THE MOL SOLUTION FOR THE ONE-DIMENSIONAL HEAT EQUATION SUBJECT TO NONLOCAL CONDITIONS

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Abstract

In this paper, the problem of solving the one-dimensional equation $\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2} = g(x, t)$ subject to given initial and nonlocal conditions by the method of lines (MOL). Firstly the theory of application of MOL on heat equation is presented. Secondly we consider some examples to show the goodness of the new method. Computational results using a code based on our method are presented for these examples.

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1 Introduction

A problem which arises in the quasi-static theory of thermoelasticity (Day[10],[9]) requires solving the one-dimensional heat equation

$$\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2} = g(x, t) \quad \text{for } x \in (0, 1) \quad \text{and } t > 0, \quad (1)$$

subject to the initial condition

$$\varphi(x, 0) = u(x) \quad \text{for } x \in [0, 1], \quad (2)$$

and the nonlocal conditions

$$\left. \begin{aligned} \varphi(0, t) &= \int_0^1 p(x) \varphi(x, t) dx + r(t) \\ \varphi(1, t) &= \int_0^1 q(x) \varphi(x, t) dx + v(t) \end{aligned} \right\} \quad \text{for } t > 0, \quad (3)$$

where x and t are the spatial and time coordinates respectively, φ is the unknown function of x and t to be determined and g, u, p, q, r and v are suitably prescribed function.

Investigators like Ang [11] had studied the problem and introduced the Laplace transform domain methods solving it numerically. An approach related to finite difference method, which has found interesting applications in the numerical treatment of partial differential equations, is the method of lines. It makes use of the finite difference method in the construction of discrete-analytical model of a given partial differential equation in which all but one of its independent variables discretized. This procedure leads to a system of ordinary differential equations in the remaining variable.

2 Semi discrete models for heat equation

To solve (1)-(3), the problem is discretized in respect to x by finite difference method. For this we define a uniform partition P as:

$$P : a = x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_{2n+1} = b, \quad (4)$$

which a constant spacing $h = x_{k+1} - x_k$, $k = 1, 2, \dots, 2n$.

We use an $O(h^4)$ central difference scheme to estimate $\frac{\partial^2 \varphi}{\partial x^2}$ on internal nodes.

Hence we have the following difference scheme :

for $x = x_1$

$$\frac{\partial^2 \varphi}{\partial x^2} \Big|_{x=x_1} \cong \frac{1}{12h^2} (45y_1 - 154y_2 + 214y_3 - 156y_4 + 61y_5 - 10y_6) = E_1(t), \quad (5)$$

for $x = x_2$

$$\frac{\partial^2 \varphi}{\partial x^2} \Big|_{x=x_2} \cong \frac{1}{24h^2} (20y_1 - 30y_2 - 8y_3 + 28y_4 - 12y_5 + 2y_6) = E_2(t), \quad (6)$$

for $x = x_i$

$$\frac{\partial^2 \varphi}{\partial x^2} \Big|_{x=x_i} \cong \frac{1}{12h^2} (-2y_{i-2} + 32y_{i-1} - 60y_i + 32y_{i+1} - 2y_{i+2}) = E_i(t), \quad (7)$$

for $x = x_{2n}$

$$\frac{\partial^2 \varphi}{\partial x^2} \Big|_{x=x_{2n}} \cong \frac{1}{24h^2} (20y_{2n+1} - 30y_{2n} - 8y_{2n-1} + 28y_{2n-2} - 12y_{2n-3} + 2y_{2n-4}) = E_{2n}(t), \quad (8)$$

for $x = x_{2n+1}$

$$\frac{\partial^2 \varphi}{\partial x^2} \Big|_{x=x_{2n+1}} \cong \frac{1}{12h^2} (45y_{2n+1} - 154y_{2n} + 214y_{2n-1} - 156y_{2n-2} + 61y_{2n-3} - 10y_{2n-4})$$

$$= E_{2n+1}(t), \quad (9)$$

where $y_k = y_k(t) \cong \varphi(x_k, t)$, $k = 1, 2, \dots, 2n$.

By (5)-(9) the following system of ordinary differential equations is obtained :

$$y'_i(t) = E_i(t) + g_i(t) \quad , \quad k = 1, 2, \dots, 2n \quad (10)$$

where $g_i(t) = g_i = g(x_i, t)$ whit following condition:

$$y_i(0) = \varphi(x_i, 0) = u(x_i) = u_i. \quad (11)$$

If we use the simpson rule to conditions (3) respect to x_i by definition of partition P as (4) we have:

$$\begin{aligned} \varphi(0, t) \cong & \frac{h}{3} [p(x_1)\varphi(x_1, t) + 4p(x_2)\varphi(x_2, t) + \dots \\ & + 4p(x_{2n})\varphi(x_{2n}, t) + p(x_{2n+1})\varphi(x_{2n+1}, t)] + r(t), \end{aligned} \quad (12)$$

and

$$\begin{aligned} \varphi(1, t) \cong & \frac{h}{3} [q(x_1)\varphi(x_1, t) + 4q(x_2)\varphi(x_2, t) + \dots \\ & + 4q(x_{2n})\varphi(x_{2n}, t) + q(x_{2n+1})\varphi(x_{2n+1}, t)] + v(t). \end{aligned} \quad (13)$$

We solve the (12)-(13) respect to y_1 and y_{2n+1} as following:

$$\begin{aligned} y_{2n+1} = & \frac{1}{A} \left[\frac{h^2 q_1}{9-3hp_1} (4p_2 y_2 + \dots + 4p_{2n} y_{2n} + \frac{3r(t)}{3-hp_1}) \right. \\ & \left. + \frac{h}{3} (4q_2 y_2 + \dots + 4q_{2n} y_{2n}) \right] = \frac{B}{A}, \end{aligned} \quad (14)$$

$$y_1 = \frac{h}{3-hp_1} (4p_2 y_2 + \dots + 4p_{2n} y_{2n} + p_{2n+1} \times \frac{B}{A}) + \frac{3r(t)}{3-hp_1}, \quad (15)$$

where $p_i = p(x_i)$, $q_i = q(x_i)$, $i = 1, 2, \dots, 2n$ and

$$A = 1 - \frac{q_1 h^2 p_{2n+1}}{9-3hp_1} - \frac{hq_{2n+1}}{3},$$

$$B = \frac{h^2 q_1}{9-3hp_1} (4p_2 y_2 + \dots + 4p_{2n} y_{2n} + \frac{3r(t)}{3-hp_1}) + \frac{h}{3} (4q_2 y_2 + \dots + 4q_{2n} y_{2n}),$$

Then we can combine the relations (10),(14),(15) we have the following system of algebric differential equations:

$$\begin{cases} y'(t) = f(t, y) \\ y(0) = y_0 \end{cases} \quad (16)$$

Consider system (16) and suppose that it is stiff. In the backward differentiation formula (BDF), we have

$$y_n = \sum_{j=1}^k \alpha_j y_{n-j} + h\beta_k f(t_n, y_n), \quad (17)$$

where k is the order of scheme and the values of α_j and β_k are given in [10]. From (17) we have the following system of nonlinear equations to be solved:

$$F(x_n) = x_n - hf(t_n, \alpha_n + \beta_k x_n) \quad (18)$$

Where $\alpha_n = \sum_{j=1}^k \alpha_j y_{n-j}$ and $x_n = hf(t_n, y_n)$, see [4]. Usually, to solve a system like (18), a modified Newton method is used. Then a direct method is usually used to solve any resulting system of linear equations. Hence in each step, we need to obtain the jacobian matrix and its related decomposed matrices. Therefore causing more computational and running time. In matrix free (MF-BDF) method, the jacobian matrix is not used explicitly, because the exact Newton method is used and then the IOM algorithm, see [12], is applied to solve the resulting system of linear equations. Cash, modified the BDF method, this modification led to the EMBDF method [3], and MEBDF method [2]. In [7], authors applied the Cash's modification to the MF-BDF scheme and obtained a new method to solve ODE's and PDE's. They called their method MF-MEBDF.

In this study we want to apply the MF-BDF and MF-MEBDF methods solve system of DAE's.

3 Numerical examples

In this section, we apply the method of solution described above to solve the problem defined by (1)-(3) for particular g, u, p, q, r and v .

Example 1. The first problem is selected from [11]. In this example, we take

$$g(x, t) = -x(x - 1) \exp(-t)$$

$$u(x) = x(x - 1) - 2$$

$$p(x) = \frac{12}{13}, \quad q(x) = \frac{12}{13}$$

$$r(t) = 0, \quad v(t) = 0.$$

The analytical solution of the problem is

$$\varphi(x, t) = [x(x - 1) - 2] \exp(-t).$$

Numerical results by 21 line is presented in Table 1.

	$MF - MEBDF$	$MF - MEBDF$	$MF - BDF$	$MF - BDF$
t	$MaxErr$	$MinErr$	$MaxErr$	$MinErr$
0.2	$1.45E - 10$	$1.29E - 10$	$2.34E - 9$	$1.25E - 9$
0.4	$1.50E - 5$	$1.37E - 5$	$7.67E - 4$	$5.89E - 4$
0.6	$1.23E - 5$	$1.12E - 5$	$6.58E - 4$	$2.34E - 4$
0.8	$3.65E - 10$	$3.27E - 10$	$5.78E - 9$	$3.05E - 9$
1.0	$2.73E - 10$	$2.54E - 10$	$4.35E - 9$	$2.15E - 9$

Table1

Example 2. The second problem is selected from [11]. In this example, we take

$$g(x, t) = \frac{-2(x^2 + t + 1)}{(t + 1)^3}$$

$$u(x) = x^2$$

$$p(x) = x, \quad q(x) = x$$

$$r(t) = -\frac{1}{4(t+1)^2}, \quad v(t) = \frac{3}{4(t+1)^2}.$$

The analytical solution of the problem is

$$\varphi(x, t) = \left(\frac{x}{t+1}\right)^2.$$

Numerical results by 21 line is presented in Table 2.

	$MF - MEBDF$	$MF - MEBDF$	$MF - BDF$	$MF - BDF$
t	$MaxErr$	$MinErr$	$MaxErr$	$MinErr$
0.2	$2.99E - 10$	$1.45E - 11$	$4.32E - 9$	$3.25E - 10$
0.4	$6.57E - 5$	$1.82E - 7$	$4.35E - 5$	$1.99E - 6$
0.6	$4.43E - 5$	$1.22E - 7$	$8.97E - 5$	$2.88E - 6$
0.8	$6.88E - 10$	$5.91E - 12$	$1.23E - 9$	$3.55E - 10$
1.0	$1.56E - 9$	$1.48E - 11$	$2.17E - 8$	$8.99E - 9$

Table2

where

$$MaxErr = \max |y_k(t) - \varphi(x_k, t)|, \quad k = 1, 2, \dots, 2n$$

$$MinErr = \min |y_k(t) - \varphi(x_k, t)|, \quad k = 1, 2, \dots, 2n$$

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