SEQUENCES FROM INTEGER TETRAHEDRONS

T. Aaron Gulliver

Department of Electrical and Computer Engineering
University of Victoria, P.O. Box 3055, STN CSC
Victoria, BC, Canada V8W 3P6
agullive@ece.uvic.ca

Abstract

This paper presents a number of sequences based on integers arranged in a three-dimensional array which is a tetrahedron. This approach provides a simple derivation of some well known sequences. In addition, a number of new integer sequences are obtained.

Keywords: integer arrays, integer sequences

Mathematics Subject Classification: 11Y55

1. Introduction

In a previous paper [1], several well-known sequences (and many new sequences), were derived from two-dimensional arrays of integers. In this paper, we consider the case of a three-dimensional array which is a three-sided pyramid or tetrahedron of integers. For example, the number of elements in the tetrahedron is

$$s_n = 1+3+6+10+15+\dots+n(n+1)/2 = \sum_{i=1}^n i(i+1)/2 = \frac{1}{6}n(n+1)(n+2), \quad (1)$$

where n is the number of levels. Starting from $n = 1$, we have

$$1, 4, 10, 20, 35, \dots,$$

which is sequence A000292 in the On-Line Encyclopedia of Integer Sequences [2], and is appropriately called the tetrahedral or pyramidal numbers. Additional sequences based on this integer array structure are given in the next section.

2. Tetrahedral Pyramids of Integers

A tetrahedral array of integers has the following structure for $n = 1$ to 4

In addition to (1), the following sequences are obtained from the corners of the pyramids

$$\begin{array}{l} 1, 2, 5, 11, 21, \dots \\ 1, 3, 8, 17, 31, \dots \end{array}$$

The first of these is sequence A050407, and is given by

$$s_n = \frac{1}{6}(n+2)(n^2 - 2n + 3),$$

which is just one more than the tetrahedral numbers. The second is sequence A105163 (with the leading 1 removed), given by

$$s_n = \frac{1}{6}(n^3 + 3n^2 - 4n + 6).$$

In this case, the sequence $s_n - 1$ is the x^3 term in $(1+x+x^2)^n$ (A005581 in the Encyclopedia). The sum of the elements along the edges give three sequences

$$\begin{array}{l} 1, 3, 8, 19, 40, \dots \\ 1, 4, 12, 29, 60, \dots, \\ 1, 5, 15, 35, 70, \dots \end{array}$$

with

$$\begin{aligned} s_n &= \frac{1}{24}n(n^3 + 2n^2 - n + 22) \\ s_n &= \frac{1}{24}n(n^3 + 6n^2 - n + 18) \quad , \\ s_n &= \frac{1}{24}n(n+1)(n+2)(n+3) \end{aligned} \quad (2)$$

respectively. The third sequence is A000332, and is easily recognized as the binomial coefficient $\binom{n}{4}$. The second is A086274, while the first sequence is new. Unless indicated otherwise, all subsequent sequences are new. Taking the sum of the three sequences in (2) gives

$$3, \quad 12, \quad 35, \quad 83, \quad 170, \quad \dots ,$$

with

$$s_n = \frac{1}{24}n(3n^3 + 14n^2 + 9n + 46) . \quad (3)$$

The sum of the elements on the edges is then (3) -2

$$1, \quad 10, \quad 33, \quad 81, \quad 168, \quad \dots ,$$

with

$$s_n = \frac{1}{24}(3n^4 + 14n^3 + 9n^2 + 46n - 48) . \quad (4)$$

The sum of the rows on the faces of the pyramid give the sequences

$$\begin{aligned} &1, \quad 5, \quad 19, \quad 54, \quad 125, \quad \dots \\ &1, \quad 6, \quad 22, \quad 60, \quad 135, \quad \dots , \\ &1, \quad 7, \quad 27, \quad 74, \quad 165, \quad \dots \end{aligned}$$

with

$$\begin{aligned} s_n &= \frac{1}{6}n(n^3 + n^2 - n + 5) \\ s_n &= \frac{1}{6}n(n+1)(n^2 + 2) \quad , \\ s_n &= \frac{1}{6}n(n^3 + 3n^2 - n + 3) \end{aligned} \quad (5)$$

respectively. From (5), we can obtain the sum of all elements on each face which are the sequences

$$\begin{aligned} &1, \quad 6, \quad 25, \quad 79, \quad 204, \quad \dots \\ &1, \quad 7, \quad 29, \quad 89, \quad 224, \quad \dots , \\ &1, \quad 8, \quad 35, \quad 109, \quad 274, \quad \dots \end{aligned}$$

with

$$\begin{aligned} s_n &= \frac{1}{120}n(n+1)(4n^3 + 11n^2 - n + 4) \\ s_n &= \frac{1}{120}n(n+1)(n+2)(4n^2 + 3n + 13) , \\ s_n &= \frac{1}{120}n(n+1)(4n^3 + 21n^2 + 9n + 26) \end{aligned} \quad (6)$$

respectively. Summing the sequences in (6) gives the sequence for the sum of the elements on all three faces (if the apex is counted once for each face)

$$3, \ 21, \ 89, \ 277, \ 702, \ \dots ,$$

with the terms given by the formula

$$s_n = \frac{1}{120}n(n+1)(12n^3 + 43n^2 + 27n + 98). \quad (7)$$

Adding the elements on a level gives

$$1, \ 9, \ 45, \ 155, \ 420, \ \dots$$

with

$$s_n = \frac{1}{24}n(n+1)(n+2)(2n^2 - n + 3). \quad (8)$$

This is sequence A095166, the natural numbers arranged in triangles. Note that the formula in [2] differs from that above because the sequence here begins at $n = 1$, as opposed to $n = 0$. The sum of all elements is the sum for 1 to n levels which is

$$1, \ 10, \ 55, \ 210, \ 630, \ \dots ,$$

with terms

$$s_n = \frac{1}{72}n(n+1)(n+2)(n+3)(n^2 + 2). \quad (9)$$

If one takes the sum of the faces (7) and subtracts the sum of the edges (3), we obtain

$$0, \ 9, \ 54, \ 194, \ 532, \ \dots$$

with

$$s_n = \frac{1}{30}n(n-1)(3n^3 + 13n^2 + 13n + 33). \quad (10)$$

Adding one to this total to include the apex (as it has been added three times and subtracted three times), gives

$$1, \ 10, \ 55, \ 195, \ 533, \ \dots \quad (11)$$

which is the sum of the elements on the surface of the tetrahedron. The corresponding sequence terms are given by the formula

$$s_n = \frac{1}{30}(3n^5 + 10n^4 + 20n^2 - 33n + 30). \quad (12)$$

The sum of the edge elements on each level of the pyramid gives the sequence

$$3, \quad 9, \quad 23, \quad 48, \quad 87, \quad \dots$$

with

$$s_n = \frac{1}{2}(n^3 + 2n^2 - n + 4) \quad (13)$$

Note that the apex is counted three times as it is part of each edge. The sum of the face elements on each level is the sequence

$$3, \quad 18, \quad 68, \quad 188, \quad 425, \quad \dots$$

with

$$s_n = \frac{1}{6}n(3n^3 + 5n^2 + 10) \quad (14)$$

where again the apex is counted three times. Taking the difference (14)-(13) gives the sequence

$$0, \quad 9, \quad 45, \quad 140, \quad 338, \quad \dots$$

with

$$s_n = \frac{1}{6}(n-1)(3n^3 + 5n^2 - n + 12) \quad (15)$$

Summing these elements starting from level 1 and adding 1 for the apex gives (11), as expected.

References

- [1] T.A. Gulliver, *Sequences from Arrays of Integers*, Int. Math. Journal **1** 323–332 (2002).
- [2] N.J.A. Sloane, On-Line Encyclopedia of Integer Sequences, <http://www.research.att.com/~njas/sequences/index.html>.

Received: May 12, 2005