MULTIENDOMORPHISMS OF HYPERGROUPOIDS

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Abstract

We introduce the notion of multiendomorphism in a hypergroupoid and the notion of G-semiring. We show that, if $(H, *)$ is a commutative semi-hypergroup, these multiendomorphisms form a G-semiring $(E, +, \circ, \leq)$, where the operation $+$ is induced by $*$, $\circ$ is the usual composition of maps $\circ$ and $\leq$ is the usual inclusion of maps. Moreover, we show under which conditions the G-semiring $(E, +, \circ, \leq)$ is, in fact, a semiring.

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1 Definitions and recalls

First of all we introduce some definitions of hypergroup theory (see [1] for more details).

Definition 1.1 *A hypergroupoid $(H, *)$ is a non-empty set H equipped with a hyperoperation $*$, that is a map $*$: $H \times H \rightarrow \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the set of non-empty subsets of H . If $x, y \in H$, we will denote by $x * y$ the hyperproduct of x and y . For $A, B \subseteq H$ we define, $A \circ B = \bigcup_{a \in A, b \in B} a * b$.*

Definition 1.2 *A hypergroupoid $(H, *)$ is said to be semi-hypergroup if and only if:*

$$\forall (x_1, x_2, x_3) \in H^3, x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3.$$

A hypergroupoid $(H, *)$ is said to be commutative if and only if:

$$\forall x, y \in H, x * y = y * x.$$

A hypergroupoid $(H, *)$ is said to be reproductive (or quasi-hypergroup) if and only if:

$$\forall x \in H, x * H = H = H * x.$$

A reproductive semi-hypergroup is said to be a hypergroup.

Example 1.1 Let's consider the hypergroupoid $(H, *)$ defined by the following table:

$*$	1	2	3
1	1	1,2	1,2,3
2	1	1,2	1,2,3
3	1	1,2	3

$(H, *)$ is a semi-hypergroup, neither commutative nor reproductive.

Example 1.2 The hypergroupoid $(H, *)$ defined by the following table:

$*$	1	2
1	1	1,2
2	2	1

is a quasi-hypergroup, neither commutative nor associative.

Example 1.3 The hypergroupoid $(H, *)$ defined by the following table:

$*$	1	2	3
1	1	1,2	1,3
2	1,2	2	2,3
3	1,3	2,3	3

is a commutative hypergroup.

Definition 1.3 Let $(H, *)$, $(H', *')$ be two hypergroupoids; a map $f : H \rightarrow H'$ is said to be a homomorphism if and only if:

$$\forall x, y \in H, \quad f(x * y) \subseteq f(x) *' f(y);$$

f is said to be a good homomorphism if and only if:

$$\forall x, y \in H, \quad f(x * y) = f(x) *' f(y).$$

If $(H, *) = (H', *')$, f is said to be an endomorphism (resp. good endomorphism). Moreover, we will denote by $E(H, *)$, or simply $E(H)$, the set of all endomorphisms of $(H, *)$ and by $E_g(H, *)$, or simply $E_g(H)$, the set of all good endomorphisms of $(H, *)$.

Now we generalize the previous notion of homomorphism and that one of multihomomorphism given in [2] by the following:

Definition 1.4 Let $(H, *)$, $(H', *')$ be two hypergroupoids; a map $f : H \rightarrow \mathcal{P}^*(H')$ is said to be a multihomomorphism if and only if:

$$\forall x, y \in H, \quad f(x * y) \subseteq f(x) *' f(y);$$

f is said to be a good multihomomorphism if and only if:

$$\forall x, y \in H, \quad f(x * y) = f(x) *' f(y).$$

If $(H, *) = (H', *')$, f is said to be a multiendomorphism (resp. good multiendomorphism). Moreover, we will denote by $\mathcal{E}(H, *)$, or simply $\mathcal{E}(H)$, the set of all multiendomorphisms of $(H, *)$ and by $\mathcal{E}_g(H, *)$, or simply $\mathcal{E}_g(H)$, the set of all good multiendomorphisms of $(H, *)$.

Remark 1.1 If $(H, *)$ is hypergroupoid with $H = \{h_1, h_2, \dots, h_n\}$, the multiendomorphism $f \in \mathcal{E}(H)$ such that:

$$\begin{cases} f(h_1) = A_1; \\ f(h_2) = A_2; \\ \vdots \\ f(h_n) = A_n; \end{cases}$$

with $A_i \subseteq H, \forall i = 1, \dots, n$, will be denoted by:

$$(A_1, A_2, \dots, A_n).$$

If $A_i = \{a\}$ we will write a instead of $\{a\}$.

Remark 1.2 Clearly any good homomorphism (resp. good multihomomorphism) is a homomorphism (resp. multihomomorphism); in particular, we obtain that in any hypergroupoid $(H, *)$ we have:

$$E_g(H) \subseteq E(H), \quad \mathcal{E}_g(H) \subseteq \mathcal{E}(H)$$

Moreover, by identifying the map $f : x \mapsto f(x)$ with the map $f : x \mapsto \{f(x)\}$, it is clear that any homomorphism (resp. good homomorphism) is a multihomomorphism (resp. good multihomomorphism); so, the following inclusions are true:

$$E_g(H) \subseteq \mathcal{E}_g(H), \quad E(H) \subseteq \mathcal{E}(H).$$

Then, in any hypergroupoid $(H, *)$ we have:

$$E_g(H) \subseteq E(H) \subseteq \mathcal{E}(H), \quad E_g(H) \subseteq \mathcal{E}_g(H) \subseteq \mathcal{E}(H).$$

Notice that, in general,

$$E(H) \not\subseteq \mathcal{E}_g(H), \quad \mathcal{E}_g(H) \not\subseteq E(H)$$

The preceding inclusions are not invertible as we can see in the following example.

Example 1.4 Let $(H, *)$ be the hypergroupoid defined by the following table:

$*$	1	2
1	1, 2	1
2	1	2

We have:

$$\begin{aligned}
 E_g(H) &= \{(1, 2), (2, 2)\} \\
 E(H) &= \{(1, 1), (1, 2), (2, 2)\} = E_g(H) \cup \{(1, 1)\} \\
 \mathcal{E}_g(H) &= \{(1, 2), (2, 2), (H, 2), (H, H)\} = E_g(H) \cup \{(H, 2), (H, H)\} \\
 \mathcal{E}(H) &= \{(1, 1), (1, 2), (1, H), (2, 2), (H, 1), (H, 2), (H, H)\} = \\
 &= \mathcal{E}_g(H) \cup \{(1, 1), (1, H), (H, 1)\} = \\
 &= E(H) \cup \{(1, H), (H, 1), (H, 2), (H, H)\}
 \end{aligned}$$

Moreover, $E(H)$ and $\mathcal{E}_g(H)$ are not comparable.

2 The G-semirings

In this section we recall the definition of semiring (see [5] for more details) and we give the notion of G -semiring.

Definition 2.1 An algebraic structure $(S, +, \times)$ is said to be a semiring if and only if the following conditions are satisfied:

1. $(S, +)$ is a semigroup;
2. $(S, \times)$ is a semigroup;
3. The distributivity law holds for any triple of elements of S .

If $(S, +)$ has an identity 0 and $(S, \times)$ has an identity 1 , with $0 \neq 1$, the semiring is said to be with identities.

Example 2.1 The most natural example of semiring is $(\mathbf{N}, +, \times)$, where $\mathbf{N}$ is the set of positive integers, $+$ and $\times$ are the natural addition and multiplication.

We recall the following definition (see [6]).

Definition 2.2 *An algebraic structure $(G, \cdot, \leq)$ is said to be a partially ordered groupoid if and only if:*

1. $(G, \cdot)$ is a groupoid;
2. $(G, \leq)$ is a partially ordered set;
3. $a \leq b \Rightarrow a \cdot c \leq b \cdot c, c \cdot a \leq c \cdot b, \forall c \in G$.

If the operation $\cdot$ is associative $(G, \cdot, \leq)$ is said to be a partially ordered semigroup.

Now we can give the following notion.

Definition 2.3 *An algebraic structure $(S, +, \times, \leq)$ is said to be a G -semiring if and only if the following conditions are satisfied:*

- S1. $(S, +, \leq)$ is a partially ordered semigroup;
- S2. $(S, \times, \leq)$ is a partially ordered semigroup;
- S3. $\forall x, y, z \in S, \quad \begin{cases} x \times (y + z) \leq (x \times y) + (x \times z) \\ (x + y) \times z \leq (x \times z) + (y \times z) \end{cases}$.

If $(S, +)$ has an identity 0 and $(S, \times)$ has an identity 1, with $0 \neq 1$, the G -semiring is said to be with identities.

Remark 2.1 *Clearly any semiring (with identities) is a G -semiring (with identities), where the partial order is the equality. The converse is not true; an example will be given in the next section.*

Remark 2.2 *If in a G -semiring $(S, +, \times, \leq)$ the following conditions are satisfied:*

$$\forall x, y, z \in S, \quad \begin{cases} (x \times y) + (x \times z) \leq x \times (y + z) \\ (x \times z) + (y \times z) \leq (x + y) \times z \end{cases},$$

then $(S, +, \times)$ is a semiring.

3 The G -semiring of multiendomorphisms

In this section we will construct a G -semiring involving multiendomorphisms of commutative semi-hypergroups, showing that, in general, this G -semiring is not a semiring.

Lemma 3.1 *Let $(H, *)$ be a hypergroupoid, $A, B \subseteq H, f \in \mathcal{E}(H)$. Then $f(A * B) \subseteq f(A) * f(B)$.*

Proof.

$$\begin{aligned}
 f(A * B) &= f\left(\bigcup_{a \in A, b \in B} a * b\right) = \bigcup_{a \in A, b \in B} f(a * b) \subseteq \\
 &\subseteq \bigcup_{a \in A, b \in B} (f(a) * f(b)) = \left(\bigcup_{a \in A} f(a)\right) * \left(\bigcup_{b \in B} f(b)\right) = \\
 &= f(A) * f(B).
 \end{aligned}$$

◇

In a similar way one can prove the following lemma.

Lemma 3.2 *Let $(H, *)$ be a hypergroupoid, $A, B \subseteq H, f \in \mathcal{E}_g(H)$. Then $f(A * B) = f(A) * f(B)$.*

Now we can prove the following proposition.

Proposition 3.3 *Let $(H, *)$ be a hypergroupoid, $f, g \in \mathcal{E}(H)$. Then $f \circ g \in \mathcal{E}(H)$, where $\circ$ is the usual composition of maps.*

Proof. From $f, g \in \mathcal{E}(H)$ it follows that:

$$\forall x, y \in H, \quad f(x * y) \subseteq f(x) * f(y), \quad g(x * y) \subseteq g(x) * g(y),$$

and then:

$$f \circ g(x * y) = f(g(x * y)) \subseteq f(g(x) * g(y)).$$

By Lemma 3.1 it follows that:

$$f(g(x) * g(y)) \subseteq f(g(x)) * f(g(y)) = (f \circ g)(x) * (f \circ g)(y),$$

that is $f \circ g \in \mathcal{E}(H)$. ◇

By Lemma 3.2 and using a similar argument one can prove the following proposition.

Proposition 3.4 *Let $(H, *)$ be a hypergroupoid, $f, g \in \mathcal{E}_g(H)$. Then $f \circ g \in \mathcal{E}_g(H)$.*

Now we can prove that.

Theorem 3.5 *$(\mathcal{E}(H), \circ)$ and $(\mathcal{E}_g(H), \circ)$ are semigroups with identity.*

Proof. By the previous propositions $(\mathcal{E}(H), \circ)$ and $(\mathcal{E}_g(H), \circ)$ are groupoids. Clearly, the associative law holds for any triple of (good) multiendomorphisms. Moreover, if we consider the map $1_H : x \mapsto \{x\}$ we have:

$$\forall x, y \in H, \quad 1_H(x * y) = x * y = 1_H(x) * 1_H(y),$$

that is 1_H is a good multiendomorphism (and so a multiendomorphism). Finally:

$$\forall f \in \mathcal{E}(H), \forall x \in H, (f \circ 1_H)(x) = f(1_H(x)) = f(x) = 1_H(f(x)) = (1_H \circ f)(x),$$

and so 1_H is the identity in $(\mathcal{E}(H), \circ)$ (and therefore in $(\mathcal{E}_g(H), \circ)$ also). $\diamond$

Let $(H, *)$ be a hypergroupoid. We introduce in $\mathcal{E}(H)$ (and therefore in $\mathcal{E}_g(H)$) another operation $+$, called "sum", defined by setting:

$$\forall f, g \in \mathcal{E}(H), \forall x \in H, \quad (f + g)(x) = f(x) * g(x).$$

In general, the sum of multiendomorphisms is not a multiendomorphism as we can see in the following example.

Example 3.1 Let $(H, *)$ be the hypergroupoid defined by the following table:

$*$	1	2
1	2	1, 2
2	1, 2	1

We have:

$$\mathcal{E}(H) = \{(1, 1), (1, 2), (2, 1), (2, 2), (H, H)\}.$$

If we consider $f = (1, 1)$ and $g = (1, 2)$, we obtain that:

$$\begin{aligned} (f + g)(1) &= f(1) * g(1) = 1 * 1 = 2; \\ (f + g)(2) &= f(2) * g(2) = 1 * 2 = H; \end{aligned}$$

that is:

$$f + g = (2, H)$$

and so $f + g \notin \mathcal{E}(H)$.

In which case the sum of multiendomorphisms is a multiendomorphism? The answer is given in the following proposition.

Proposition 3.6 Let $(H, *)$ be a commutative semi-hypergroup, $f, g \in \mathcal{E}(H)$. Then $f + g \in \mathcal{E}(H)$.

Proof. Let $x, y \in H, f, g \in \mathcal{E}(H)$. Then, by definition of $+$ we have:

$$(f + g)(x * y) = \bigcup_{z \in x * y} (f + g)(z) = \bigcup_{z \in x * y} f(z) * g(z);$$

but, clearly:

$$\bigcup_{z \in x * y} f(z) * g(z) \subseteq \bigcup_{t, z \in x * y} f(t) * g(z) = \left(\bigcup_{t \in x * y} f(t) \right) * \left(\bigcup_{z \in x * y} g(z) \right);$$

the last hyperproduct is $f(x * y) * g(x * y)$; now, by hypotesis we have:

$$f(x * y) * g(x * y) \subseteq (f(x) * f(y)) * (g(x) * g(y));$$

by hypotesis, it follows that:

$$(f(x) * f(y)) * (g(x) * g(y)) = (f(x) * g(x)) * (f(y) * g(y));$$

the last hyperproduct is, by definition, $(f + g)(x) * (f + g)(y)$. By the previous equalities and inclusions we obtain:

$$(f + g)(x * y) \subseteq (f + g)(x) * (f + g)(y),$$

that is $f + g \in \mathcal{E}(H)$. $\diamond$

Remark 3.1 *Unlike multiendomorphisms, in general the sum of good multiendomorphisms is not a good multiendomorphism also in the case that $(H, *)$ is a commutative semi-hypergroup, as we can see in the following example.*

Example 3.2 *Let $(H, *)$ be the commutative hypergroup defined by the following table:*

$*$	1	2	3
1	1	2	3
2	2	1	3
3	3	3	1, 2

The map $1_H = (1, 2, 3)$ is clearly a good multiendomorphism; let $f = 1_H + 1_H$; we have:

$$\begin{aligned} f(1) &= (1_H + 1_H)(1) = 1_H(1) * 1_H(1) = 1 * 1 = 1; \\ f(2) &= (1_H + 1_H)(2) = 1_H(2) * 1_H(2) = 2 * 2 = 1; \\ f(3) &= (1_H + 1_H)(3) = 1_H(3) * 1_H(3) = 3 * 3 = \{1, 2\}; \end{aligned}$$

that is:

$$f = (1, 1, \{1, 2\}).$$

This map is not a good multihomomorphism because, for instance

$$f(3 * 3) = 1,$$

while:

$$f(3) * f(3) = \{1, 2\}.$$

So, $1_H \in \mathcal{E}_g(H)$, while $1_H + 1_H \notin \mathcal{E}_g(H)$. Direct calculations show that the set of all good multihomomorphisms is:

$$\mathcal{E}_g(H) = \left\{ \begin{array}{l} (1, 1, 1), (1, 1, 2), (1, 2, 3), (1, 2, \{1, 2\}), (\{1, 2\}, 3, H), \\ (\{1, 2\}, \{1, 2\}, H), (\{1, 2\}, \{1, 2\}, \{1, 2\}), (H, H, H) \end{array} \right\}.$$

We recall the following definition (see [1]).

Definition 3.1 Let $(H, *)$ be a hypergroupoid. An element $e \in H$ is said to be an identity if:

$$\forall x \in H, \quad x * e \ni x \in e * x;$$

e is said to be a scalar identity if:

$$\forall x \in H, \quad x * e = \{x\} = e * x.$$

Now we can prove the following theorem.

Theorem 3.7 Let $(H, *)$ be a commutative semi-hypergroup. Then $(\mathcal{E}(H), +)$ is a commutative semigroup. Moreover, if $(H, *)$ has a scalar identity, then $(\mathcal{E}(H), +)$ has an identity.

Proof. By the previous proposition $(\mathcal{E}(H), +)$ is a groupoid. Let $f, g, k \in \mathcal{E}(H), x \in H$; we have:

$$(f + (g + k))(x) = f(x) * (g + k)(x) = f(x) * (g(x) * k(x)),$$

and:

$$((f + g) + k)(x) = ((f + g)(x)) * k(x) = (f(x) * g(x)) * k(x);$$

since $(H, *)$ is a semi-hypergroup, it follows that:

$$f(x) * (g(x) * k(x)) = (f(x) * g(x)) * k(x),$$

and so $(\mathcal{E}(H), +)$ is a semigroup. Moreover:

$$(f + g)(x) = f(x) * g(x) = g(x) * f(x) = (g + f)(x),$$

and so $(\mathcal{E}(H), +)$ is commutative. Finally let e be the scalar identity of $(H, *)$ and let $0_H : H \rightarrow \mathcal{P}^*(H)$ defined by setting:

$$\forall x \in H, \quad 0_H(x) = \{e\};$$

we have:

$$\forall x, y \in H, \quad 0_H(x * y) = e = e * e = 0_H(x) * 0_H(y),$$

and so 0_H is a (good) multiendomorphism; moreover:

$$\forall f \in \mathcal{E}(H), \forall x \in H, \quad (0_H + f)(x) = 0_H(x) * f(x) = e * f(x) = f(x),$$

and, by the commutativity:

$$\forall f \in \mathcal{E}(H), \forall x \in H, \quad (f + 0_H)(x) = f(x).$$

So, 0_H is the identity of $(\mathcal{E}(H), +)$. $\diamond$

Now we introduce in $\mathcal{E}(H)$ (and so in $\mathcal{E}_g(H)$) a relation $\leq$ by setting:

$$\forall f, g \in \mathcal{E}(H), \quad f \leq g \Leftrightarrow f(x) \subseteq g(x), \forall x \in H.$$

Remark 3.2 *It is clear that $\leq$ is a partial order in $\mathcal{E}(H)$, that is $(\mathcal{E}(H), \leq)$ is a partially ordered set.*

Now we can show the main theorem.

Theorem 3.8 *Let $(H, *)$ be a commutative semi-hypergroup. Then $(\mathcal{E}(H), +, \circ, \leq)$ is a G -semiring. Moreover, if $(H, *)$ has a scalar identity, then $(\mathcal{E}(H), +, \circ, \leq)$ is with identities.*

Proof. By Theorem 3.7 it follows that $(\mathcal{E}(H), +)$ is a semigroup; moreover, if $(H, *)$ has a scalar identity, then $(\mathcal{E}(H), +)$ has an identity 1_H . Now let $f \leq g$ that is:

$$\forall x \in H, \quad f(x) \subseteq g(x);$$

we have:

$$\forall k \in \mathcal{E}(H), \forall x \in H, \quad f(x) * k(x) \subseteq g(x) * k(x);$$

that is:

$$\forall k \in \mathcal{E}(H), \quad f + k \leq g + k;$$

by the commutativity of $+$ it follows that:

$$\forall k \in \mathcal{E}(H), \quad k + f \leq k + g.$$

This shows that $(\mathcal{E}(H), +, \leq)$ is a partially ordered semigroup.

By Theorem 3.5 it follows that $(\mathcal{E}(H), \circ)$ is a semigroup with identity 0_H . Now let $f \leq g$ that is:

$$\forall x \in H, \quad f(x) \subseteq g(x);$$

we have:

$$\forall k \in \mathcal{E}(H), \forall x \in H, \quad k(f(x)) \subseteq k(g(x)), \quad f(k(x)) \subseteq g(k(x));$$

that is:

$$\forall k \in \mathcal{E}(H), \quad k \circ f \leq k \circ g, \quad f \circ k \leq g \circ k;$$

This shows that $(\mathcal{E}(H), \circ, \leq)$ is a partially ordered semigroup.

Clearly $0_H \neq 1_H$. So, it remains to show S.3.

Let $f, g, k \in \mathcal{E}(H)$; we have:

$$(f \circ (g + k))(x) = f((g + k)(x)) = f(g(x) * k(x));$$

by Lemma 3.1 it follows that:

$$f(g(x) * k(x)) \subseteq f(g(x)) * f(k(x));$$

since:

$$f(g(x)) * f(k(x)) = (f \circ g)(x) * (f \circ k)(x) = ((f \circ g) + (f \circ k))(x)$$

we obtain that:

$$(f \circ (g + k))(x) \subseteq ((f \circ g) + (f \circ k))(x)$$

that is:

$$(f \circ (g + k)) \leq ((f \circ g) + (f \circ k)).$$

Moreover:

$$\begin{aligned} ((f + g) \circ k)(x) &= (f + g)(k(x)) = \bigcup_{z \in k(x)} ((f + g)(z)) = \bigcup_{z \in k(x)} (f(z) * g(z)) \subseteq \\ &\subseteq \bigcup_{z, t \in k(x)} (f(z) * g(t)) = \left(\bigcup_{z \in k(x)} f(z) \right) * \left(\bigcup_{t \in k(x)} g(t) \right) = \\ &= f(k(x)) * g(k(x)) = (f \circ k)(x) * (g \circ k)(x) = \\ &= ((f \circ k) + (g \circ k))(x) \end{aligned}$$

that is:

$$((f + g) \circ k) \leq ((f \circ k) + (g \circ k)).$$

So, the theorem is proved. $\diamond$

The G -semiring obtained in Theorem 3.8 will be called the G -semiring of multiendomorphisms.

The following example shows that there exist G -semirings that are not semirings.

Example 3.3 Let $(H, *)$ be the hypergroupoid defined by the following table:

$*$	1	2
1	1	1
2	1	1, 2

We have:

$$\mathcal{E}(H) = \{(1, 1), (1, 2), (1, H), (2, 2), (2, H), (H, 2), (H, H)\}.$$

We consider $f = (2, H), g = (H, 2), k = (1, H), t = (H, H)$. We have:

$$\begin{aligned} (g + k)(1) &= g(1) * k(1) = H * 1 = 1; \\ (g + k)(2) &= g(2) * k(2) = 2 * H = H. \end{aligned}$$

that is:

$$g + k = k,$$

and so:

$$f \circ (g + k) = f \circ k = f.$$

On the other hand:

$$(f \circ k) + (g \circ k) = f + t = t.$$

Since, clearly, $f < t$ we have that:

$$f \circ (g + k) < (f \circ g) + (f \circ k),$$

and so $(\mathcal{E}(H), +, \circ, \leq)$ is a G -semiring that is not a semiring.

4 End-closed and End_g -closed semi-hypergroups

In general, if $(H, *)$ is a commutative semi-hypergroup, the structure $(\mathcal{E}(H), +, \circ)$ is not a semiring, since in general the distributive law does not holds (see example 3.3). Now we give the following definition.

Definition 4.1 A commutative semi-hypergroup $(H, *)$ is said to be *End-closed* if and only if $(\mathcal{E}(H), +, \circ)$ is a semiring.

By Remark 2.2 it follows that:

Proposition 4.1 A commutative semi-hypergroup $(H, *)$ is *End-closed* if and only if:

$$\forall f, g, k \in \mathcal{E}(H), \quad \begin{cases} (f \times g) + (f \times k) \leq f \times (g + k) \\ (f \times k) + (g \times k) \leq (f + g) \times k \end{cases}$$

Analogously, in general, $(\mathcal{E}_g(H), +, \circ)$ is not a semiring since, in general, $\mathcal{E}_g(H)$ is not closed under $+$ (see example 3.2). Now we give the following definition:

Definition 4.2 *A commutative semi-hypergroup $(H, *)$ is said to be End_g -closed if and only if $(\mathcal{E}_g(H), +, \circ)$ is a semiring.*

By using a similar argument as in Theorem 3.8, and by observing that in this case all the inclusions are, in fact, identities, we can prove that:

Proposition 4.2 *A commutative semi-hypergroup $(H, *)$ is End_g -closed if and only if $(\mathcal{E}_g(H), +)$ is a groupoid.*

Clearly, if $(H, *)$ is a commutative semigroup (i.e. if $*$ is single-valued), in the proof of Theorem 3.8 all the inclusions are, in fact, identities, and so we obtain the following corollary.

Corollary 4.3 *Any commutative semigroup is End -closed and End_g -closed.*

Now we give a class of semi-hypergroups (in fact, hypergroups) that are End -closed and End_g -closed.

Lemma 4.4 *Let $(H, *)$ be a commutative semi-hypergroup. If $\mathcal{E}_g(H) = \mathcal{E}(H)$ then $(H, *)$ is End -closed and End_g -closed, that is $(\mathcal{E}_g(H), +, \circ) \equiv (\mathcal{E}(H), +, \circ)$ is a semiring.*

Proof. By Proposition 3.7 we know that $(\mathcal{E}(H), +)$ (and so $(\mathcal{E}_g(H), +)$) is a groupoid. The lemma follows now by Proposition 4.3. $\diamond$

Theorem 4.5 *Let $(H, *)$ be defined by:*

$$\forall x, y \in H, \quad x * y = \{x, y\}.$$

Then $(H, *)$ is End -closed and End_g -closed.

Proof. First of all it is clear that $(H, *)$ is a commutative hypergroup. Now it is enough to show that in this case $\mathcal{E}_g(H) = \mathcal{E}(H)$. We will show that, in fact, any map $f : H \rightarrow \mathcal{P}^*(H)$ is a good multiendomorphism. Let $f : H \rightarrow \mathcal{P}^*(H)$; we have:

$$\forall x, y \in H, \quad f(x * y) = f(\{x, y\}) = f(x) \cup f(y) = f(x) * f(y),$$

that is f is a good multiendomorphism. The theorem follows now from Lemma 4.4. $\diamond$

We recall now that a k -Steiner system is a geometric space $(H, \mathcal{L})$, where H is a non-empty set, whose elements are called points, and $\mathcal{L}$ is a family of non-empty subsets of H , whose elements are called lines, such that every line has k points and any two distinct points are contained in a unique line. In [7] the following definition was introduced:

Definition 4.3 A hypergroupoid $(H, *)$ is said to be a k -Steiner hypergroupoid ($k \geq 2$) (or simply Steiner hypergroupoid), if and only if the following conditions are satisfied:

- 1) $\forall x, y \in H, \quad \{x, y\} \subseteq x * y;$
- 2) $\forall x, y \in H, \quad |x * y| = \begin{cases} 1 & \text{if } x = y \\ k & \text{if } x \neq y \end{cases};$
- 3) the associativity holds for any triple of points not all distinct.

We can prove that if $(H, \mathcal{L})$ is a k -Steiner system then it is possible to define uniquely a k -Steiner hypergroupoid $(H, *)$ (see [7], [3], [4] for more details); this k -Steiner hypergroupoid is said to be associated to $(H, \mathcal{L})$; in particular, if $k = 2$, then the 2-Steiner systems are complete graphs and the 2-Steiner hypergroupoids verify the condition:

$$\forall x, y \in H, \quad x * y = \{x, y\}.$$

Therefore we can rewrite the previous theorem as follows:

Theorem 4.6 Any Steiner hypergroupoid associated to a complete graph is End-closed and End_g -closed.

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