Neighborhoods of a Class of Analytic Functions with Negative Coefficients

H. Özlem Güney and S. Sümer Eker

University of Dicle
Faculty of Science and Arts, Department of Mathematics
21280, Diyarbakır, Turkey
ozlemg@dicle.edu.tr & sevtaps@dicle.edu.tr

Abstract

We making use of the familiar concept of neighborhoods of analytic functions, we prove several inclusion relations associated with the \((n, \delta)\)-neighborhoods of various subclass of univalent functions with negative coefficients that is convex of order \(\alpha\).

1. Introduction

Let \(A(n)\) denote the class of functions \(f(z)\) of the form

\[
f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\})
\]  

which are analytic in the open unit disk \(U = \{z \in \mathbb{C} : |z| < 1\}\).

For any \(f(z) \in A(n)\) and \(\delta \geq 0\) we define

\[
N_{n,\delta}(f) = \{g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta\}
\]

which was called \((n, \delta)\)-neighborhoods of \(f(z)\). So, for \(e(z) = z\), we see that

\[
N_{n,\delta}(e) = \{g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k|b_k| \leq \delta\}
\]

The concept of neighborhoods was firstly by A.W.Goodman[1] and then generalized by ST.Ruscheweyh [2]. The main object of the present paper is
to investigate the neighborhoods of the following subclasses of class \( \mathcal{A}(n) \) of univalent functions with negative coefficients that is convex of order \( \alpha \).

A function \( f(z) \) is said to be in the class \( \mathcal{C}(n, \lambda, \alpha) \) if it satisfies

\[
\Re \left\{ z \frac{\lambda z^2 f'''(z) + (2 \lambda + 1)zf''(z) + f'(z)}{\lambda z^2 f''(z) + zf'(z)} \right\} > \alpha
\]

for some \( 0 \leq \alpha \leq 1 \), \( 0 \leq \lambda \leq 1 \) and for all \( z \in U \) \[3\]. We note that \( \mathcal{C}(1, 0, \alpha) \equiv \mathcal{C}(\alpha) \) is the generalization of \( \mathcal{C}(\alpha) \) by H.Silverman \[4\].

2. A Set of Inclusion Relations Involving \( \mathcal{N}_{n,\delta}(e) \)

In our investigation of the inclusion relations involving \( \mathcal{N}_{n,\delta}(e) \), we shall require the following Lemma which was proved in \[3\].

**Lemma**  Let \( \mathcal{A}(n) \) denote the class of functions of the form

\[
f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\})
\]

that are analytic in the unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). A function \( f(z) \in \mathcal{A}(n) \) is in the class \( \mathcal{C}(n, \lambda, \alpha) \) if and only if

\[
\sum_{k=n+1}^{\infty} k (k - \alpha) (\lambda k - \lambda + 1) |a_k| \leq 1 - \alpha.
\]  \(2.1\)

Our first inclusion relation involving \( \mathcal{N}_{n,\delta}(e) \) is given by the following:

**Theorem 1**  Let

\[
\delta = \frac{1 - \alpha}{(n + 1 - \alpha)(\lambda n + 1)}
\]

then

\[
\mathcal{C}(n, \lambda, \alpha) \subset \mathcal{N}_{n,\delta}(e).
\]

**Proof**  For \( f \in \mathcal{C}(n, \lambda, \alpha) \), Lemma immediately yields,

\[
(n + 1 - \alpha)(\lambda n + 1) \sum_{k=n+1}^{\infty} k |a_k| \leq 1 - \alpha
\]

so that

\[
\sum_{k=n+1}^{\infty} k |a_k| \leq \frac{1 - \alpha}{(n + 1 - \alpha)(\lambda n + 1)} = \delta,
\]

which, in the view \(1.3\), proves Theorem 1.
3. Neighborhoods for the class $\mathcal{C}^{(\beta)}(n, \lambda, \alpha)$

In this section, we determine the neighborhoods for the class $\mathcal{C}^{(\beta)}(n, \lambda, \alpha)$ which we define as follows. A function $f(z) \in \mathcal{A}(n)$ is said to be in the class $\mathcal{C}^{(\beta)}(n, \lambda, \alpha)$ if there exists a function $g \in \mathcal{C}(n, \lambda, \alpha)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \beta$$

for $\beta(0 \leq \beta \leq 1)$ and $z \in U$.

**Theorem 2** If $g \in \mathcal{C}(n, \lambda, \alpha)$ and

$$\beta = 1 - \frac{\delta(n + 1 - \alpha)(\lambda n + 1)}{n[(n + 2 - \alpha)(\lambda n + 1) + (1 - \alpha)\lambda]},$$

then

$$\mathcal{N}_{n,\delta}(g) \subset \mathcal{C}^{(\beta)}(n, \lambda, \alpha).$$

**Proof** Suppose that $f \in \mathcal{N}_{n,\delta}(g)$. Then we find from (1.2) that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta$$

which readily implies the coefficients inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1}, \quad n \in \mathbb{N}.$$ 

Next, since $g \in \mathcal{C}(n, \lambda, \alpha)$, we have

$$(n + 1)(n + 1 - \alpha)(\lambda n + 1) \sum_{k=n+1}^{\infty} a_k \leq 1 - \alpha$$

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{1 - \alpha}{(n + 1)(n + 1 - \alpha)(\lambda n + 1)}.$$ 

Therefore,

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k} \leq \frac{\delta(n + 1 - \alpha)(\lambda n + 1)}{n[(n + 2 - \alpha)(\lambda n + 1) + (1 - \alpha)\lambda]} = 1 - \beta$$

provided that $\beta$ is given precisely by (3.2). Thus, by definition of $\mathcal{C}^{(\beta)}(n, \lambda, \alpha)$, $f \in \mathcal{C}^{(\beta)}(n, \lambda, \alpha)$ for $\beta$ given by (3.2), which evidently completes our proof of Theorem 2.
References


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