A central limit theorem for moving average process with negatively associated innovation

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Abstract
In this paper we derive the central limit theorem for \(\sum_{i=1}^{n} a_{ni} \xi_{i}\), where \(\{a_{ni}, 1 \leq i \leq n\}\) is a triangular array of nonnegative numbers such that \(\sup_{n} \sum_{i=1}^{n} a_{ni}^{2} < \infty\), \(\max_{1 \leq i \leq n} a_{ni} \to 0\) as \(n \to \infty\) and the innovation \(\{\xi_{i}\}\) is a centered sequence of negatively associated random variables. We also apply this result to obtain the asymptotic behavior of a partial sum of a moving average process \(X_{n} = \sum_{j=-\infty}^{\infty} a_{k+j} \xi_{j}\) where \(\{a_{j}\}\) is a sequence of nonnegative numbers such that \(\sum_{j=-\infty}^{\infty} a_{j} < \infty\).

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1. Introduction
The sequence \(\{\xi_{i}, i \in N\}\) is called negatively associated(NA) if for every pair of disjoint subsets \(A, B\) of \(N\) and any pair of coordinatewise increasing functions \(f(\xi_{i}, i \in A), g(\xi_{j}, j \in B)\) with \(Ef^{2}(\xi_{i}, i \in A) < \infty\) and \(Eg^{2}(\xi_{j}, j \in B) < \infty\), it holds that
\[
\text{Cov}(f(\xi_{i}, i \in A), g(\xi_{j}, j \in B)) \leq 0.
\]
The concept of NA was introduced by Joag-Dev and Proschan (1983). As pointed out and proved by Joag-Dev and Proschan (1983), a number of well-known multivariate distributions possess the NA property, such as multinomial distribution, multivariate hypergeometric distribution, negatively correlated normal distribution and joint distribution of ranks. Because of their wide applications in multivariate statistical analysis and reliability theory, the concepts of negatively associated random variables have received extensive attention recently. We refer to Joag-Dev and the Proschan (1983) for fundamental properties and we refer to Newman (1984) for a moment inequality and the convergence in distribution.

Let \( \{\xi_i\} \) be a centered sequence of random variables and let \( \{a_{ni}, 1 \leq i \leq n\} \) be a triangular array of numbers. Many statistical procedures produce estimators of the type

\[
S_n = \sum_{i=1}^{n} a_{ni} \xi_i. \tag{1.1}
\]

We will derive the convergence in distribution of the type (1.1), where \( \xi_i' \)'s are NA sequence.

Define a moving average process by

\[
X_n = \sum_{j=-\infty}^{\infty} a_{k+j} \xi_j. \tag{1.2}
\]

where the innovation \( \{\xi_k\} \) is a centered sequence of random variables and \( \{a_k\} \) is a sequence of real numbers. In time-series analysis, this process is of great importance. Many important time-series models, such as the causal ARMA process (Brockwell and Davis, 1987, p. 89), have type (1.2).

We shall see that the central limit theorem of NA variables of the form (1.2), where \( \xi_i' \)'s are NA random variables, can be obtained by the study of the weighted sum of NA random variables of the form (1.1).

Our paper organized in the following way: Section 2 contains the main results and the statement of some known results which will be used in the proof and Section 3 contains the proof of the main results.

2. Results

**Theorem 2.1** Let \( \{a_{ni}, 1 \leq i \leq n\} \) be a triangular array of nonnegative numbers such that

\[
\sup_{n} \sum_{i=1}^{n} a_{ni}^2 < \infty, \tag{2.1}
\]
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and

\[ \max_{1 \leq i \leq n} a_{ni} \to 0 \text{ as } n \to \infty. \] (2.2)

Let \( \{\xi_i\} \) be a centered sequence of negatively associated random variables such that

\( \{\xi_i^2\} \) is an uniformly integrable family \hspace{1cm} (2.3)

and

\[ \text{Var}(\sum_{i=1}^{n} a_{ni} \xi_i) = 1. \] (2.4)

Assume

\[ \sum_{j : |i-j| \geq u} \text{Cov}(\xi_i, \xi_j) \to 0 \text{ as } u \to \infty \text{ uniformly in } i \geq 1. \] (2.5)

\text{(see Cox and Grimmett(1984))}

Then

\[ \sum_{i=1}^{n} a_{ni} \xi_i \xrightarrow{D} N(0,1) \text{ as } n \to \infty. \]

Remark Theorem 2.1 extends the Newman’s(1984) central limit theorem for strictly stationary negatively associated sequence from equal weights to general weights, weakening at the same time the assumption of stationarity.

Corollary 2.1 Let \( \{\xi_i\} \) be a negatively associated sequence of centered random variables such that \( \{\xi_i^2\} \) is an uniformly integrable family and let \( \{a_{ni}, 1 \leq i \leq n\} \) be a triangular array of nonnegative numbers such that

\[ \sup_n \sum_{i=1}^{n} \frac{a_{ni}^2}{\sigma_n^2} < \infty, \] (2.6)

\[ \max_{1 \leq i \leq n} \frac{a_{ni}}{\sigma_n} \to 0 \text{ as } n \to \infty. \] (2.7)

where \( \sigma_n^2 = \text{Var}(\sum_{i=1}^{n} a_{ni} \xi_i) \). If (2.5) holds, then, as \( n \to \infty \)

\[ \frac{1}{\sigma_n} \sum_{i=1}^{n} a_{ni} \xi_i \xrightarrow{D} N(0,1). \] (2.8)
Theorem 2.2  Let \( \{a_j, j \in \mathbb{Z}\} \) be a sequence of nonnegative numbers such that \( \sum_j a_j < \infty \) and let \( \{\xi_j, j \in \mathbb{Z}\} \) be a centered sequence of negatively associated random variables which is uniformly integrable in \( L_2 \) and satisfies (2.5). Let

\[
X_k = \sum_{j=-\infty}^{\infty} a_{k+j} \xi_j \quad \text{and} \quad S_n = \sum_{i=1}^{n} X_i.
\]

Assume

\[
\inf_{n \geq 1} n^{-1} \sigma_n^2 > 0 \tag{2.9}
\]

where \( \sigma_n^2 = \text{Var}(S_n) \). Then

\[
\frac{S_n}{\sigma_n} \xrightarrow{p} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty. \tag{2.10}
\]

This result is an extension of Theorem 18.6.5 in Ibragimov and Linnik(1971) from the i.i.d. to the negative association by adding the condition (2.5).

3. Proofs

We starts with the following lemma.

Lemma 3.1 (Newman, 1980) Let \( \{Z_i, 1 \leq i \leq n\} \) be a sequence of negatively associated random variables with finite second moments. Then

\[
|E \exp(it \sum_{j=1}^{n} Z_j) - \prod_{j=1}^{n} E \exp(it Z_j)| \leq C t^2 |\text{Var}(\sum_{j=1}^{n} Z_j) - \sum_{j=1}^{n} \text{Var}(Z_j)|
\]

for all \( t \in \mathbb{R} \), where \( C > 0 \) is an arbitrary constant, not depending on \( n \).

Proof of Theorem 2.1  Without loss of generality we assume that \( a_{ni} = 0 \) for all \( i > n \) and \( \sup E \xi_n^2 = 1 \). For every \( 1 \leq a < b \leq n \) and \( 1 \leq u \leq b-a \) we have, after a simple manipulations,

\[
0 \leq \sum_{i=a}^{b-u} a_{ni} \sum_{j=i+u}^{b} a_{nj} \text{Cov}(\xi_i, \xi_j)^-
\]

\[
\leq \sup_k \left( \sum_{j:|k-j| \geq u} \text{Cov}(\xi_k, \xi_j)^- \right) \left( \sum_{i=a}^{b} a_{ni}^2 \right). \tag{3.1}
\]

In particular, by (2.5) and (3.1) there exists a positive constant \( M \), for every \( 1 \leq a \leq b \leq n \),

\[
\text{Var}(\sum_{i=a}^{b} a_{ni} \xi_i) \leq M \sum_{i=a}^{b} a_{ni}^2. \tag{3.2}
\]
We shall construct now a triangular array of random variables \( \{Z_{ni}, 1 \leq i \leq n\} \) for which we shall make use of Lemma 3.1. Fix a small positive \( \epsilon \) and find a positive integer \( u = u_\epsilon \) such that, for every \( n \geq u + 1 \)

\[
0 \leq \left( \sum_{i=1}^{n-u} a_{ni} \sum_{j=i+u}^{b} a_{nj} \text{Cov}(\xi_i, \xi_j) \right) \leq \epsilon. \tag{3.3}
\]

This is possible because of (3.1) and (2.5). Denote by \([x]\) the integer part of \( x \) and define

\[
K = \left[ \frac{1}{\epsilon} \right],
\]

\[
Y_{nj} = \sum_{i=uj+1}^{u(j+1)} a_{ni}\xi_i, \quad j = 0, 1, \cdots,
\]

\[
A_j = \{i : 2Kj \leq i < 2Kj + K, \text{Cov}(Y_{ni}, Y_{n,i+1}) \leq \frac{2Kj+K}{K} \sum_{i=2Kj} \text{Var}(Y_{ni}) \}.
\]

Since \( 2\text{Cov}(Y_{ni}, Y_{n,i+1}) \leq \text{Var}(Y_{ni}) + \text{Var}(Y_{n,i+1}) \), we get that for every \( j \) the set \( A_j \) is not empty. Now we define the integers \( m_1, m_2, \cdots, m_n \) recurrently. Let \( m_0 = O \) and

\[
m_{j+1} = \text{min}\{m ; m > m_j, m \in A_j\}
\]

and define

\[
Z_{nj} = \sum_{i=m_j+1}^{m_{j+1}} Y_{ni}, \quad j = 0, 1, \cdots,
\]

\[
\Delta_j = \{u(m_j + 1) + 1, \cdots, u(m_{j+1} + 1)\}.
\]

We observe that

\[
Z_{nj} = \sum_{k \in \Delta_j} a_{nk}\xi_k, \quad j = 0, 1, \cdots,
\]

By definition of NA \( Z'_{nj}s \) are negatively associated. From the fact that \( m_j \geq 2K(j - 1) \) and \( m_{j+1} \leq K(2j + 1) \) every set \( \Delta_j \) contains no more than \( 3Ku \) elements and \( m_{j+1}/m_j \to 1 \) as \( j \to \infty \). Hence, for every fixed positive \( \epsilon \) by (2.1) - (2.4) the array \( \{Z_{nj} : i = 1, \cdots, n; \ n \geq 1\} \) satisfies the Lindeberg’s condition (see Petrov(1975), Theorem 22, p.100). It remains to observe that by Lemma 3.1 and construction,
\[ |E \exp(it \sum_{j=1}^{n} Z_{nj}) - \prod_{j=1}^{n} E \exp(iZ_{nj})| \]
\[ \leq c t^2 \{ Var(\sum_{j=1}^{n} Z_{nj}) - \sum_{j=1}^{n} Var(Z_{nj}) \} \]
\[ \leq c t^2 \{ 2(\sum_{i=1}^{n} Cov(Z_{ni}, Z_{n,i+1})^-) + 2(\sum_{i=1}^{n-2} \sum_{j=i+2}^{n} Cov(Z_{ni}, Z_{nj})^-) \} \]
\[ \leq c t^2 \{ 4 \sum_{i=1}^{n-u} a_{ni} \sum_{j=i+u}^{n} a_{nj} Cov(\xi_i, \xi_j)^- + 2 \sum_{j=1}^{n} Cov(Y_{n,m_j}, Y_{n,m_j+1})^- \} \]
\[ \leq c t^2 \{ 4\epsilon + \frac{8}{K} \sum_{i=1}^{n} Var(Y_{ni}) \} \]
\[ = c t^2 \{ 4\epsilon + \frac{8}{K} \sum_{j=1}^{n} Var(\sum_{i=uj+1}^{u(j+1)} a_{ni}\xi_i) \} \]
\[ \leq c t^2 \{ 4\epsilon + \frac{8M}{K} \sum_{j=1}^{n} \sum_{i=uj+1}^{u(j+1)} a_{ni}^2 \} \text{ by (3.2)} \]
\[ \leq c_1 t^2 \epsilon \{ 1 + \sup_{n} \sum_{i=1}^{n} a_{ni}^2 \} \]
\[ \leq c_2 t^2 \epsilon. \]

Now the proof is complete by Theorem 4.2 in Billingsley(1968).

**Proof of Corollary 2.1**: Let \( A_{ni} = \frac{a_{ni}}{\sigma_n} \). Then we have
\[
\max_{1 \leq i \leq n} A_{ni} \to 0 \quad \text{as} \quad n \to \infty,
\]
\[
\sup_{n} \sum_{i=1}^{n} A_{ni}^2 < \infty,
\]
\[
Var(\sum_{i=1}^{n} A_{ni}\xi_i) = 1.
\]
Hence, by Theorem 2.1 the desired result (2.10) follows.

**Proof of Theorem 2.2**: First note that \( \sum_j a_j^2 < \infty \) and without restricting the generality, we can assume \( \sup EC_k^2 = 1 \). Let
\[
S_n = \sum_{k=1}^{n} X_k = \sum_{j=-\infty}^{\infty} (\sum_{k+j}^{n} a_{k+j})\xi_j.
\]
In order to apply Theorem 2.1, we fix \( W_n \) such that \( \sum_{|j| > W_n} a_j^2 < n^{-3} \) and take \( k_n = W_n + n \). Then

\[
\frac{S_n}{\sigma_n} = \sum_{|j| \leq k_n} \frac{\sum_{k=1}^{n} a_{k+j}}{\sigma_n} \xi_j + \sum_{|j| > k_n} \frac{\sum_{k=1}^{n} a_{k+j}}{\sigma_n} \xi_j
\]

\[
= T_n + U_n \text{(say)}.
\]

and by Cauchy Schwarz inequality and assumptions we have the following estimate

\[
Var(U_n) \leq \sum_{|j| > k_n} \left( \sum_{k=1}^{n} a_{k+j}/\sigma_n \right)^2 \xi_j^2
\]

\[
\leq n \sigma_n^{-2} \sum_{|j| > k_n} \left( \sum_{k=1}^{n} a_{k+j}^2 \right)
\]

\[
\leq n^2 \sigma_n^{-2} \sum_{|j| > W_n} a_j^2
\]

\[
\leq n^2 \sigma_n^{-2} \sum_{|j| > W_n} a_j^2
\]

which yields

\[
U_n \to 0 \text{ in probability as } n \to \infty. \quad (3.4)
\]

By Theorem 4.1 of Billingsley(1968) it remains only to prove that \( T_n \xrightarrow{D} N(0,1) \). Put

\[
a_{nk} = \frac{\sum_{j=1}^{n} a_{k+j}}{\sigma_n}.
\]

(3.5)

From assumption \( \sum_j a_j < \infty \) \( (a_j > 0) \), (2.9) and (3.5) we obtain

\[
\sup_{-\infty < k < \infty} \frac{\sum_{j=1}^{n} a_{k+j}}{\sigma_n} \to 0 \quad \text{as } n \to \infty,
\]

\[
\max_{1 \leq k \leq n} a_{nk} \to 0 \quad \text{as } n \to \infty,
\]

\[
\sup_n \sum_{k=1}^{n} a_{nk}^2 < \infty.
\]
Hence, by Theorem 2.1
\[ T_n \xrightarrow{D} N(0, 1) \quad (3.6) \]
and from (3.4) and (3.6) the desired result (2.10) follows.

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