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Neighborhoods of a Class of Analytic Functions with Negative Coefficients

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Abstract

We making use of the familiar concept of neighborhoods of analytic functions, we prove several inclusion relations associated with the (n, δ)-neighborhoods of various subclass of univalent functions with negative coefficients that is convex of order α.

1. Introduction

Let $\mathcal{A}(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

For any $f(z) \in \mathcal{A}(n)$ and $\delta \geq 0$ we define

$$\mathcal{N}_{n,\delta}(f) = \{g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta\} \tag{1.2}$$

which was called (n, δ)-neighborhoods of $f(z)$. So, for $e(z) = z$,we see that

$$\mathcal{N}_{n,\delta}(e) = \{g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k|b_k| \leq \delta\} \tag{1.3}$$

The concept of neighborhoods was firstly by A.W.Goodman[1] and then generalized by ST.Ruscheweyh [2]. The main object of the present paper is to investigate the neighborhoods of the following subclasses of class $\mathcal{A}(n)$ of univalent functions with negative coefficients that is convex of order α.

A function $f(z)$ is said to be in the class $\mathcal{C}(n, \lambda, \alpha)$ if it satisfies

$$Re \left\{ z \frac{\lambda z^2 f'''(z) + (2\lambda + 1) z f''(z) + f'(z)}{\lambda z^2 f''(z) + z f'(z)} \right\} > \alpha$$

for some $\alpha(0 \leq \alpha \leq 1), \lambda(0 \leq \lambda \leq 1)$ and for all $z \in U$ [3]. We note that $\mathcal{C}(1, 0, \alpha) \equiv \mathcal{C}(\alpha)$ is the generalization of $\mathcal{C}(\alpha)$ by H.Silverman [4].

2. A Set of Inclusion Relations Involving $\mathcal{N}_{n,\delta}(e)$

In our investigation of the inclusion relations involving $\mathcal{N}_{n,\delta}(e)$, we shall require the following Lemma which was proved in [3].

Lemma Let$\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\})$$

that are analytic in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}(n)$ is in the class $\mathcal{C}(n, \lambda, \alpha)$ if and only if

$$\sum_{k=n+1}^{\infty} k (k - \alpha) (\lambda k - \lambda + 1) |a_k| \leqq 1 - \alpha. \tag{2.1}$$

Our first inclusion relation involving $\mathcal{N}_{n,\delta}(e)$ is given by the following:

Theorem 1 Let

$$\delta = \frac{1 - \alpha}{(n + 1 - \alpha)(\lambda n + 1)}$$

then

$$\mathcal{C}(n, \lambda, \alpha) \subset \mathcal{N}_{n,\delta}(e).$$

Proof For $f \in \mathcal{C}(n, \lambda, \alpha)$, Lemma immediately yields,

$$(n + 1 - \alpha)(\lambda n + 1) \sum_{k=n+1}^{\infty} k\, a_k| \leqq 1 - \alpha$$

so that

$$\sum_{k=n+1}^{\infty} k\, a_k| \leqq \frac{1 - \alpha}{(n + 1 - \alpha)(\lambda n + 1)} = \delta,$$

which, in the view (1.3), proves Theorem 1.

3. Neighborhoods for the class $\mathcal{C}^{(\beta)}(n, \lambda, \alpha)$

In this section , we determine the neighborhoods for the class $\mathcal{C}^{(\beta)}(n, \lambda, \alpha)$ which we define as follows. A function $f(z) \in \mathcal{A}(n)$ is said to be in the class $\mathcal{C}^{(\beta)}(n, \lambda, \alpha)$ if there exists a function $g \in \mathcal{C}(n, \lambda, \alpha)$ such that

$$|\frac{f(z)}{g(z)} - 1| < 1 - \beta \tag{3.1}$$

for $\beta(0 \leq \beta \leq 1)$ and $z \epsilon U$.

Theorem 2 If $g \in \mathcal{C}(n, \lambda, \alpha)$ and

$$\beta = 1 - \frac{\delta(n+1-\alpha)(\lambda n+1)}{n[(n+2-\alpha)(\lambda n+1)+(1-\alpha)\lambda]}, \tag{3.2}$$

then

$$\mathcal{N}_{n,\delta}(g) \subset \mathcal{C}^{(\beta)}(n, \lambda, \alpha).$$

Proof Suppose that$f \in \mathcal{N}_{n,\delta}(g)$. Then we find from (1.2) that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta$$

which readily implies the coefficients inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1}, \ n \in \mathbb{N}.$$

Next, since $g \in \mathcal{C}(n, \lambda, \alpha)$, we have

$$(n+1)(n+1-\alpha)(\lambda n+1) \sum_{k=n+1}^{\infty} a_k \leqq 1 - \alpha$$

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{1-\alpha}{(n+1)(n+1-\alpha)(\lambda n+1)}.$$

Therefore,

$$|\frac{f(z)}{g(z)} - 1| < \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k} \leq \frac{\delta(n+1-\alpha)(\lambda n+1)}{n[(n+2-\alpha)(\lambda n+1)+(1-\alpha)\lambda]} = 1 - \beta$$

provided that β is given precisely by (3.2). Thus, by definition of $\mathcal{C}^{(\beta)}(n, \lambda, \alpha)$, $f \in \mathcal{C}^{(\beta)}(n, \lambda, \alpha)$ for β given by (3.2),which evidently completes our proof of Theorem 2.

References

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