beta_0 \right) \right), \]
\[ v = \varphi_0(t) + \frac{4a_1}{b} \pm \frac{\left| c_1 \right|}{c_1} (\varphi_0(t) + \frac{2a_1}{b}) \sin \left( \sqrt{\frac{2}{\beta}} x \mp \left| c_1 \right| t - \beta_0 \right), \]  
(3)

where

\[ \varphi_0(t) = \frac{1}{3b} \left\{ \frac{2}{2} t_0 - t - 6c_1, \quad (t_0 - t) - a_1 - 3c_1, \quad a_1 \neq 3c_1, \right. \]
\[ b_1, b_2, c_1, c_2, h_1, h_2, \beta_0 \text{ and } t_0 \text{ are arbitrary constants such that, } b = b_1 = b_2 > 0, \]
\[ a_2 = a_1 - 6c_1, \quad c_2 = -c_1, \quad h_1 = (2c_1 a_1 - 6c_1^2)/b, \quad \text{and } h_2 = h_1 + \frac{4a_1}{b}(3c_1 - a_1). \]

The Adomian decomposition method have been shown to solve easily and more accurately a large class of system of partial differential equations with approximates that converges rapidly to accurate solutions [1,2,3,13]. The implementation of the method has shown reliable results in that few terms are needed to obtain either exact solution or to find an approximate solution of a reasonable degree of accuracy in real physical models. Moreover, no linearization or perturbation is required in the method.

The paper is organized as follows: In Section 2, we present the analysis of the Adomian decomposition method applied to nonlinear coupled system. In Section 3, we apply the Adomian decomposition method to obtain analytic approximate solution for system (2) and the numerical experiment is introduced to obtain the results for comparison purposes. Section 4 is devoted for the conclusions.
2 The analysis of the Adomian decomposition method

In this Section, we introduce the main steps of the Adomian decomposition method [3]. We define the linear operator

\[ L_t = \frac{\partial}{\partial t} \quad \text{and} \quad L_t^{-1} = \int_0^t (.) dt, \]  

(5)

where \( L_t^{-1} \) is the inverse operator of \( L_t \).

Using (5), system (2) can be written as:

\[ L_t u = (uu_x)_x + u(a_1 + b_1u) + h_1 + c_1v, \]
\[ L_t v = (vv_x)_x + v(a_2 + b_2v) + h_2 + c_2u. \]  

(6)

Applying the inverse operator to both sides of the above system, we get

\[ u(x, t) = f(x) + L_t^{-1} [a_1u + c_1v + h_1 + F(u)], \]
\[ v(x, t) = g(x) + L_t^{-1} [a_2v + c_2u + h_2 + G(v)], \]  

(7)

where \( F(u) = (uu_x)_x + b_1u^2, \quad G(v) = (vv_x)_x + b_2v^2. \)  

(8)

are the nonlinear terms in (6), \( u(x, 0) = f(x) \) and \( v(x, 0) = g(x) \).

According to the decomposition method [3], we assume that a series solution of the unknown functions \( u(x,t) \) and \( v(x,t) \) are given by

\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \quad v(x, t) = \sum_{n=0}^{\infty} v_n(x, t). \]  

(9)

The nonlinear terms \( F(u) \) and \( G(v) \) can be decomposed into the infinite series of polynomials given as

\[ F(x, t) = \sum_{n=0}^{\infty} A_n, \quad G(x, t) = \sum_{n=0}^{\infty} B_n. \]  

(10)

where the components \( u_n(x,t) \) and \( v_n(x,t) \) will be determined recursively, and the \( A_n \)'s, \( B_n \)'s are the so called Adomian Polynomials of \( u_n \)'s and \( v_n \)'s respectively.

Specific algorithms were set in [3,13] for calculating Adomian’s polynomials for nonlinear terms.

\[ A_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} F \left( \sum_{k=0}^{n} \lambda^k u_k \right) \right]_{\lambda=0}, \quad n \geq 0, \]  

(11)

\[ B_n(v_0, v_1, \cdots, v_n) = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} G \left( \sum_{k=0}^{n} \lambda^k v_k \right) \right]_{\lambda=0}, \quad n \geq 0. \]  

(12)
Thus,

\[
A_0 = bu_0^2 + u_0u_{0xx} + (u^2)_x,
A_1 = bu_0u_1 + u_1u_{0xx} + u_0u_{1xx} + 2u_0u_{1x},
A_2 = \frac{1}{2} \left[ b(2u_1^2 + 4u_0u_2) + 2u_2u_{0xx} + u_1u_{1xx} + 2u_0u_{2xx} + 2(u_{1x})^2 + 4u_0u_{2x} \right],
A_3 = \frac{1}{6} \left[ b(12u_1u_2 + 12u_0u_3) + 6u_3u_{0xx} + 6u_2u_{1xx} + 6u_1u_{2xx} + 6u_0u_{3xx} + 12u_1u_{2x} + 12u_0u_{3x} \right], (13)
\]

and

\[
B_0 = bv_0^2 + v_0v_{0xx} + (v^2)_x,
B_1 = bv_0v_1 + v_1v_{0xx} + v_0v_{1xx} + 2v_0v_{1x},
B_2 = \frac{1}{2} \left[ b(2v_1^2 + 4v_0v_2) + 2v_2v_{0xx} + v_1v_{1xx} + 2v_0v_{2xx} + 2(v_{1x})^2 + 4v_0v_{2x} \right],
B_3 = \frac{1}{6} \left[ b(12v_1v_2 + 12v_0v_3) + 6v_3v_{0xx} + 6v_2v_{1xx} + 6v_1v_{2xx} + 6v_0v_{3xx} + 12v_1v_{2x} + 12v_0v_{3x} \right], (14)
\]

and so on. The components \( u_n \) and \( v_n \) for \( n \geq 0 \) are given by the following recursive relationships:

\[
\begin{align*}
 u_0 &= u(x, 0) = f(x), \\
 v_0 &= v(x, 0) = g(x), \\
 u_1 &= L_t^{-1} [a_1u_0 + c_1v_0 + h_1 + A_0], \\
 v_1 &= L_t^{-1} [a_2v_0 + c_2u_0 + h_2 + B_0], \\
 &\vdots \\
 u_{n+1} &= L_t^{-1} [a_1u_n + c_1v_n + h_1 + A_n], \quad n \geq 0, \\
 v_{n+1} &= L_t^{-1} [a_2v_n + c_2u_n + h_2 + B_n], \quad n \geq 0. (15)
\end{align*}
\]

Using the above recursive relationships, we construct the solutions \( u(x, t) \) and \( v(x, t) \) as

\[
\begin{align*}
 u(x, t) &= \lim_{n \to \infty} \psi_n(x, t), \quad v(x, t) = \lim_{n \to \infty} \varphi_n(x, t), (16)
\end{align*}
\]
where

\[
\psi_n(x, t) = \sum_{i=0}^{n-1} u_i(x, t), \quad \varphi_n(x, t) = \sum_{i=0}^{n-1} v_i(x, t), \quad n \geq 1. \tag{17}
\]

It is interesting to note that, we obtain the series solution by using the initial conditions only.

3 The test problem for the Adomian decomposition method

The main purpose of the work reported in this section is the testing of the Adomian decomposition method based on the method introduced in Section 2. The initial conditions can be obtained from the exact solution (3) and (4). Applying the recurrence relation (15), we get the following:

Case 1: For \(a_1 = 3c_1, \varphi_0(t) = \frac{1}{3b}(\frac{2}{t_0} - 6c_1), a_2 = a_1 - 6c_1, h_1 = h_2 = 0, c_1 = 1, b_1 = b_2 = b, c_2 = -1,\)

\[
u_0 = \frac{1}{3b}\left(\frac{2}{t_0} - 6\right) + \left(\frac{1}{3b}\left(\frac{2}{t_0} - 6\right) + \frac{2}{b}\right) \cos\left(\sqrt{\frac{b}{2}} x - \beta_0\right), \tag{18}
\]

\[
\nu_0 = \frac{1}{3b}\left(\frac{2}{t_0} - 6\right) + \left(\frac{1}{3b}\left(\frac{2}{t_0} - 6\right) + \frac{2}{b}\right) \sin\left(\sqrt{\frac{b}{2}} x - \beta_0\right), \tag{19}
\]

\[
u_1 = bt\left(\frac{2}{t_0} - 6c_1 + \left(\frac{2}{t_0} + \left(\frac{2}{b} - 6\right)c_1\right) \cos^2\left(\sqrt{\frac{b}{2}} x - \beta_0\right) \right) - \frac{1}{bt_0}\left(\frac{t}{2}(2t_0c_1 + B(2 - 6t_0c_1)) \cos\left(\sqrt{\frac{b}{2}} x - \beta_0\right) \right) + \frac{1}{bt_0}\left(2t_0c_1 + (2t_0c_1 + b(2 - 6t_0c_1)) \cos\left(\sqrt{\frac{b}{2}} x - \beta_0\right) \right) + \frac{bt}{2}\left(\frac{2}{t_0} + \left(\frac{2}{b} - 6\right)c_1\right)^2 \sin^2\left(\sqrt{\frac{b}{2}} x - \beta_0\right) + tc_1\left(\frac{2}{t_0} - 6c_1 + \frac{4c_1}{b} + \frac{1}{c_1}\left(\frac{2}{t_0} + \left(\frac{2}{b} - 6\right)c_1\right) \sin\left(\sqrt{\frac{b}{2}} x - \beta_0\right) \right) - t\left(\frac{2}{t_0} - 6c_1 + \left(\frac{2}{t_0} + \left(\frac{2}{b} - 6\right)c_1\right) \cos\left(\sqrt{\frac{b}{2}} x - \beta_0\right) \right) a_1 + th_1, \tag{20}
\]
\[ v_1 = \frac{t}{2} \left( \frac{1}{c_1^2} \left( b \left( \frac{2}{t_0} - \frac{2}{b} - 6 \right) c_1 \right)^2 \right) \left( \frac{\sqrt{b}}{2} x - \beta_0 \right) - \\
2c_1 \left( \frac{2}{t_0} - 6c_1 + \left( \frac{2}{b} + \frac{2}{b} - 6 \right) c_1 \cos \left( \frac{\sqrt{b}}{2} x - \beta_0 \right) \right) + \\
2b \left( \frac{2}{t_0} - 6c_1 - \frac{4c_1}{b} \right) + \frac{1}{c_1} \left( \frac{2}{t_0} + \frac{2}{b} - 6 \right) c_1 \\
\left| c_1 \sin \left( \frac{\sqrt{b}}{2} x - \beta_0 \right) \right|^2 - \frac{1}{2bt_0^2c_1^2} \left( 2t_0 c_1 - \\
b(2 - 6t_0 c_1) \right) \left| c_1 \sin \left( \frac{\sqrt{b}}{2} x - \beta_0 \right) \right) (c_1 (4t_0 c_1 - \\
b(2 - 6t_0 c_1)) + (2t_0 c_1 + b(2 - 6t_0 c_1)) \\
\left| c_1 \sin \left( \frac{\sqrt{b}}{2} x - \beta_0 \right) \right) + 2 \left( \frac{2}{t_0} - 6c_1 - \frac{4c_1}{b} \right) - \\
\frac{1}{c_1} \left( \frac{2}{t_0} + \frac{2}{b} - 6 \right) c_1 \right) \left| c_1 \sin \left( \frac{\sqrt{b}}{2} x - \beta_0 \right) \right) a_2 + 2h_2, \quad (21) \]

In order to prove numerically whether the Adomian decomposition method for system (2) leads to higher accuracy, we evaluate the approximate solution using the 6-terms approximations

\[ \psi_6 = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + u_4(x, t) + u_5(x, t), \quad (22) \]
\[ \varphi_6 = v_0(x, t) + v_1(x, t) + v_2(x, t) + v_3(x, t) + v_4(x, t) + v_5(x, t). \quad (23) \]

Table 1 and Table 2 shows the exact solution, \( u(x, t), v(x, t) \), the approximate solutions, \( \psi_6, \varphi_6 \), the absolute error, \( |u - \psi_6|, |v - \varphi_6| \) where \( t = 0.6, t_0 = 4, b = 2, \beta_0 = 3, a_1 = 3, a_2 = -3, h_1 = h_2 = 0, c_1 = 1. \)

| \( x \) | \( u_E \) | \( u_A \) | \( |u_E - u_A| \) |
|---|---|---|---|
| -8 | -0.846246 | -0.846249 | 2.64963 \times 10^{-6} |
| -6 | -0.998499 | -0.998501 | 2.50145 \times 10^{-6} |
| -4 | -0.877327 | -0.877326 | 1.56867 \times 10^{-6} |
| -2 | -0.825925 | -0.825927 | 1.96704 \times 10^{-6} |
| 0 | -0.989878 | -0.989881 | 3.09441 \times 10^{-6} |
| 2 | -0.904823 | -0.904822 | 1.37961 \times 10^{-6} |
| 4 | -0.811661 | -0.811662 | 1.21672 \times 10^{-6} |
| 6 | -0.974254 | -0.974258 | 3.52984 \times 10^{-6} |
| 8 | -0.932091 | -0.93209 | 9.91685 \times 10^{-6} |
Taking the Adomian decomposition method, we get

| Case 2: For \(a_1 \neq 3c_1\), \(\varphi_0(t) = \frac{1}{36}(3c_1 - a_1) \tanh \left(\frac{3c_1 - a_1}{2}(t_0 - t)\right) - a_1 - 3c_1\). Taking \(a_1 = 3\), \(h_1 = 1\), \(c_1 = 1.2\), \(b = 3\), \(c_2 = -1.2\) and applying recurrence relation (13) we get |

\[
\begin{array}{|c|c|c|c|}
\hline
x_i & v_E & v_A & |v_E - v_A| \\
\hline
-8 & 1.17871 & 1.17871 & 1.10767265 \times 10^{-6} \\
-6 & 1.11513 & 1.11513 & 3.43627205 \times 10^{-6} \\
-4 & 1.00315 & 1.00314 & 1.41058665 \times 10^{-6} \\
-2 & 1.15993 & 1.15993 & 1.44739291 \times 10^{-6} \\
0 & 1.14142 & 1.14142 & 2.95697079 \times 10^{-6} \\
2 & 1.00004 & 1.00004 & 2.14922631 \times 10^{-6} \\
4 & 1.13622 & 1.13622 & 1.58285677 \times 10^{-6} \\
6 & 1.16426 & 1.16426 & 2.33107685 \times 10^{-6} \\
8 & 1.00474 & 1.00474 & 2.80561774 \times 10^{-6} \\
\hline
\end{array}
\]

\[\bar{u}_0 = -0.6975 + 0.1024 \cos(3 - \sqrt{\frac{3}{2}}x), \quad (24)\]

\[\bar{v}_0 = 0.9024 - 0.1024 \sin(3 - \sqrt{\frac{3}{2}}x), \quad (25)\]

\[\bar{u}_1 = t(1.4500 - 0.0142) \cos(3 - 1.22x) + 0.0157 \cos^2(3 - 1.22x) - 0.1229 \sin(3 - 1.22x) + 0.0157 \sin^2(3 - 1.22x), \quad (26)\]

\[\bar{v}_1 = t(1.4500 - 0.1229) \cos(3 - 1.22x) + 0.0157 \cos^2(3 - 1.22x) + 0.1423 \sin(3 - 1.22x) + 0.0157 \sin^2(3 - 1.22x), \quad (27)\]

\[\bar{u}_2 = t(1 + 0.0107t + t(0.0001 + 0.0036 \cos(3 - 1.22x)) \cos^2(3 - 1.22x) + 0.0107t \sin(3 - 1.22x) - 0.0020t \sin^2(3 - 1.22x) + t \cos(3 - 1.22x) \times (-0.0727 - 0.0021 \cos(3 - 1.22x)) + (0.0189 + 2.612 \times 10^{-18} \sin(3 - 1.22x)) \sin(3 - 1.22x) + t \cos(3 - 1.22x)(0.334 - 0.0189 \sin(3 - 1.22x) + 0.0036 \sin^2(3 - 1.22x))), \quad (28)\]

\[\bar{v}_2 = t(1.96 + 0.0107t + t \cos^2(3 - 1.22x)(0.0001 - \]
$0.0036 \sin(3 - 1.22x) + t \cos(3 - 1.22x)(0.0170 + \\
\cos(3 - 1.22x)(-0.002 - 2.612 \times 10^{-18} \sin(3 - 1.22x)) + \\
0.0189 \sin(3 - 1.22x) - 0.334t \sin(3 - 1.22x) + \\
t(0.0001 - 0.0036 \sin(3 - 1.22x)) \sin^2(3 - 1.22x) + \\
t \sin(3 - 1.22x)(0.0727 + (0.0184 \cos(3 - 1.22x) - \\
(0.0048 \sin(3 - 1.22x))))$, \hspace{1cm} (29)

Table 3 and Table 4 shows the exact solutions $u$ and $v$. The approximate solutions, $\tilde{\psi}_6, \tilde{\varphi}_6$, the absolute error, $|u - \tilde{\psi}_6|$, $|v - \tilde{\varphi}_6|$ and the relative error where $t = 0.001$, $a_1 = 3$, $h_1 = 1$, $c_1 = 1.2$, $b = 3$, $t_0 = 4$, $\beta_0 = 3$.

| $x_i$ | $u_E$ | $u_A$ | $|u_E - u_A|$ |
|-------|-------|-------|----------------|
| -8    | -0.59785 | -0.59236 | 5.482 \times 10^{-3} |
| -6    | -0.75921 | -0.75373 | 5.481 \times 10^{-3} |
| -4    | -0.70227 | -0.69679 | 5.481 \times 10^{-3} |
| -2    | -0.62858 | -0.62310 | 5.482 \times 10^{-3} |
| 0     | -0.79899 | -0.79351 | 5.480 \times 10^{-3} |
| 2     | -0.61029 | -0.60480 | 5.482 \times 10^{-3} |
| 4     | -0.73045 | -0.72497 | 5.481 \times 10^{-3} |
| 6     | -0.73412 | -0.72864 | 5.481 \times 10^{-3} |
| 8     | -0.60830 | -0.60282 | 5.482 \times 10^{-3} |

| $x_i$ | $v_E$ | $v_A$ | $|v_E - v_A|$ |
|-------|-------|-------|----------------|
| -8    | 0.878821 | 0.888141 | 9.321 \times 10^{-3} |
| -6    | 0.984272 | 0.993595 | 9.323 \times 10^{-3} |
| -4    | 0.800109 | 0.809429 | 9.320 \times 10^{-3} |
| -2    | 0.978233 | 0.987556 | 9.323 \times 10^{-3} |
| 0     | 0.888119 | 0.897441 | 9.321 \times 10^{-3} |
| 2     | 0.848754 | 0.858075 | 9.320 \times 10^{-3} |
| 4     | 0.999483 | 1.008811 | 9.323 \times 10^{-3} |
| 6     | 0.806754 | 0.816074 | 9.320 \times 10^{-3} |
| 8     | 0.952792 | 0.962114 | 9.322 \times 10^{-3} |
4 Conclusions

We have presented a scheme used to obtain analytic approximate solution of the nonlinear reaction diffusion system of Lotka-Volterra type by using the Adomian decomposition method. The absolute error, exact and numerical results are presented and compared each other in tables for some values of $x$ and fixed $t$. As expected from the tables the analytic approximate solution clearly indicates that how the Adomian decomposition method obtains efficient results much closer to the accurate solutions.

References


Received: April 12, 2006