Some Class of Morphisms on Finsler Pro- C^* -Modules

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Abstract

The present paper deals with some classes of morphisms between Finsler and Hilbert modules over a pro- C^* -algebra and C^* -algebra. Especially, we prove that the set of all continuous $\mathbb{C}$ - and A -linear maps over a Finsler pro- C^* -module is a complete LMC -algebra. Also, the polar decomposition theorem for Hilbert pro- C^* -modules is discussed. Finally, we introduce the Hilbert-Schmidt operators on Hilbert C^* -modules.

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1. INTRODUCTION

The operator theory of C^* -algebras and Hilbert C^* -modules and investigation of basic properties of morphisms between them, are interesting and important in mathematics. Recently, this theory between pro- C^* -algebras and Hilbert pro- C^* -modules was developed by several researchers. For example, by A. Mallios in [7], N. C. Phillips in [9]. On the other hand, some of the results

about the morphisms between Hilbert spaces and Hilbert C^* -modules have been proved and these results did not generalize to the case of pro- C^* -modules, see [1]. We introduced in [5], the notion of Finsler pro- C^* -modules as a generalization of Finsler C^* -modules and Hilbert pro- C^* -modules. In this section, we recall the definition of Finsler pro- C^* -module. In section 2, we introduce the notion of φ -morphism between Finsler pro- C^* -modules and investigate some of its basic properties. Especially, in Theorem 2.10, we prove that ${}_AB(X)$, the algebra of all bounded A - and $\mathbb{C}$ -linear maps on a Finsler pro- C^* -module X is a complete LMC -algebra. In section 3, we introduce the Hilbert-Schmidt operators on Hilbert C^* -modules and then, we get some important results about them.

Definition 1.1. A *pro- C^* -algebra* is a complete Hausdorff topological $*$ -algebra, with a topology induced by its continuous C^* -seminorms, in the sense that a net $(a_\lambda)_\lambda$ converges to 0 if and only if $p(a_\lambda) \rightarrow 0$ for every continuous C^* -seminorm p on A .

Definition 1.2. Let A be a pro- C^* -algebra and $S(A)$ denotes the set of all continuous C^* -seminorms on A . Also, suppose A^+ denotes the set of positive elements of A . A *pre-Finsler module* over A is a left A -module X which is equipped with a map $\rho_A : X \rightarrow A^+$ such that

- (1) The map $\bar{p} : x \mapsto p(\rho_A(x))$ for every $p \in S(A)$ is a seminorm on X ,
- (2) $\rho_A(ax)^2 = a\rho_A(x)^2a^*$ for $a \in A$ and $x \in X$,
- (3) $\rho_A(x) = 0$ if and only if $x = 0$.

If in addition, X is complete with respect to the family of seminorms $\{\bar{p}\}_{p \in S(A)}$, X is called a *Finsler pro- C^* -module* over A . It is obvious that if $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert module over pro- C^* -algebra A , then H with $\rho_A(x) = \langle x, x \rangle^{1/2}$ is a Finsler A -module.

We denote the linear span of $E \subseteq A$ by $\langle E \rangle$. A Finsler module (X, ρ_A) over pro- C^* -algebra A is called *full*, if the linear span of

$$\rho_A(X)^2 = \{\rho_A(x)^2, x \in X\},$$

is dense in A . In other words,

$$A = \overline{\langle \rho_A(X)^2 \rangle},$$

where bar is the closure with respect to the locally convex topology of A .

2. φ -MORPHISMS OF FINSLER MODULES

Definition 2.1. Let (X, ρ_A) and (Y, ρ_B) be Finsler modules over pro- C^* -algebras A and B , respectively, and let $\varphi : A \longrightarrow B$ be a morphism of pro- C^* -algebras. A linear operator $\Phi : X \longrightarrow Y$ is called a φ -morphism of Finsler modules if the following conditions are satisfied:

- (1) $\Phi(ax) = \varphi(a)\Phi(x)$,
- (2) $\rho_B(\Phi(x)) = \varphi(\rho_A(x))$,

for all $x \in X$ and $a \in A$. If $X = Y$ and $A = B$ and Φ is an id_A -morphism, then Φ is called an *isometry* on X .

Let X, Y and Z be Finsler modules over pro- C^* -algebras A, B and C , respectively, and let $\varphi_1 : A \longrightarrow B$ and $\varphi_2 : B \longrightarrow C$ be $*$ -morphisms of pro- C^* -algebras. If $\Phi_1 : X \longrightarrow Y$ and $\Phi_2 : Y \longrightarrow Z$ are φ_1 - and φ_2 -morphism of Finsler modules, respectively, it is obvious that $\Phi_2\Phi_1 : X \longrightarrow Z$ is a $\varphi_2\varphi_1$ -morphism.

Theorem 2.2. [5, Proposition 1.23] *Let (X, ρ_A) be a Finsler module over pro- C^* -algebra A and I be a two sided ideal of A . Then*

$$< IX > = \text{span}\{ax : a \in I, x \in X\} = \{x \in X : \rho_A(x) \in I\}.$$

Theorem 2.3. *Let X and Y be Finsler modules over pro- C^* -algebras A and B , respectively, let $\varphi : A \longrightarrow B$ be a $*$ -morphism of pro- C^* -algebras and let $\Phi : X \longrightarrow Y$ be a φ -morphism of Finsler modules. Then*

- (1) *If $\ker \varphi = I$, then $\ker \Phi = < IX >$. Especially, if φ is injective, then Φ is injective.*
- (2) *If X is a full Finsler A -module and Φ is injective, then φ is injective.*

Proof. (1) Let $z \in IX$, and $z = ax$ for some $a \in I$ and $x \in X$. Therefore

$$\Phi(z) = \Phi(ax) = \varphi(a)\Phi(x) = 0,$$

so, $z \in \ker \Phi$ and because $\ker \Phi$ is a closed submodule of X , then $< IX > \subseteq \ker \Phi$. Conversely, if $z \in \ker \Phi$, then $\Phi(z) = 0$, therefore,

$$\varphi(\rho_A(z)) = \rho_B(\Phi(z)) = 0,$$

hence $\rho_A(z) \in \ker \varphi$. By Theorem 2.2, we have $z \in < IX >$, therefore $\ker \Phi \subseteq < IX >$.

(2) Let Φ be injective and $I = \ker \varphi$. Then $\ker \Phi = \langle IX \rangle = 0$, hence $IX = \{0\}$. If $a \in \ker \varphi$, then for any $x \in X$, we have $ax = 0$. Fullness of X implies that for every $b \in A$,

$$b = \lim_{\lambda} \sum_{i=1}^{k_{\lambda}} C_{i,\lambda} \rho_A(x_{i,\lambda})^2, C_{i,\lambda} \in \mathbb{C}, x_{i,\lambda} \in X, k_{\lambda} \in \mathbf{N}.$$

Then,

$$\begin{aligned} aba^* &= \lim_{\lambda} \sum_{i=1}^{k_{\lambda}} C_{i,\lambda} \rho_A(x_{i,\lambda})^2 \\ &= \lim_{\lambda} \sum_i C_{i,\lambda} \rho_A(ax_{i,\lambda})^2 = 0. \end{aligned}$$

Now by substituting b by a^*a , we have

$$(aa^*)^2 = 0.$$

Then for all $p \in S(A)$, $p(aa^*) = 0$, since p is a C^* -seminorm, $p(a) = 0$ and so $a = 0$. $\diamond$

Theorem 2.4. *Let X and Y be Finsler modules over pro- C^* -algebras A and B , respectively. Let $\varphi : A \rightarrow B$ be a morphism of pro- C^* -algebras and $\Phi : X \rightarrow Y$ be a continuous φ -morphism. Then*

(1) *$\text{Im}\Phi$ is a pre-Finsler module over the pro- C^* -algebra $\text{Im}\varphi \subset B$ such that*

$$\langle \rho_{\text{Im}\varphi}(\text{Im}\Phi)^2 \rangle = \varphi(\langle \rho_A^2(X) \rangle).$$

(2) *If X is a full A -module, then $\text{Im}\Phi$ is also a full $\text{Im}\varphi$ -module.*

(3) *If Φ is surjective and Y is a full B -module, φ is also surjective.*

Proof. (1) By continuity of Φ , $\text{Im}\Phi$ is a closed submodule of Y . Also we have $\varphi(a)\Phi(x) = \Phi(ax) \in \text{Im}\Phi$ and $\rho_{\text{Im}\varphi}(\Phi(x)) = \varphi(\rho_A(x))$. Then $\text{Im}\Phi$ is a Finsler module over pro- C^* -algebra $\text{Im}\varphi$ and $\rho_{\text{Im}\varphi}(\text{Im}\Phi) = \varphi(\rho_A(X))$, and hence

$$\langle \rho_{\text{Im}\varphi}^2(\text{Im}\Phi) \rangle = \varphi(\langle \rho_A^2(X) \rangle).$$

(2) If X is full, then $\langle \rho_{\text{Im}\varphi}^2(\text{Im}\Phi) \rangle = \text{Im}\varphi$, therefore, $\text{Im}\Phi$ is a full Finsler- $\text{Im}\varphi$ module.

(3) Let Φ be surjective and let Y be a full B -module. We have

$$\varphi(\langle \rho_A^2(X) \rangle) = \langle \rho_B^2(\text{Im}\Phi) \rangle = \langle \rho_B^2(Y) \rangle = B,$$

then $B \subseteq \varphi(A) \subseteq B$, hence φ is surjective. $\diamond$

Corollary 2.5. *Let X be a full Finsler A -module, and let φ and Φ be surjective, then Y is a full Finsler B -module.*

Definition 2.6. Let A and B be pro- C^* -algebras, and let X and Y be Finsler modules over A and B , respectively. We say that a map $\Phi : X \longrightarrow Y$ is a *unitary* if there exists an injective morphism $\varphi : A \longrightarrow B$ such that Φ is a surjective φ -morphism.

Remark 2.7. (1) Every unitary map between Finsler pro- C^* -algebras is an isomorphism (by Theorem 2.3.).

(2) Let Y be a full Finsler B -module. Since Φ is surjective, then by Theorem 2.4, φ is also surjective and hence is an isomorphism between pro- C^* -algebras.

(3) Let X be a Finsler module over pro- C^* -algebra A and let $\varphi : A \longrightarrow B$ be an isomorphism between pro- C^* -algebras. Then we regard X as a Finsler module over B . Now the identity map between these Finsler modules is necessarily a unitary map.

Conversely, let X and Y be Finsler modules over pro- C^* -algebras A and B , respectively, let Y be a full module and let Φ be a unitary map between X and Y . Then A and B are isomorphic (by part 2).

(4) Unitary equivalence of full Finsler pro- C^* -modules is an equivalence relation.

(5) Let $\varphi : A \longrightarrow B$ be an isomorphism between pro- C^* -algebras, let X and Y be full Finsler modules over A and B , respectively, and let $\Phi : X \longrightarrow Y$ be a unitary φ -morphism. Then $\Phi^{-1} : Y \longrightarrow X$ is also a unitary φ^{-1} -morphism. Because, Φ by part (1), is an isomorphism and then Φ^{-1} is a linear operator. Moreover, if $b \in B$ and $y \in Y$, then there exist an element $a \in A$ and $x \in X$ such that $\varphi(a) = b$, $\Phi(x) = y$. Then

$$\Phi^{-1}(by) = \Phi^{-1}(\varphi(a)\Phi(x)) = \Phi^{-1}(\Phi(ax)) = ax = \varphi^{-1}(b)\Phi^{-1}(y).$$

Also, we have

$$\varphi(\rho_A(x)) = \rho_B(\Phi(x)) = \rho_B(y),$$

then

$$\rho_A(\Phi^{-1}(y)) = \varphi^{-1}(\rho_B(y)). \diamond$$

Definition 2.8. Let X be a Finsler module over a pro- C^* -algebra A and let $S(A)$ denotes the set of all continuous seminorms on A . An A and $\mathbb{C}$ - linear map $\varphi : X \longrightarrow X$ is called *bounded* if

$$(1) \quad \varphi(ax) = a\varphi(x), \quad a \in A, \quad x \in X. \quad (A\text{-linearity of } \varphi)$$

- (2) For every $p \in S(A)$, there is a positive real number c_p such that $\bar{p}(\varphi(x)) \leq c_p \bar{p}(x)$ for any $x \in X$.

The set of all bounded A - and $\mathbb{C}$ -linear maps over X is denoted by ${}_AB(X)$.

Lemma 2.9. *For every $p \in S(A)$,*

$$\tilde{p}(\varphi) = \sup_{\bar{p}(x) \leq 1} \bar{p}(\varphi(x)) \quad (\varphi \in {}_AB(X)),$$

is a seminorm on ${}_AB(X)$.

Proof. Obvious. ◇

Theorem 2.10. *The set ${}_AB(X)$ is a complete LMC-algebra, i.e. an algebra which is complete, separable and locally convex with respect to the set $\{\tilde{p}\}_{p \in S(A)}$ and any $\tilde{p}$ is submultiplicative.*

Proof. Let $\varphi_1, \varphi_2 \in {}_AB(X)$. Then for any $p \in S(A)$ and $x \in X$, we have

$$\begin{aligned} \bar{p}((\varphi_1 + \varphi_2)(x)) &= \bar{p}(\varphi_1(x) + \varphi_2(x)) \leq (\bar{p}(\varphi_1(x)) + \bar{p}(\varphi_2(x))) \\ &\leq (\tilde{p}(\varphi_1) + \tilde{p}(\varphi_2))\bar{p}(x). \end{aligned}$$

Then

$$\tilde{p}(\varphi_1 + \varphi_2) \leq \tilde{p}(\varphi_1) + \tilde{p}(\varphi_2) < \infty.$$

Also

$$\begin{aligned} \bar{p}((\varphi_1 \varphi_2)(x)) &= \bar{p}(\varphi_1(x) \varphi_2(x)) \\ &\leq \bar{p}(\varphi_1(x)) \bar{p}(\varphi_2(x)) \\ &\leq \tilde{p}(\varphi_1) \tilde{p}(\varphi_2) \bar{p}(x)^2. \end{aligned}$$

Then $\tilde{p}(\varphi_1 \varphi_2) \leq \tilde{p}(\varphi_1) \tilde{p}(\varphi_2)$. Therefore, ${}_AB(X)$ is an algebra and each $\tilde{p}$ is submultiplicative.

Suppose that $\tilde{p}(\varphi) = 0$ for all $p \in S(A)$. Then for every $x \in X$, $\tilde{p}(\varphi(x)) = 0$ and hence $\varphi(x) = 0$. Therefore, $\varphi = 0$ which proves that the set $\{\tilde{p}\}_{p \in S(A)}$ is separating. Now, we prove that ${}_AB(X)$ is complete with respect to $\{\tilde{p}\}_{p \in S(A)}$, then by Lemma 2.9, the theorem is proved. Let $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ be a Cauchy net in ${}_AB(X)$. In other words, for $\epsilon > 0$ and $p \in S(A)$, there is a $\lambda_0 \in \Lambda$ such that for every $\lambda, \mu \in \Lambda$ with $\lambda, \mu \geq \lambda_0$

$$\tilde{p}(\varphi_\lambda - \varphi_\mu) < \epsilon.$$

If $x \in X$, we have

$$\bar{p}((\varphi_\lambda - \varphi_\mu)(x)) \leq \tilde{p}(\varphi_\lambda - \varphi_\mu)\bar{p}(x) < \epsilon\bar{p}(x). \quad (*)$$

Therefore, for any $x \in X$, the net $\{\varphi_\lambda(x)\}_{\lambda \in \Lambda}$ is Cauchy in X , and hence converges to an element $\varphi(x) \in X$. It is obvious that the map $\varphi : X \longrightarrow Y$ is $\mathbb{C}$ -linear and $\varphi(ax) = a\varphi(x)$ for $a \in A$ and $x \in X$. We have

$$|\tilde{p}(\varphi_\lambda) - \tilde{p}(\varphi_\mu)| \leq \tilde{p}(\varphi_\lambda - \varphi_\mu),$$

for $\lambda, \mu \in \Lambda$. Then the net $\{\tilde{p}(\varphi_\lambda)\}_{\lambda \in \Lambda}$ is Cauchy in $\mathbb{R}$ and hence is convergent to an element c_p . Therefore,

$$\bar{p}(\varphi(x)) = \lim_{\lambda} \bar{p}(\varphi_\lambda(x)) \leq (\lim_{\lambda} \tilde{p}(\varphi_\lambda))\bar{p}(x) = c_p \cdot \bar{p}(x),$$

for $x \in X$, and we have $\varphi \in {}_A B(X)$. Now by (*) and for any X , we have

$$\bar{p}((\varphi_\lambda - \varphi)(x)) \leq \epsilon\bar{p}(x).$$

Then $\tilde{p}(\varphi_\lambda - \varphi) \leq \epsilon$. Therefore, $\varphi_\lambda \longrightarrow \varphi$ in ${}_A B(X)$ which proves the theorem. $\diamond$

3. THE HILBERT-SCHMIDT OPERATORS ON HILBERT C^* -MODULES

Let H_1, H_2 be Hilbert pro- C^* -modules and let $\varphi \in B(H_1, H_2)$. Then φ is adjointable with adjoint $\varphi^* \in B(H_2, H_1)$ if we have

$$\langle x, \varphi(y) \rangle = \langle \varphi^*(x), y \rangle, \quad x \in H_2, y \in H_1.$$

We denote the set of adjointable elements of $B(H_1, H_2)$ by $B(H_1, H_2)_a$. If $H_1 = H_2 = H$, then $B(H)_a$ is a pro- C^* -algebra (See [9, Theorem 2.4.]).

Let M be a submodule of H_1 and $M^\perp$ denotes the orthogonal complement of M in H_1 . It is easy to see that $M^\perp$ is a closed submodule of H_1 ($M^\perp = \{x \in H_1 : \langle x, y \rangle = 0 \text{ for any } y \in M\}$). Suppose $\varphi \in B(H_1, H_2)_a$, then $\overline{(Im\varphi^*)} = (\ker \varphi)^\perp$, where bar is the closure with respect to the locally convex topology of H_1 .

Definition 3.1. Suppose $\varphi : H_1 \longrightarrow H_2$ is a continuous morphism of Hilbert modules over pro- C^* -algebra A . We call φ a partial isometry if

$$\langle \varphi(x), \varphi(x) \rangle = \langle x, x \rangle, \quad x \in (\ker \varphi)^\perp.$$

In other words, φ is an isometry on $(\ker \varphi)^\perp$.

Theorem 3.2. *Let H_1, H_2 be Hilbert modules over pro- C^* -algebra A and $\varphi \in B(H_1, H_2)_a$. Then the following statements are equivalent, if $\ker \varphi$ is orthogonally complemented in H_1 :*

- (1) $\varphi = \varphi\varphi^*\varphi$;
- (2) $\varphi^*\varphi$ is a projection;
- (3) $\varphi\varphi^*$ is a projection;
- (4) φ is a partial isometry.

Proof. (2) $\implies$ (3): If $a = \varphi\varphi^* \in B(H_2)_a$, then by Corollary 1.7 of [9] we have

$$\sigma(a) \cup \{0\} = \sigma(\varphi^*\varphi) \cup \{0\} \subseteq \{0, 1\},$$

so $\sigma(a) \subseteq \{0, 1\}$. Now by continuous functional calculus in a [9, Proposition 1.8] and for $f(z) = z^2 - z$, we have

$$f(z) = z^2 - z \in C(\sigma(a)),$$

then $f(a) = 0$. Therefore, a and consequently $\varphi\varphi^*$ is a projection.

(3) $\implies$ (2): By the preceding argument and symmetry.

(2) $\implies$ (1): If $\varphi^*\varphi$ is a projection, then for every $p \in S(A)$,

$$\begin{aligned} \bar{p}(\varphi(z)) &= p(\langle \varphi(z), \varphi(z) \rangle)^{1/2} \\ &= p(\langle \varphi^*\varphi(z), z \rangle)^{1/2} \\ &= p(\langle \varphi^*\varphi(z), \varphi^*\varphi(z) \rangle)^{1/2} \\ &= \bar{p}(\varphi^*\varphi(z)). \end{aligned}$$

Now for

$$z = (1 - \varphi^*\varphi)(x),$$

we have $\bar{p}(\varphi(1 - \varphi^*\varphi)(x)) = 0$, for all $p \in S(A)$ and for any $x \in H_1$. Therefore $\varphi(1 - \varphi^*\varphi) = 0$ and so $\varphi = \varphi\varphi^*\varphi$.

(1) $\implies$ (4): If $\varphi = \varphi\varphi^*\varphi$, then $\varphi^*\varphi$ is a projection on $(\ker \varphi)^\perp$, because

$$(\ker \varphi)^\perp = (Im \varphi^*)^- = (\varphi^*(H_2))^- = \varphi^*\varphi(H_1).$$

Then $\varphi^*\varphi = id$ on $(\ker \varphi)^\perp$, and hence for $p \in S(A)$ and $x \in (\ker \varphi)^\perp$

$$\langle \varphi(x), \varphi(x) \rangle = \langle \varphi^*\varphi(x), x \rangle = \langle x, x \rangle.$$

(4) $\implies$ (2): Suppose that φ is a partial isometry and π is the projection on $(\ker \varphi)^\perp$. For $x \in (\ker \varphi)^\perp$

$$\langle \pi(x), x \rangle = \langle x, x \rangle = \langle \varphi(x), \varphi(x) \rangle = \langle \varphi^*\varphi(x), x \rangle.$$

If $x \in \ker \varphi$, we have $\varphi(x) = 0$ and then

$$\langle \pi(x), x \rangle = \langle 0, x \rangle = \langle \varphi^* \varphi(x), x \rangle.$$

Because $H_1 = \ker \varphi \oplus (\ker \varphi)^\perp$, we have $\langle \pi(x), x \rangle = \langle \varphi^* \varphi(x), x \rangle$ for all $x \in H_1$. Then by polarization identity

$$\langle \pi(x), y \rangle = \langle \varphi^* \varphi(x), y \rangle, x, y \in H_1.$$

Therefore, $\pi = \varphi^* \varphi$. ◇

Theorem 3.3. (Polar Decomposition) *Let H be a Hilbert pro- C^* -module and let $\varphi \in B(H)_a$ such that the closure of $|\varphi|(H)$ is orthogonally complemented in H . Then there is a unique partial isometry $\psi \in B(H)_a$ such that*

$$\varphi = \psi |\varphi|, \quad \ker \varphi = \ker \psi.$$

Moreover, $\psi^* \varphi = |\varphi|$. (Note that $|\varphi| = (\varphi^* \varphi)^{1/2}$, by Proposition 2.1 of [4] exists.)

Proof. As in the Proof of Theorem 2.3.4 of [8] with this remark, what we means isometry, a map φ such that $\langle \varphi(x), \varphi(y) \rangle = \langle x, y \rangle$ for $x, y \in H$. ◇

We recall that if H is a Hilbert module over a unital C^* -algebra A , then a set of *generators* in H is a subset $\{x_i\}_{i \in I}$ of H such that the A -linear hull of $\{x_i : i \in I\}$ is norm-dense in H . The Hilbert module H is *finitely (countably) generated* if there exists a finite (countable) set of generators in H . A finite (countable) set $\mathcal{F} = \{\xi_j : j \in \mathcal{I}\}$ in H is a *frame*, if there are real constants C and D , $0 < C \leq D$ such that

$$C \langle x, x \rangle \leq \sum_{i \in I} \langle x, x_i \rangle \langle x_i, x \rangle \leq D \langle x, x \rangle, \quad (1)$$

for every $x \in H$. If the middle sum in (1) is norm convergent in A , $\mathcal{F}$ is called *standard*. Also, if $C = D$ in (1), $\mathcal{F}$ is called *tight* and if $C = D = 1$, $\mathcal{F}$ is called *normalized*. Therefore, $\mathcal{F}$ will be a standard normalized tight frame if for all $x \in H$

$$\langle x, x \rangle = \sum_{i \in I} \langle x, x_i \rangle \langle x_i, x \rangle, \quad (2)$$

and the sum is norm convergent.

It is proved in Example 3.5 of [2] and Theorem 1 of [4] that any finitely or countably Hilbert C^* -module over a unital C^* -algebra A , possesses a standard

normalized tight frame $\{x_i : i \in I\}$ and the following *reconstruction formula*, is valid:

$$x = \sum_{i \in I} \langle x, x_i \rangle x_i \quad (x \in H). \quad (3)$$

Now suppose that A is commutative and $\{y_j : j \in J\}$ is another standard normalized tight frame for H and $\varphi \in B(H)_a$. Then if both of the series

$$\sum_{i \in I} \langle \varphi(x_i), \varphi(x_i) \rangle, \quad \sum_{j \in J} \langle \varphi(y_j), \varphi(y_j) \rangle$$

are norm convergent in A , their sums are equal. Because, if $\{z_\lambda\}_{\lambda \in \Lambda}$ is a standard normalized tight frame of H , then we have

$$\begin{aligned} & \sum_{i \in I} \langle \varphi(x_i), \varphi(x_i) \rangle \\ &= \sum_{i \in I} \sum_{\lambda \in \Lambda} \langle \varphi(x_i), z_\lambda \rangle \langle z_\lambda, \varphi(x_i) \rangle \\ &= \sum_{i \in I} \sum_{\lambda \in \Lambda} \langle x_i, \varphi^*(z_\lambda) \rangle \langle \varphi^*(z_\lambda), x_i \rangle \\ &= \sum_{\lambda \in \Lambda} \sum_{i \in I} \langle \varphi^*(z_\lambda), x_i \rangle \langle x_i, \varphi^*(z_\lambda) \rangle \\ &= \sum_{\lambda \in \Lambda} \langle \varphi^*(z_\lambda), \varphi^*(z_\lambda) \rangle. \end{aligned}$$

Repeating the preceding relations for $\{y_j : j \in J\}$ gives

$$\sum_{j \in J} \langle \varphi(y_j), \varphi(y_j) \rangle = \sum_{i \in I} \langle \varphi(x_i), \varphi(x_i) \rangle. \quad (4)$$

Lemma 3.4 *Suppose that A is a unital commutative Banach algebra, H is a Hilbert A -module and*

$$\rho(x) = \langle x, x \rangle^{1/2}$$

is the A -valued norm on H . Then

$$\rho(x + y) \leq \rho(x) + \rho(y),$$

for every $x, y \in H$.

Proof By Proposition 2. of [10] and for commutative $A = C_0(X)$, we have

$$\rho(x + y) \leq \rho(x) + \rho(y).$$

Now for a unital commutative Banach algebra, $\Omega(A)$, the spectrum of A , is compact and the Gelfand transform $\phi : A \longrightarrow C(\Omega(A))$ is an isometry and $*$ -isomorphism. Now for $a \in A$, we have

$$a \geq 0 \iff a = c^2 \text{ for an element } c \in A \iff \phi(a) = \phi(c)^2 \iff \phi(a) \geq 0.$$

Therefore, ϕ and ϕ^{-1} are positive maps. By the above argument,

$$\phi(\rho(x + y)) \leq \phi(\rho(x)) + \phi(\rho(y)).$$

Then by positivity of ϕ^{-1} ,

$$\rho(x + y) \leq \rho(x) + \rho(y).$$

◇

Definition 3.5 Suppose that τ belongs to $\Omega(A)$, the spectrum of A , and $\varphi \in B(H)_a$. Then φ is called a *Hilbert-Schmidt operator on H associated to τ* , if

$$\sum_{i \in I} \tau(|\varphi(x_i)|^2) < \infty,$$

Where,

$$|x| = \langle x, x \rangle^{1/2}$$

is the A -valued norm of A , and $\{x_i\}_{i \in I}$ is a standard normalized tight frame of H . For such φ , we define

$$\|\varphi\|_\tau = \left(\sum_{i \in I} \tau(|\varphi(x_i)|^2) \right)^{1/2}.$$

By the above argument, this definition is independent of the choice of $\{x_i\}_{i \in I}$. If we denote the set of such operators by $L^2(H, \tau)$, then it follows from (4) that

$$\|\varphi\|_\tau = \|\varphi^*\|_\tau$$

and so $L^2(H, \tau)$ is self-adjoint.

By Lemma 3.4, the A -valued norm $|\cdot|$ satisfies the triangle inequality and then for any finite set $J \subseteq I$, we have the Minkovski inequality

$$\left(\sum_{i \in J} |X_i + Y_i|^2 \right)^{1/2} \leq \left(\sum_{i \in J} |X_i|^2 \right)^{1/2} + \left(\sum_{i \in J} |Y_i|^2 \right)^{1/2},$$

for $X_i, Y_i \in H$. Then, for

$$X_i = \tau(|\varphi(x_i)|)^2$$

and

$$Y_i = \tau(|\varphi(y_i)|)^2,$$

we have

$$\|\varphi + \psi\|_\tau \leq \|\varphi\|_\tau + \|\psi\|_\tau.$$

Moreover, for any $\psi \in B(H)_a$ and $\varphi \in L^2(H, \tau)$, by Proposition 1.2 of [6], we have

$$\begin{aligned} \|\psi\phi\|_\tau^2 &= \tau\left(\sum_{i \in I} |\psi\phi(x_i)|^2\right) \leq \tau(\|\psi\|^2 \sum_{i \in I} |\varphi(x_i)|^2) \\ &= \|\psi\|^2 \cdot \|\varphi\|_\tau^2. \end{aligned}$$

Then,

$$\|\psi\varphi\|_\tau \leq \|\psi\| \cdot \|\varphi\|_\tau$$

and hence, we have

$$\|\varphi\psi\|_\tau = \|(\varphi\psi)^*\|_\tau = \|\psi^*\varphi^*\|_\tau \leq \|\psi^*\| \cdot \|\varphi^*\|_\tau = \|\psi\| \cdot \|\varphi\|_\tau.$$

Therefore, $L^2(H, \tau)$ is a self-adjoint two-sided ideal of $B(H)_a$, which is a semi-normed space and complex algebra.

Now, for $\varphi \in B(H)_a$, we define

$$\|\varphi\|_2 = \sup_{\tau \in \Omega(A)} \|\varphi\|_\tau.$$

we call φ a *Hilbert-Schmidt* operator on H , if

$$\|\varphi\|_2 < \infty.$$

We denote the set of this operators by $L^2(H)$.

Theorem 3.6 *The set $L^2(H)$ is a normed complex $*$ -algebra which is a two-sided ideal of $B(H)_a$.*

Proof (1)

$$\|\lambda\varphi\|_2 = |\lambda| \cdot \|\varphi\|_2$$

for $\lambda \in \mathbb{C}$ and $\varphi \in L^2(H)$.

(2)

$$\begin{aligned} \|\varphi + \psi\|_2 &= \sup_\tau \|\varphi + \psi\|_\tau \\ &\leq \sup_\tau (\|\varphi\|_\tau + \|\psi\|_\tau) \\ &\leq \sup_\tau \|\varphi\|_\tau + \sup_\tau \|\psi\|_\tau \\ &= \|\varphi\|_2 + \|\psi\|_2, \end{aligned}$$

for $\varphi, \psi \in L^2(H)$.

(3) Suppose that for $\varphi \in L^2(H)$, $\|\varphi\|_2 = 0$. Then for any $\tau \in \Omega(A)$ $\|\varphi\|_\tau = 0$ and hence, for any $i \in I$, $\tau(|\varphi(x_i)|) = 0$. Therefore, $|\varphi(x_i)| = 0$ and then $\varphi(x_i) = 0$. By (3), and for $x \in H$, we have $\varphi(x) = \sum_{i \in I} \langle x, x_i \rangle \varphi(x_i)$, and then $\varphi = 0$.

(4) It is clear that $\|\psi\varphi\|_2 \leq \|\psi\| \cdot \|\varphi\|_2$ and $\|\varphi\psi\|_2 \leq \|\varphi\|_2 \cdot \|\psi\|$ for $\varphi \in L^2(H)$ and $\psi \in B(H)_a$. Moreover, $\|\varphi\|_2 = \|\varphi^*\|_2$. Therefore, the proof is completed. $\diamond$

Remark 3.7. (1) Suppose that in Definition 3.5, $A = \mathbb{C}$. Then H is a Hilbert space and the above definition of Hilbert-Schmidt operator coincides with definition for Hilbert spaces. (See [8])

(2) We can introduce a set $L_1^2(H)$, which consists of operators φ such that

$$\|\varphi\|'_2 = \sum_{\tau \in \Omega} \|\varphi\|_\tau < \infty.$$

It is obvious that $L_1^2(H) \subseteq L^2(H)$.

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