On Apéry Sets of Symmetric Numerical Semigroups

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Abstract

In this paper, we give some results on Apéry sets of Symmetric Numerical Semigroups with $e(S) = 2$. Also, we rewrite the definitions $n(S)$ and $H(S)$ by means of Apéry sets of S .

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1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \dots, n, \dots\}$ and $S \subseteq \mathbb{N}$. S is called a numerical semigroup if S is sub-semigroup of $(\mathbb{N}, +)$ with $0 \in S$.

It is known that every numerical semigroup is finitely generated, i.e. there exist elements of S , say $n_0, n_1, \dots, n_p$ such that $n_0 < n_1 < \dots < n_p$ and

$$S = \langle n_0, n_1, \dots, n_p \rangle = \left\{ \sum_{i=0}^p k_i n_i : k_i \in \mathbb{N} \right\}$$

and

$$G.C.D.(n_0, n_1, \dots, n_p) = 1 \Leftrightarrow Card(\mathbb{N} \setminus S) < \infty$$

by [1].

For S numerical semigroup, we define the following:

$g(S) = \max\{x \in \mathbb{Z} : x \notin S\}$ is called the Frobenius number of S , where $\mathbb{Z}$ is the integer set. Thus, S numerical semigroup is $S = \{0, n_0, n_1, \dots, g(S) + 1 \rightarrow \dots\}$ (The arrow " $\rightarrow$ " means that every integer which is greater then

$g(S) + 1$ belongs to S). If S has unique minimal system of generators which is $\{n_0 < n_1 < \cdots < n_p\}$, then the numbers n_0 and $e(S) = p + 1$ are called the multiplicity and the embedding dimension of S , respectively.

We say that a numerical semigroup is symmetric if for every $x \in \mathbb{Z} \setminus S$, we have $g(S) - x \in S$.

For $n \in S \setminus \{0\}$, we define the Apéry set of the element n as the set

$$Ap(S, n) = \{s \in S : s - n \notin S\}.$$

It can easily be proved that $Ap(S, n)$ is formed by the smallest elements of S belonging to the different congruence classes $\text{mod } n$. Thus, $\#(Ap(S, n)) = n$ and $g(S) = \max(Ap(S, n)) - n$, where $\#(A)$ stands for *cardinality*(A) by [2].

2. Apéry Sets of S Symmetric Numerical Semigroups with $e(S) = 2$

Now, we give some theorems for $e(S) = 2$.

Theorem 1. If $S = \langle n_0, n_1 \rangle$ is symmetric numerical semigroup, then

$$Ap(S, n_0) = \{(n_0 - r)n_1 : r = 1, 2, \dots, n_0\}.$$

and

$$Ap(S, n_1) = \{(n_1 - r)n_0 : r = 1, 2, \dots, n_1\}.$$

Proof. $x \in Ap(S, n_0) \Rightarrow x - n_0 \notin S, x \in S \Rightarrow x \neq n_0$. Because, if $x = n_0$, then we find that $x - n_0 = 0 \notin S$. This is a contradiction.

In order to obtain the required result, it suffices to show that $x = kn_1$ for $k = 0, 1, 2, \dots, n_0 - 1$.

Firstly, we must show that $n_1 \in Ap(S, n_0)$. We assume that $n_1 \notin Ap(S, n_0)$, i.e., $n_1 - n_0 \in S$. Then, there exist elements $p, q \in \mathbb{N}$ such that $n_1 - n_0 = qn_0 + pn_1$. Therefore, we find $n_1(1 - p) = n_0(q + 1) \in \mathbb{N}$. In this case, $p = 0$ or $p = 1$. If $p = 0$, then $n_1 = n_0(q + 1)$. Thus, $n_0 \mid n_1$. This is a contradiction. If $p = 1$, then we write $-n_0 = qn_0 \in \mathbb{N}$, which is a contradiction. Thus we find that $(0 \leq k \leq n_0 - 1) \quad kn_1 \in Ap(S, n_0)$ for $n_1 \in Ap(S, n_0)$. On the other hand, we can put $r = 0, 1, 2, \dots, n_0$, since $(\#Ap(S, n_0)) = n_0$.

Hence, we write

$$\begin{aligned} Ap(S, n_0) &= \{(n_0 - 1)n_1, (n_0 - 2)n_1, \dots, (n_0 - (n_0 - 1)n_1), (n_0 - n_0)n_1\}. \\ &= \{(n_0 - r)n_1 : r = 1, 2, \dots, n_0\}. \end{aligned}$$

If we make some operations for the set $Ap(S, n_1)$, we obtain that

$$Ap(S, n_1) = \{(n_1 - r)n_0 : r = 1, 2, \dots, n_1\}.$$

Theorem 2. If $S_1 = \langle n_0, n_1 \rangle$ and $S_2 = \langle kn_0, n_1 \rangle$ are two symmetric numerical semigroups for $k > 1$ and $k \in \mathbb{N}$, then $S_2 \subset S_1$ and $Ap(S_1, n_0) \subset Ap(S_2, kn_0)$.

Proof. It is obvious that S_2 is subset of S_1 by their definitions. On the other hand, if $x \in Ap(S_1, n_0)$, then we write $(x - n_0) \notin S_1, x \in S_1$. thus, we find $(x - kn_0) \notin S_2$. Therefore, we obtain $Ap(S_2, kn_0)$.

Example 1. Let $S_1 = \langle 2, 5 \rangle = \{0, 2, 4, 5, 6, \rightarrow, \dots\}$ and $S_2 = \langle 4, 5 \rangle = \{0, 4, 5, 8, 9, 10, 12, \rightarrow, \dots\}$. Hence, we find that

$$Ap(S_1, 2) = \{5(2 - r) : r = 1, 2\} = \{0, 5\},$$

$$Ap(S_1, 5) = \{2(5 - r) : r = 1, 2, 3, 4, 5\} = \{0, 2, 4, 6, 8\}$$

and

$$Ap(S_2, 4) = \{5(4 - r) : r = 1, 2, 3, 4\} = \{0, 5, 10, 15\},$$

$$Ap(S_2, 5) = \{4(5 - r) : r = 1, 2, 3, 4, 5\} = \{0, 4, 8, 12, 16\}$$

for S_1 and S_2 , respectively. It is clear that $S_2 \subset S_1$ and $Ap(S_1, 2) \subset Ap(S_2, 4)$.

Now, we give new definitions $n(S)$ and $H(S)$ by means of Apéry sets of S .

Theorem 3. If $S = \langle n_0, n_1 \rangle$ is a symmetric numerical semigroup and, its Frobenius number is $g(S)$, and $n(S) = \#(\{0, 1, 2, \dots, g(S)\} \cap S)$, then we obtain

$$n(S) = \frac{[\#(Ap(S, n_0)) - 1][\#(Ap(S, n_1)) - 1]}{2}.$$

Proof. We known that $n(S) = \frac{g(S)+1}{2}$ by [4], since $S = \langle n_0, n_1 \rangle$ is symmetric, On the other hand, we have

$$g(S) = \max(Ap(S, n_0)) - n_0 = (n_0 - 1)n_1 - n_0 = n_0n_1 - n_1 - n_0.$$

Hence, we find

$$n(S) = \frac{n_0n_1 - n_1 - n_0 + 1}{2} = \frac{(n_0 - 1)(n_1 - 1)}{2} = \frac{[\#(Ap(S, n_0)) - 1][\#(Ap(S, n_1)) - 1]}{2}.$$

Example 2. Let S_1 and S_2 be defined as Example 1 above. Then,

$$n(S_1) = \frac{(2 - 1)(5 - 1)}{2} = 2$$

and

$$n(S_2) = \frac{(4 - 1)(5 - 1)}{2} = 6.$$

In fact,

$$n(S_1) = \sharp(\{0, 1, 2, 3\} \cap \{0, 2, 4, 5, \dots\}) = 2$$

and

$$n(S_2) = \sharp(\{0, 1, 2, 3, \dots, 10, 11\} \cap \{0, 4, 5, 8, 9, 10, 12, \dots\}) = 6.$$

Theorem 4. If $S = \langle n_0, n_1 \rangle$ is a symmetric numerical semigroup and, its Frobenius number is $g(S)$, and $H(S) = \{x : x \in \mathbb{N} \setminus S\}$ whose elements are called gaps of S by [5], then we write

$$H(S) = \{0, 1, 2, \dots, g(S)\} \setminus (Ap(S, n_0) \cup Ap(S, n_1) \cup \{n_0 + n_1\}).$$

Proof. Let $B = \{0, 1, 2, \dots, g(S)\}$ and $A = (Ap(S, n_0) \cup Ap(S, n_1) \cup \{n_0 + n_1\})$. Now, we show that $H(S) = B \setminus A$. If $x \in H(S) = (\mathbb{N} \setminus S)$, then $x \in \mathbb{N}$ and $x \notin S$. Thus, we write $x \in \mathbb{N}$ and $x \notin A$. Therefore, we find that $x \in B$ and $x \notin A$. Because if $x \notin B$, then $x > g(S)$ and $x \in S$, but this is a contradiction.

Conversely, if $y \in B \setminus A$, then $y \in B$ and $y \notin A$. Thus, we write $y \in B$ and $y \notin S$. Because if $y \in S$ then $y \in B \cap S$, and therefore, we find a contradiction $S \subset B$. For this reason, $y \in \mathbb{N}$ and $y \notin S$, and we obtain $y \in H(S)$.

Example 3. Let $S = \langle 4, 5 \rangle = \{0, 4, 5, 8, 9, 10, 12, \dots\}$. Then

$$Ap(S, 4) = \{5(4 - r) : r = 1, 2, 3, 4\} = \{0, 5, 10, 15\},$$

$$Ap(S, 5) = \{4(5 - r) : r = 1, 2, 3, 4, 5\} = \{0, 4, 8, 12, 16\}$$

and $g(S) = 11$. Hence, we write

$$H(S) = \{0, 1, 2, \dots, 11\} \setminus (Ap(S, 4) \cup Ap(S, 5) \cup \{9\}) = \{1, 2, 3, 6, 7, 11\}.$$

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