On the Existence of Faithful Characters¹

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Abstract

In this paper, we show that if a finite group G has exactly two minimal normal subgroups, then G has a faithful irreducible character.

1 Introduction

Let G be a finite group and let us denote by $\text{Irr}(G)$ the set of all complex irreducible characters of G . As we know, the intersection of kernels of all irreducible characters of G is trivial and so if N is a normal subgroup of G , then it is easy to see that N is the intersection of kernels of all irreducible characters of G whose kernel containing N . Thus, there is a one-to-one correspondence between irreducible characters of G/N and irreducible characters of G with kernel containing N . Under this correspondence, $\text{Irr}(G/N)$ can be viewed as a subset of $\text{Irr}(G)$, that is $\text{Irr}(G/N) = \{\chi \in \text{Irr}(G) \mid N \subseteq \ker \chi\}$.

2 Main Results

We first state an apparent result about groups which have a unique minimal normal subgroup.

Theorem 1. Let G be a finite group which has a unique minimal normal subgroup. Then G has a faithful irreducible character.

Proof. Let N be the unique minimal normal subgroup of G and suppose that G has not a faithful irreducible character. Therefore, for every irreducible

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character χ , we have $\ker \chi \neq 1$ and so N is contained in $\ker \chi$. But we know that the intersection of kernels of all irreducible characters of G is trivial, so $N = 1$ and this is a contradiction.

The following is an immediate result of the above theorem.

Corollary. Let G be a finite p -group with a cyclic center. Then G has a faithful irreducible character.

Proof. We know that if a finite p -group has a cyclic center, then it has a unique minimal normal subgroup of order p , and so the result follows from theorem 1.

Now we are ready to prove our main theorem.

Theorem 2. Let G be a finite group with exactly two minimal normal subgroups. Then G has a faithful irreducible character.

Proof. Suppose that A and B are the only minimal normal subgroups of G and G has not a faithful irreducible character. Then, for every irreducible character χ , we have $\ker \chi \neq 1$ and so either A or B is contained in $\ker \chi$. Thus, we have $\text{Irr}(G) = \text{Irr}(G/A) \cup \text{Irr}(G/B)$. Now we have,

$$|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = \sum_{\chi \in \text{Irr}(G/A)} \chi(1)^2 + \sum_{\chi \in \text{Irr}(G/B)} \chi(1)^2 - \sum_{\chi \in \text{Irr}(G/A) \cap \text{Irr}(G/B)} \chi(1)^2.$$

But it is easy to see that $\text{Irr}(G/A) \cap \text{Irr}(G/B) = \text{Irr}(G/AB)$. Then we can deduce that

$$|G| = |G/A| + |G/B| - |G/AB|,$$

and so

$$|AB| = |B| + |A| - 1 = |B| + |A| - |A \cap B| = |A \cup B|.$$

Thus, we have that $AB = A \cup B$ and so either $A \subseteq B$ or $B \subseteq A$ and this is a contradiction.

Corollary. Let G be a finite nilpotent group with a cyclic center and $|G|$ has only two prime divisors. Then G has a faithful irreducible character.

Proof. Suppose that G is a finite nilpotent group whose order divisible by only two primes. Thus, $G = P \times Q$ where P and Q are the only Sylow p -subgroup and Sylow q -subgroup of G , respectively. Then we have $Z(G) = Z(P) \times Z(Q)$ and so $Z(G)$ contains exactly two minimal normal subgroups of G . Since every minimal normal subgroup of a nilpotent group is contained in its center, therefore G has only two minimal normal subgroups and theorem 2 can be applied.

Remark. It should be noted that the above corollary is true for any finite nilpotent group with a cyclic center without any hypothesis on the order of group, but its proof depends on the direct product of characters.

References

- [1] **I. M. Isaacs**, *Character Theory of Finite Groups*, Dover, New York, 1994.

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