A computing method for the Moore-Penrose inverse of some special matrices

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Abstract

A number of methods have been proposed for computing the Moore - Penrose (M-P) inverse, X^+ , of an arbitrary $m \times n$ matrix of rank $r \leq m \leq n$. In this paper we have given a simple procedure to find the explicit form of M-P inverse of a matrix that has special form and will consider the ANOVA models. Also computational algorithms for constructing and computing M-P inverse of these special matrices are given.

Keywords: Direct (Kronecker) Product; Factorial Design; Moore - Penrose Inverse.

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1 Introduction

The application of generalized inverse matrices to linear statistical models has been of considerable interest since early 1960's. Generalized inverse of a matrix plays significant role in mathematical statistics, also Moore-Penrose (M-P) inverse. The M-P inverse of a matrix is the basis for much of the modern development in regression analysis and analysis of variance. For an extensive discussion of the theory of M-P inverses, we refer, e.g. to the books by Boullion and Odell [3], Pringle and Rayner [8], Rao and Mitra [9], Albert [1], Ben Israel and Greville [2], Searle [10], Graybill [4]. Many statisticians contributed to the derivation of the M-P inverse X^+ for the matrices appearing in special statistical models. But it is not easy to compute, especially when has many columns. Most matrices occurring in the linear model can be represented as

direct products of identity matrices and matrices with all elements equal to 1. Using this regularity, Kim and Lee [5] provided general explicit forms X^+ for the classificatory models. Sunwoo [12] described an algorithm for computing sums of square without storing the design matrix. He also showed that such matrices can be obtained by a simple rule.

In this paper we shall give a simple procedure to compute the M-P inverse of a matrix that has a special form. This method can be applied to one, and multi - way factorial designs. We will also derive computational algorithms for constructing direct (Kronecker) product and computing the M-P inverse of matrix that has a special form.

2 The Moore - Penrose Inverse and Background Information

Definition 2.1 (Searle [10], p.212, Schott [11], p.171) The $n \times m$ matrix A^+ is called M-P inverse of $m \times n$ matrix A if satisfies the following four conditions:

i) $AA^+A = A$, ii) $A^+AA^+ = A^+$, iii) AA^+ is symmetric , iv) A^+A is symmetric.

If a matrix is satisfied only condition (i) it is called a generalized inverse.

Theorem 2.1 (Graybill [4], p.112). If A is an $m \times n$ matrix, then $A^+ = A'(AA')^+ = (A'A)^+$.

Definition 2.2 Let A be an $m_1 \times n_1$ matrix and let B be an $m_2 \times n_2$ matrix then the direct product of A and B , which we write $A \otimes B$, is a matrix C of size $m_1 m_2 \times n_1 n_2$ defined by $C = [a_{ij}B]$, $i = 1, 2, \dots, m_1$, $j = 1, 2, \dots, n_1$.

Actually the definition is a right direct product (see Graybill [4], Harville [5]). Direct product has many useful properties some of which are as follows:

- (1) $(A \otimes B)' = A' \otimes B'$
- (2i) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, (2ii) $(A \otimes B)^+ = A^+ \otimes B^+$
- (3) $(A \otimes B)(C \otimes D) = AC \otimes BD$
- (4) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$.

Theorem 2.2 (Graybill [4], p.115). If $A = A_1 + A_2 + \dots + A_t$ and $A_i A_j' = \emptyset$, $A_i' A_j = \emptyset$ for all $i, j = 1, 2, \dots, t$, $i \neq j$. Then $A^+ = A_1^+ + A_2^+ + \dots + A_t^+$.

3 Main Results

Theorem 3.1 Suppose that the matrix X can be partitioned into sub-matrices $X = [X_0, X_1, X_2, \dots, X_m]$, where $X_i = j_{n_1} \otimes j_{n_2} \otimes \dots \otimes I_{n_i} \otimes \dots \otimes j_{n_{k-1}} \otimes j_{n_k}$ for $i = 1, 2, \dots, m$, $m \leq k$, that is each consist of the direct product of k matrices such that i^{th} matrix is identity matrix other ones are vectors with all elements equal to 1, and $X_0 = j_N = j_{n_1} \otimes j_{n_2} \otimes \dots \otimes j_{n_{k-1}} \otimes j_{n_k}$ then the M-P inverse of the matrix X is given by

$$X^+ = \begin{bmatrix} \frac{n_1}{N} X'_1 + (\frac{a}{n_1} - 1) \frac{1}{N} J_{n_1} \otimes j'_{n_2} \otimes \dots \otimes j'_{n_{k-1}} \otimes j'_{n_k} \\ \vdots \\ \frac{n_m}{N} X'_m + (\frac{a}{n_m} - 1) \frac{1}{N} J_{n_1} \otimes j'_{n_2} \otimes \dots \otimes J_{n_m} \otimes \dots \otimes j'_{n_k} \end{bmatrix} \quad (3.1)$$

where $a = \frac{1}{\sum_{i=0}^m \frac{1}{n_i}}$, $n_0 = 1$, $N = \prod_{i=1}^k n_i$, j_p is the $p \times 1$ vector of ones, $J_p = j_p \otimes j'_p$

is the $p \times p$ matrix of ones, and j'_p is the transpose of j_p .

Proof : $XX' = J_{n_1} \otimes J_{n_2} \otimes \dots \otimes J_{n_k} + \sum_{i=0}^m J_{n_1} \otimes J_{n_2} \otimes \dots \otimes I_{n_i} \otimes \dots \otimes J_{n_k}$

If we write

$$XX' = \left(\sum_{i=0}^m \frac{1}{n_i} \right) J_{n_1} \otimes J_{n_2} \otimes \dots \otimes J_{n_k} + \sum_{i=0}^m J_{n_1} \otimes J_{n_2} \otimes \dots \otimes (I_{n_i} - \frac{1}{n_i} J_{n_i}) \otimes \dots \otimes J_{n_k},$$

the matrix XX' satisfies the conditions in Theorem 2.2. So we get $XX' =$

$\frac{a}{N^2} J_{n_1} \otimes J_{n_2} \otimes \dots \otimes J_{n_k} + \frac{1}{N^2} \sum_{i=0}^m n_i^2 J_{n_1} \otimes J_{n_2} \otimes \dots \otimes (I_{n_i} - \frac{1}{n_i} J_{n_i}) \otimes \dots \otimes J_{n_k}$ Then, using Theorem 2.1 we get equation (3.1). This completes the proof.

Theorem 3.2 Let $X = [X_0, X_1, X_2, \dots, X_k]$ be partitioning of X into $k + 1$ sub-matrices, where sub-matrices are as follows:

X_i is a block matrix consist of $\binom{k}{i}$ matrices, where $i = 1, 2, \dots, k$. In each block of X_i is the direct product of i I_{n_j} identity matrices and $(k - i)$ j_{n_j} vectors ($j = 1, 2, \dots, k$). Then the M-P inverse of the partitioned matrix is given by

$$X^+ = \begin{bmatrix} X'_0(XX')^+ \\ X'_1(XX')^+ \\ \vdots \\ X'_k(XX')^+ \end{bmatrix} = \prod_{i=1}^k \frac{1}{n_i + 1} \begin{bmatrix} \tilde{X}_0 \\ \tilde{X}_1 \\ \tilde{X}_2 \\ \vdots \\ \tilde{X}_{k-1} \\ \tilde{X}_k \end{bmatrix} \quad (3.2)$$

where $\tilde{X}_i = X'_i(XX')^+$ for $i = 0, 1, 2, \dots, k$ and X^+ is a block matrix of $k + 1$ sub-matrices. Here each $\tilde{X}_i$ is the transpose of the matrix obtained from X_i by replacing (in each block) I_{n_j} by $[(n_j + 1)I_{n_j} - J_{n_j}]$.

Proof: It is obvious that matrix X has 2^k block matrices. After the calculation we obtain

$$\begin{aligned} XX' &= (I_{n_1} + J_{n_1}) \otimes (I_{n_2} + J_{n_2}) \otimes \dots \otimes (I_{n_k} + J_{n_k}) \\ &= (1 + \frac{1}{n_1})J_{n_1} \otimes (I_{n_2} + J_{n_2}) \otimes \dots \otimes (I_{n_k} + J_{n_k}) + \\ &\quad (I_{n_1} - \frac{1}{n_1})J_{n_1} \otimes (I_{n_2} + J_{n_2}) \otimes \dots \otimes (I_{n_k} + J_{n_k}) \end{aligned}$$

The matrix XX' satisfies the conditions in Theorem 2.1. Hence we get

$$\begin{aligned} (XX')^+ &= \frac{1}{n_1(n_1 + 1)}J_{n_1} \otimes (I_{n_2} - \frac{1}{n_2 + 1}J_{n_2}) \otimes \dots \otimes (I_{n_k} - \frac{1}{n_k + 1}J_{n_k}) + \\ &\quad (I_{n_1} - \frac{1}{n_1})J_{n_1} \otimes (I_{n_2} - \frac{1}{n_2 + 1}J_{n_2}) \otimes \dots \otimes (I_{n_k} - \frac{1}{n_k + 1}J_{n_k}) \end{aligned}$$

where $(I_{n_i} + J_{n_i})^{-1} = I_{n_i} - \frac{1}{n_i + 1}J_{n_i}$. Thus, using Theorem 2.1 we get equation (3.2). This completes the proof.

For example, second block of X is

$$X_1 = [I_{n_1} \otimes j_{n_2} \otimes \dots \otimes j_{n_k}, j_{n_1} \otimes I_{n_2} \otimes \dots \otimes j_{n_k}, \dots, j_{n_1} \otimes j_{n_2} \otimes \dots \otimes I_{n_k}]$$

and X_1 has $\binom{k}{i} = k$ sub-matrices. Using Theorem 3.2, we get

$$\tilde{X}_1 = X'_1(XX')^+ = \begin{bmatrix} [(n_1 + 1)I_{n_1} - J_{n_1}] \otimes j'_{n_2} \otimes \dots \otimes j'_{n_k} \\ j'_{n_1} \otimes [(n_2 + 1)I_{n_2} - J_{n_2}] \otimes j'_{n_3} \otimes \dots \otimes j'_{n_k} \\ \vdots \\ j'_{n_1} \otimes j'_{n_2} \otimes \dots \otimes j'_{n_{k-1}} \otimes [(n_k + 1)I_{n_k} - J_{n_k}] \end{bmatrix}$$

Note that $\tilde{X}_1$ is the transpose of the matrix obtained from X_1 by replacing I_{n_j} by $[(n_j + 1)I_{n_j} - J_{n_j}]$ for $j = 1, 2, \dots, k$.

Theorem 3.1 can be applied to find the M-P inverse of the design matrix of one-way classification model, and multi-way factorial designs with no interactions. For the multi-way factorial designs with interactions Theorem 3.2 can be applied. We will consider the three-way classification model without interaction and one observation per cell, and three-way classification with interactions and one observation per cell as examples.

Example 1 Consider the three-way classification model without interaction and one observation per cell

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \varepsilon_{ijk} \quad (3.3)$$

where $i = 1, 2, \dots, n_1$; $j = 1, 2, \dots, n_2$; $k = 1, 2, \dots, n_3$, y_{ijk} is the response for the treatment when factor A is at level i^{th} , factor B is at the j^{th} level, and factor C is at the k^{th} level. μ is the general mean, α_i is the effect on the response of the level of factor A , β_j is the effect on the response of the level of factor B , γ_k is the effect on the response of the level factor C and ε_{ijk} is the error term. In matrix form Equation (3.3) can be written as

$$Y = X\beta + e \quad (3.4)$$

where $Y' = (y_{111}, y_{112}, \dots, y_{11n_3}, y_{121}, y_{122}, \dots, y_{12n_3}, \dots, y_{1n_21}, y_{1n_22}, \dots, y_{1n_2n_3}, \dots, y_{n_1n_21}, \dots, y_{n_1n_2n_3})$, $\beta' = (\mu, \alpha_1, \dots, \alpha_{n_1}, \beta_1, \dots, \beta_{n_2}, \gamma_1, \dots, \gamma_{n_3})$,

$X = (X_0, X_1, X_2, X_3)$, $X_0 = j_{n_1} \otimes j_{n_2} \otimes j_{n_3}$, $X_1 = I_{n_1} \otimes j_{n_2} \otimes j_{n_3}$, $X_2 = j_{n_1} \otimes I_{n_2} \otimes j_{n_3}$ and $X_3 = j_{n_1} \otimes j_{n_2} \otimes I_{n_3}$. If we take $k = 3$ and $m = 3$ in Theorem 3.1 we get

$$X = (j_{n_1} \otimes j_{n_2} \otimes j_{n_3}, I_{n_1} \otimes j_{n_2} \otimes j_{n_3}, j_{n_1} \otimes I_{n_2} \otimes j_{n_3}, j_{n_1} \otimes j_{n_2} \otimes I_{n_3}).$$

So we have a design matrix for three-way classification without interaction and one observation per cell. Thus, from Theorem 2.1 we obtain the M-P inverse of X

$$X^+ = \begin{bmatrix} \frac{a}{N} j'_{n_1} \otimes j'_{n_2} \otimes j'_{n_3} \\ [I_{n_1} + (\frac{a}{n_1} - 1)J_{n_1}] \otimes \frac{1}{n_2} j'_{n_2} \otimes \frac{1}{n_3} j'_{n_3} \\ \frac{1}{n_1} j'_{n_1} \otimes [I_{n_2} + (\frac{a}{n_2} - 1)J_{n_2}] \otimes \frac{1}{n_3} j'_{n_3} \\ \frac{1}{n_1} j'_{n_1} \otimes \frac{1}{n_2} j'_{n_2} \otimes [I_{n_3} + (\frac{a}{n_3} - 1)J_{n_3}] \end{bmatrix}$$

where $a = \frac{n_1 n_2 n_3}{n_1 n_2 n_3 + n_1 n_2 + n_1 n_3 + n_2 n_3}$ and $N = n_1 n_2 n_3$.

Example 2 Consider the following three way classification model with interaction and one observation per cell

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijk}$$

where $i = 1, 2, \dots, n_1$; $j = 1, 2, \dots, n_2$; $k = 1, 2, \dots, n_3$. In Theorem 3.2 if we take $k = 3$ we have the related design matrix, as $X = (X_0, X_1, X_2, X_3)$, where $X_0 = (j_{n_1} \otimes j_{n_2} \otimes j_{n_3})$, $X_1 = (I_{n_1} \otimes j_{n_2} \otimes j_{n_3}, j_{n_1} \otimes I_{n_2} \otimes j_{n_3}, j_{n_1} \otimes j_{n_2} \otimes I_{n_3})$, $X_2 = (I_{n_1} \otimes I_{n_2} \otimes j_{n_3}, I_{n_1} \otimes j_{n_2} \otimes I_{n_3}, j_{n_1} \otimes I_{n_2} \otimes I_{n_3})$ and $X_3 = (I_{n_1} \otimes I_{n_2} \otimes I_{n_3})$. Using the Theorem 3.2, we obtain the M-P inverse of as

$$X^+ = u \begin{bmatrix} j'_{n_1} \otimes j'_{n_2} \otimes j'_{n_3} \\ [(n_1 + 1)I_{n_1} - J_{n_1}] \otimes j'_{n_2} \otimes j'_{n_3} \\ j'_{n_1} \otimes [(n_2 + 1)I_{n_2} - J_{n_2}] \otimes j'_{n_3} \\ j'_{n_1} \otimes j'_{n_2} \otimes [(n_3 + 1)I_{n_3} - J_{n_3}] \\ [(n_1 + 1)I_{n_1} - J_{n_1}] \otimes [(n_2 + 1)I_{n_2} - J_{n_2}] \otimes j'_{n_3} \\ [(n_1 + 1)I_{n_1} - J_{n_1}] \otimes j'_{n_2} \otimes [(n_3 + 1)I_{n_3} - J_{n_3}] \\ j'_{n_1} \otimes [(n_2 + 1)I_{n_2} - J_{n_2}] \otimes [(n_3 + 1)I_{n_3} - J_{n_3}] \\ [(n_1 + 1)I_{n_1} - J_{n_1}] \otimes [(n_2 + 1)I_{n_2} - J_{n_2}] \otimes [(n_3 + 1)I_{n_3} - J_{n_3}] \end{bmatrix}$$

$$\text{where } u = \prod_{i=1}^3 \frac{1}{n_i + 1}$$

4 Algorithms

Let P a matrix defined to be of the form

$$P = A_1 \otimes A_2 \otimes \dots \otimes A_k \quad (4.1)$$

where each matrix A_i is either I_{n_i} or j_{n_i} . Each matrix X_j in Theorem 3.1 is in the form of (4.1). For the convenience of description of the algorithm define

$$a_i = \begin{cases} 1 & \text{if } A_i = I_{n_i} \\ 0 & \text{if } A_i = j_{n_i} \end{cases} \quad \text{for } i = 1, 2, \dots, k.$$

Algorithm 1 (Constructing the matrix P)

1. Let $P_1 := A_k$
2. For do $r = 2, 3, \dots, k$ do
 - if $a_{k-r+1} = 0$, then $(P_r)_{pq} := P_{r-1}$ for $p = 1, 2, \dots, n_{k-r+1}$
 - else $(P_r)_{pq} := P_{r-1}$ if $p = q$
 - $(P_r)_{pq} := \emptyset$ if $p \neq q$

where $(P_r)_{pq} := P_{r-1}$ means that each sub-matrix $(P_r)_{pq}$ is the same as the

principal sub-matrix P_{r-1} , and $\emptyset$ means the zero matrix with appropriate size and $P_1 = A_k$, $P_i = A_{k-i+1} \otimes \dots \otimes A_k$ for $i = 1, 2, \dots, k$, and $(P_i)_{pq}$ denotes the $(p, q)^{th}$ sub-matrix of P_i .

Note that in Theorem 3.2 each sub-matrix is the form of (4.1). So each sub-matrix of X in Theorem 3.2 is constructed by using Algorithm 1.

For the matrix X in Theorem 3.1, we have

$$XX' = \sum_{i=1}^m J_{n_1} \otimes J_{n_2} \otimes \dots \otimes (I_{n_i} - \frac{1}{n_i} J_{n_i}) \otimes \dots \otimes J_{n_k} \quad (4.2)$$

where $I_{n_0} - \frac{1}{n_0} J_{n_0}$ is defined as $\frac{1}{a} J_{n_i}$ and $a = (\sum_{i=0}^m n_i^{-1})^{-1}$.

Let Q a matrix defined to be of the form

$$Q = B_1 \otimes B_2 \otimes \dots \otimes B_k \quad (4.3)$$

where each matrix B_i is $I_{n_i} - \frac{1}{n_i} J_{n_i}$ either or J_{n_i} . Define $Q_1 = B_k$, and $Q_i = B_{k-i+1} \otimes \dots \otimes B_k$ for $i = 2, \dots, k$ then we have $Q_i = B_{k-i+1} \otimes Q_{i-1}$. We also define

$$b_i = \begin{cases} 1 & \text{if } B_i = I_{n_i} - \frac{1}{n_i} J_{n_i} \\ 0 & \text{if } B_i = J_{n_i} \end{cases} \quad \text{for } i = 1, 2, \dots, k.$$

Aim is to describe an algorithm for determining the matrix by a simple rule. Note that each matrix of sum in equation (4.2) is in form of matrix Q for $j = 0, 1, 2, \dots, m$. Let $(Q_i)_{pq}$ denotes the $(p, q)^{th}$ submatrix of Q_i .

Algorithm 2.a (Constructing the matrix Q)

1. Let $Q_1 := B_k$
2. For $r = 2, 3, \dots, k$ do
 - if $b_{k-r+1} = 0$, then $(Q_r)_{pq} := Q_{r-1}$ for all p, q
 - else $(Q_r)_{pq} := (1 - \frac{1}{n_{k-r+1}}) Q_{r-1}$ if $p = q$

$$(Q_r)_{pq} := -\frac{1}{n_{k-r+1}} Q_{r-1} \text{ if } p \neq q$$

Let $(Q_i)^+_{pq}$ denotes the $(p, q)^{th}$ submatrix of Q_i^+ .

$$c_i = \begin{cases} 1 & \text{if } B_i^+ = I_{n_i} - \frac{1}{n_i} J_{n_i} \\ 0 & \text{if } B_i^+ = \frac{1}{n_i^2} J_{n_i} \end{cases} \quad \text{for } i = 1, 2, \dots, k.$$

Note that $Q_1^+ := B_k^+$, $(Q_i)^+ := (B_{k-i+1})^+ \otimes Q_{i-1}^+$ and $(Q_i)^+ = (B_{k-i+1} \otimes \dots \otimes B_k)^+$ for $i = 2, 3, \dots, k$. Now, we will describe an algorithm for determining Q^+ by a simple rule.

Algorithm 2.b (Constructing the matrix Q^+)

1. Let $Q_1^+ := B_k^+$
2. For $r = 2, 3, \dots, k$ do
 if $c_{k-r+1} = 0$, then $(Q_r)_{pq}^+ := \frac{1}{n_{k-r+1}^2} Q_{r-1}$ for all p, q
 else $(Q_r)_{pq}^+ := (1 - \frac{1}{n_{k-r+1}}) Q_{r-1}^+$ if $p = q$
 $(Q_r)_{pq}^+ := -\frac{1}{n_{k-r+1}} Q_{r-1}^+$ if $p \neq q$

where B_k^+ is $I_{n_k} - \frac{1}{n_k} J_{n_k}$ either or $\frac{1}{n_k^2} J_{n_k}$. In Algorithm 2.a, $(Q_r)_{pq} := Q_{r-1}$ means that each submatrix $(Q_r)_{pq}$ is the same as the principal submatrix Q_{r-1} , and in Algorithm 2.b, $(Q_r)_{pq}^+ := Q_{r-1}^+$ means that each submatrix $(Q_r)_{pq}^+$ is the same as the principal submatrix Q_{r-1}^+ .

In Theorem 3.1 since $X' = (P^{(0)}, P^{(1)}, \dots, P^{(m)})'$ and $(XX')^+ := \sum_{j=0}^m (Q^{(j)})^+$ and $X^+ = X'(XX')^+ = (P^{(0)}, P^{(1)}, \dots, P^{(m)})' \sum_{j=0}^m (Q^{(j)})^+$

We get

$$X = \begin{bmatrix} (P^{(0)})'(Q^{(0)})^+ \\ (P^{(1)})'(Q^{(0)})^+ + (P^{(1)})'(Q^{(1)})^+ \\ (P^{(2)})'(Q^{(0)})^+ + (P^{(2)})'(Q^{(2)})^+ \\ \vdots \\ (P^{(m)})'(Q^{(0)})^+ + (P^{(m)})'(Q^{(m)})^+ \end{bmatrix}$$

where $P^{(0)}$ is the direct product of k J_{n_r} matrices and $(Q^{(0)})^+$ is the M-P inverse of the direct product of k J_{n_r} matrices and $(P^{(j)})$ consist of the direct product of k matrices such that j^{th} matrix is identity matrix and others are vectors with all elements equal to 1, and $(Q^{(j)})^+$ consist of the direct product of k matrices such that j^{th} matrix is $I_{n_i} - \frac{1}{n_i} J_{n_i}$ and others are J_{n_i} for $j = 1, 2, \dots, m$. It is easy to see that X^+ is the same as given in (3.1).

Now, let $T = C_1 \otimes C_2 \otimes \dots \otimes C_k$ and $T_i = C_{k-i+1} \otimes \dots \otimes C_k$ for $i = 1, 2, \dots, k$. Then $T_1 = C_k$ and $T_{i+1} = C_{k-i} \otimes T_i$ where $C_i = I_{n_i} + J_{n_i}$. We can describe algorithms for determining the matrix T and the matrix T^+ respectively.

Algorithm 3. (Constructing the matrix T or XX' in Theorem 3.2)

1. Let $T_1 := C_k$
2. For $r = 2, 3, \dots, k$ do
 $(T_r)_{pq} := 2T_{r-1}$ if $p = q$
 $(T_r)_{pq} := T_{r-1}$ if $p \neq q$

Algorithm 4. (Constructing the matrix T^+ or $(XX')^+$ in Theorem 3.2)

1. Let $T_1^+ := C_k^+$
2. For $r = 2, 3, \dots, k$ do
 $(T_r)_{pq}^+ := \frac{n_{k-r+1}}{n_{k-r+1}+1} T_{r-1}^+$ if $p = q$

$(T_r)_{pq}^+ := -\frac{1}{n_{k-r+1}+1}T_{r-1}^+$ if $p \neq q$
 where $C_k^+ = I_{n_k} - \frac{1}{n_k+1}J_{n_k}$ and $(T_{i+1})^+ = C_{k-i}^+ \otimes T_i^+$, and $(T_r)_{pq}^+ := T_{r-1}^+$ means that each sub-matrix $(T_r)_{pq}^+$ is the same as the principal sub-matrix $(T_{r-1})^+$ and $p, q = 1, 2, \dots, n_{k-r+1}$.

Example 3 Let $k = 3$ and T be a matrix of the form

$$T = C_1 \otimes C_2 \otimes C_3 = (I_{n_1} + J_{n_1}) \otimes (I_{n_2} + J_{n_2}) \otimes (I_{n_3} + J_{n_3})$$

we can construct the matrix T by using Algorithm 3 as follows :

1. Let $T_1 = C_3 = I_{n_3} + J_{n_3}$

2. For $r = 2$ we obtain

$$T_2 = C_2 \otimes T_1 = \begin{bmatrix} 2T_1 & \cdot & \cdot & T_1 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ T_1 & \cdot & \cdot & 2T_1 \end{bmatrix} = (I_{n_2} + J_{n_2}) \otimes (I_{n_3} + J_{n_3}).$$

For $r = 3$ we have

$$T = T_3 = C_1 \otimes T_2 = \begin{bmatrix} 2T_2 & \cdot & \cdot & T_2 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ T_2 & \cdot & \cdot & 2T_2 \end{bmatrix}.$$

we also construct T^+ by using Algorithm 4 as follows :

1. Let $T_1^+ := C_3^+ = I_{n_3} - \frac{1}{n_3+1}J_{n_3}$

2. For $r = 2$ we obtain

$$T_2^+ = C_2^+ \otimes T_1^+ = \begin{bmatrix} \frac{n_2}{n_2+1}T_1^+ & \cdot & \cdot & -\frac{1}{n_2+1}T_1^+ \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ -\frac{1}{n_2+1}T_1^+ & \cdot & \cdot & \frac{n_2}{n_2+1}T_1^+ \end{bmatrix}$$

and for $r = 3$ it is end of the loop and we get

$$T^+ = T_3^+ = C_1^+ \otimes T_2^+ = \begin{bmatrix} \frac{n_1}{n_1+1}T_2^+ & \cdot & \cdot & -\frac{1}{n_1+1}T_2^+ \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ -\frac{1}{n_1+1}T_2^+ & \cdot & \cdot & \frac{n_1}{n_1+1}T_2^+ \end{bmatrix} = (XX')^+.$$

Now we will illustrate computations of M-P inverse of matrix X in Theorem 3.2. Let P_i be a matrix consisting of the direct product of i I_{n_r} identity matrices and $(k-i)$ j'_{n_r} matrices for $i = 0, 1, 2, \dots, k$. It is obvious that each P_i has $\binom{k}{i}$ block matrices, and each block of P_i can be constructed by replacing j_{n_r} by j'_{n_r} in Algorithm 1. For example P_1 has k block matrices. Each block is consist of direct product of k matrices which any one is matrix I_{n_r} and other ones are matrices j'_{n_r} . The matrix P_i is transpose of the matrix X_i which is defined in

Theorem 3.2. To compute M-P inverse of matrix X in Theorem 3.2 first we find $(XX')^+$ by using Algorithm 4 then we have

$$X^+ = [(\tilde{P}_0)', (\tilde{P}_1)', \dots, (\tilde{P}_{k-1})', (\tilde{P}_k)']'$$

where $\tilde{P}_i' = P_i(XX')^+$. It is clear that $P_0(XX')^+ = (\prod_{j=1}^k \frac{1}{n_j+1})j'_{n_1} \otimes j'_{n_2} \otimes \dots \otimes j'_{n_k}$ and the matrix $P_1(XX')^+$ has k block matrices and each sub-block of $P_1(XX')^+$ is

$$(\prod_{j=1}^k \frac{1}{n_j+1})j'_{n_1} \otimes j'_{n_2} \otimes (I_{n_i} - \frac{1}{n_i+1}J_{n_i}) \otimes j'_{n_{i+1}} \otimes \dots \otimes j'_{n_k}$$

for $i = 1, 2, \dots, k$. $P_2(XX')^+$ has $\binom{k}{2}$ block matrices and each subblock of $P_2(XX')^+$ is

$$vj'_{n_1} \otimes j'_{n_2} \otimes (I_{n_i} - \frac{1}{n_i+1}J_{n_i}) \otimes j'_{n_{i+1}} \otimes \dots \otimes (I_{n_r} - \frac{1}{n_r+1}J_{n_r}) \otimes j'_{n_{r+1}} \otimes \dots \otimes j'_{n_k}$$

for $i = 1, 2, \dots, k$; $r = 1, 2, \dots, k$ and $i < r$, where $v = \prod_{j=1, j \neq i, j \neq r}^k \frac{1}{n_j+1}$. Similarly, we can construct others $\tilde{P}_i' = P_i(XX')^+$ matrices.

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