On common fixed points in fuzzy metric spaces

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Abstract

Let \((X, M, \ast)\) be a complete fuzzy metric space with \(\lim_{t \to \infty} M(x, y, t) = 1\) for all \(x, y \in X\) and \(s \ast s \geq s\) for all \(s \in [0, 1]\) and let \(A, B, S\) and \(T\) be self mappings of \(X\). In this paper we give some conditions of which \(A, B, S\) and \(T\) have a unique common fixed point in \(X\). And we characterize the conditions for continuous self mappings \(S\) and \(T\) of \(X\) have a unique common fixed point in \(X\).

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1 Introduction

The concept of fuzzy sets was introduced initially by Zadeh[14]. Since then, it was developed extensively by many authors and used in various fields. Especially, [1, 3, 8] introduced the concept of fuzzy metric spaces in different ways.

In [3, 4], George and Veeramani modified the concept of fuzzy metric space which introduced by Kramosil and Michalek[10]. They, also, obtained the Hausdorff topology for this kind of fuzzy metric spaces and showed that every metric induces a fuzzy metric.

Sessa[12] introduced a generalization of commutativity, so called weak commutativity. Further Jungck[7] introduced more generalized commutativity, which is called compatibility in metric space. He proved common fixed point theorems.

Recently, Bijendra Singh and M. S. Chauhan[13] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems.
theorems in fuzzy metric spaces in the sense of George and Veeramani with continuous \( t \)-norm \( * \) defined by \( a * b = \min\{a, b\} \) for all \( a, b \in [0, 1] \).

In this paper we modify common fixed point theorems obtained in [13] and we characterize the conditions for two continuous self mappings of complete fuzzy metric space have a unique common fixed point.

2 Preliminaries

**Definition 2.1.** [11] A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a continuous \( t \)-norm if \( ([0, 1], *) \) is an abelian topological monoid with 1 such that \( a * b \leq c * d \), whenever \( a \leq c, b \leq d \) for all \( a, b, c, d \in [0, 1] \).

Examples of \( t \)-norm are \( a * b = ab \) and \( a * b = \min\{a, b\} \).

**Definition 2.2.** [3] The 3-tuple \( (X, M, *) \) is called a fuzzy metric space if \( X \) is an arbitrary set, \( * \) is a continuous \( t \)-norm and \( M \) is a fuzzy set on \( X^2 \times (0, \infty) \) satisfying the following conditions:

1. \( M(x, y, t) > 0 \),
2. \( M(x, y, t) = 1 \) if and only if \( x = y \),
3. \( M(x, y, t) = M(y, x, t) \),
4. \( M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \),
5. \( M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1] \) is continuous, for all \( x, y, z \in X \) and \( t, s > 0 \).

Let \( (X, d) \) be a metric space, and let \( a * b = ab \) or \( a * b = \min\{a, b\} \). Let \( M(x, y, t) = \frac{t}{t + d(x, y)} \) for all \( x, y \in X \) and \( t > 0 \). Then \( (X, M, *) \) is a fuzzy metric space, and this fuzzy metric \( M \) induced by \( d \) is called the standard fuzzy metric[3].

**Definition 2.3.** [5] A sequence \( \{x_n\} \) in a fuzzy metric space \( (X, M, *) \) is said to be convergent to a point \( x \in X \) (denoted by \( \lim_{n \rightarrow \infty} x_n = x \)), if for each \( \epsilon > 0 \) and each \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x, t) > 1 - \epsilon \) for all \( n \geq n_0 \).

George and Veeramani[3] show that a sequence \( \{x_n\} \) in a fuzzy metric space \( (X, M, *) \) converges to a point \( x \in X \) if and only if \( \lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \).

A sequence \( \{x_n\} \) in a fuzzy metric space \( (X, M, *) \) is called Cauchy sequence[5] if for each \( \epsilon > 0 \) and each \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x_{n+p}, t) > 1 - \epsilon \) for all \( n \geq n_0 \) and all \( t > 0 \).

A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

George and Veeramani[3] give an example that \((\mathbb{R}, M, *)\) is not complete in the sense of [5], where \( M \) is the standard fuzzy metric with \( d(x, y) = |x - y| \), and so to make \( \mathbb{R} \) complete fuzzy metric space George and Veeramani redefine Cauchy sequence as follows.
**Definition 2.4.** [3] A sequence \( \{x_n\} \) in a fuzzy metric space \((X, M, \ast)\) is called Cauchy sequence if for each \( \epsilon > 0 \) and each \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x_m, t) > 1 - \epsilon \) for all \( n, m \geq n_0 \).

**Definition 2.5.** [13] Self mappings \( A \) and \( B \) of a fuzzy metric space \((X, M, \ast)\) is said to be compatible if \( \lim_{n \to \infty} M(ABx_n, BA x_n, t) = 1 \) for all \( t > 0 \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \) for some \( z \in X \).

From now on, let \((X, M, \ast)\) be a fuzzy metric space such that \( \lim_{t \to \infty} M(x, y, t) = 1 \) for all \( x, y \in X \) and \( s \ast s \geq s \) for all \( s \in [0, 1] \).

**Lemma 2.6.** [5] Let \((X, M, \ast)\) be a fuzzy metric space. Then for all \( x, y \in X \), \( M(x, y, \cdot) \) is nondecreasing.

**Lemma 2.7.** Let \((X, M, \ast)\) be a fuzzy metric space. If there exists \( q \in (0, 1) \) such that \( M(x, y, qt) \geq M(x, y, t) \) for all \( x, y \in X \) and \( t > 0 \), then \( x = y \).

**Proof.** Suppose that there exists \( q \in (0, 1) \) such that \( M(x, y, qt) \geq M(x, y, t) \) for all \( x, y \in X \) and \( t > 0 \). Then \( M(x, y, t) \geq M(x, y, t/q^n) \) for positive integer \( n \). Taking limit as \( n \to \infty \), \( M(x, y, t) \geq 1 \) and hence \( x = y \). \(\square\)

**Lemma 2.8.** Let \((X, M, \ast)\) be a fuzzy metric space and let \( A \) and \( S \) be continuous self mappings of \( X \) and \([A, S]\) be compatible. Let \( \{x_n\} \) be a sequence in \( X \) such that \( Ax_n \to z \) and \( Sx_n \to z \). Then \( ASx_n \to Sz \).

**Proof.** Since \( S \) is continuous, \( SAx_n \to Sz \) and so for all \( t > 0 \), \( M(SAx_n, Sz, t/2) \to 1 \). Because the pair \([A, S]\) is compatible, \( M(SAx_n, ASx_n, t/2) \to 1 \) for all \( t > 0 \). Thus \( M(ASx_n, Sz, t) \geq M(ASx_n, SAx_n, t/2) \ast M(SAx_n, Sz, t/2) \to 1 \) for all \( t > 0 \), and so \( M(ASx_n, Sz, t) \to 1 \) for all \( t > 0 \), and hence \( ASx_n \to Sz \). \(\square\)

**Lemma 2.9.** [9] The only \( t \)-norm \( \ast \) satisfying \( r \ast r \geq r \) for all \( r \in [0, 1] \) is the minimum \( t \)-norm, that is, \( a \ast b = \min \{a, b\} \) for all \( a, b \in [0, 1] \).

### 3 Common fixed point theorems

In this section, we prove some common fixed point theorems satisfying some conditions.

**Theorem 3.1.** Let \((X, M, \ast)\) be a complete fuzzy metric space and let \( A, B, S \) and \( T \) be self mappings of \( X \) such that the following conditions are satisfied:

(i) \( AX \subset TX, BX \subset SX \),
(ii) \( S \) and \( T \) are continuous,
(iii) the pairs \([A, S]\) and \([B, T]\) are compatible,
(iv) there exists \( q \in (0, 1) \) such that for every \( x, y \in X \) and \( t > 0 \),
\[
M(Ax, By, qt) \geq M(Sx, Ty, t) \ast M(Ax, Sx, t) \ast M(By, Ty, t) \ast M(Ax, Ty, t).
\]

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).
Proof. Since $AX \subset TX$ and $BX \subset SX$, for any $x_0 \in X$, there exists $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this $x_1 \in X$, there exists $x_2 \in X$ such that $Bx_1 = Sx_2$.

Inductively, we can find a sequence $\{y_n\}$ in X as follows:

$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 1, 2 \ldots$

From (iii),

$$M(y_{2n+1}, y_{2n+2}, qt) = M(Ax_{2n}, Bx_{2n+1}, qt)$$

$$\geq M(Sx_{2n}, Tx_{2n+1}, t) \ast M(Ax_{2n}, Sx_{2n}, t) \ast M(Bx_{2n+1}, Tx_{2n+1}, t) \ast M(Ax_{2n}, Tx_{2n+1}, t)$$

$$= M(y_{2n}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n+2}, t) \ast M(y_{2n+2}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n+1}, t)$$

$$\geq M(y_{2n}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n+2}, t).$$

From lemma 2.6 and 2.9, we have that

$$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t). \quad (3.1.1)$$

Similarly, we have also

$$M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t). \quad (3.1.2)$$

From (3.1.1) and (3.1.2), we have

$$M(y_{n+1}, y_{n+2}, qt) \geq M(y_n, y_{n+1}, t). \quad (3.1.3)$$

From (3.1.3),

$$M(y_n, y_{n+1}, t) \geq M(y_n, y_{n-1}, t/q) \geq M(y_{n-2}, y_{n-1}, t/q^2)$$

$$\geq \cdots \geq M(y_1, y_2, t/q^n) \to 1 \quad \text{as} \quad n \to \infty.$$

So $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$ for any $t > 0$.

For each $\epsilon > 0$ and each $t > 0$, we can choose $n_0 \in \mathbb{N}$ such that $M(y_n, y_{n+1}, t) > 1 - \epsilon$ for all $n > n_0$.

For $m, n \in \mathbb{N}$, we suppose $m \geq n$. Then we have that

$$M(y_n, y_m, t)$$

$$\geq M(y_n, y_{n+1}, t/m - n) \ast M(y_{n+1}, y_{n+2}, t/m - n) \ast \cdots \ast M(y_{m-1}, y_m, t/m - n)$$

$$\geq (1 - \epsilon) \ast (1 - \epsilon) \ast \cdots \ast (1 - \epsilon) \geq 1 - \epsilon$$

and hence $\{y_n\}$ is a Cauchy sequence in $X$.

Since $(X, M, \ast)$ is complete, $\{y_n\}$ converges to some point $z \in X$, and so $\{Ax_{2n-2}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to $z$. From lemma 2.8 and (iii),

$$ASx_{2n} \to Sz \quad (3.1.4)$$
and

\[ BT_{x_{2n-1}} \to T_z. \quad (3.1.5) \]

From (iv), we get

\[
M(AS_{x_{2n}}, BT_{x_{2n-1}}, qt) \\
\geq M(SS_{x_{2n}}, TT_{x_{2n-1}}, t) \ast M(AS_{x_{2n}}, SS_{x_{2n}}, t) \\
\ast M(BT_{x_{2n-1}}, TT_{x_{2n-1}}, t) \ast M(AS_{x_{2n}}, TT_{x_{2n-1}}, t).
\]

Taking limit as \( n \to \infty \), and using (3.1.4) and (3.1.5),

\[
M(Sz, Tz, qt) \geq M(Sz, Tz, t) \ast M(AS, Sz, t) \ast M(Bz, Tz, t) \\
\geq M(Sz, Tz, t) \ast 1 \ast M(Sz, Tz, t) \\
\geq M(Sz, Tz, t).
\]

Thus we have

\[ M(Sz, Tz, qt) \geq M(Sz, Tz, t), \text{ and hence } Sz = Tz. \quad (3.1.6) \]

Now, from (iv),

\[
M(Az, BT_{x_{2n-1}}, qt) \\
\geq M(Sz, TT_{x_{2n-1}}, t) \ast M(Az, Sz, t) \ast M(Bz, Tz, t) \ast M(Az, TT_{x_{2n-1}}, t) \ast M(Az, Tz, t) \\
\]

which implies that taking limit as \( n \to \infty \), and using (3.1.5), (3.1.6),

\[
M(Az, Tz, qt) \\
\geq M(Sz, Sz, t) \ast M(Az, Tz, t) \ast M(Tz, Tz, t) \ast M(Az, Tz, t) \\
\geq M(Az, Tz, t),
\]

and hence

\[ Az = Tz. \quad (3.1.7) \]

From (iv), (3.1.6) and (3.1.7),

\[
M(Az, Bz, qt) \\
\geq M(Sz, Tz, t) \ast M(Az, Sz, t) \ast M(Bz, Tz, t) \ast M(Az, Tz, t) \\
= M(Az, Az, t) \ast M(Az, Az, t) \ast M(Bz, Az, t) \ast M(Az, Az, t) \\
\geq M(Az, Bz, t),
\]
and so

\[ Az = Bz. \] (3.1.8)

From (3.1.6), (3.1.7) and (3.1.8),

\[ Az = Bz = Tz = Sz. \] (3.1.9)

Now, we show that \( Bz = z \).

From (iv), \( M(Ax_{2n}, Bz, qt) \)

\[ \geq M(Sx_{2n}, Tz, t) * M(Ax_{2n}, Sx_{2n}, t) * M(Bz, Tz, t) * M(Ax_{2n}, Tz, t) \]

which implies that taking limit as \( n \to \infty \), and using (3.1.6) and (3.1.7),

\[ M(z, Bz, qt) \geq M(z, Tz, t) * M(z, z, t) * M(Bz, Tz, t) * M(z, Bz, t) \]

\[ \geq M(z, Bz, t), \]

and hence \( Bz = z \). Thus from (3.1.9), \( z = Az = Bz = Tz = Sz \) and \( z \) is a common fixed point of \( A, B, S \) and \( T \).

For uniqueness, let \( w \) be another common fixed point of \( A, B, S \) and \( T \). Then

\[ M(z, w, qt) = M(Az, Bw, qt) \]

\[ \geq M(Sz, Tw, t) * M(Az, Sz, t) * M(Bw, Tw, t) * M(Az, Tw, t) \]

\[ \geq M(z, w, t). \]

From lemma 2.7, \( z = w \). This complete the proof of theorem. \( \square \)

**Corollary 3.2.** [13] Let \((X, M, \ast)\) be a complete fuzzy metric space and let \( A, B, S \) and \( T \) be self mappings of \( X \) satisfying (i) – (iii) of theorem 3.1 and there exists \( q \in (0, 1) \) such that

\[ M(Ax, By, qt) \geq M(Sx, Ty, t)*M(Ax, Sx, t)*M(By, Ty, t)*M(By, Sx, 2t)*M(Ax, Ty, t) \]

for every \( x, y \in X \) and \( t > 0 \).

Then \( A, B, S \) and \( T \) have a unique fixed point in \( X \).

**Proof.** We have \( M(Ax, By, qt) \geq M(Sx, Ty, t)*M(Ax, Sx, t)*M(By, Ty, t)*M(By, Sx, 2t)*M(Ax, Ty, t) \)

\[ \geq M(Sx, Ty, t) * M(Ax, Ty, t) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Sx, Ty, t) * M(By, Ty, t) * M(Ax, Ty, t) \]

and hence, from theorem 3.1, \( A, B, S \) and \( T \) have a unique fixed point in \( X \). \( \square \)
Corollary 3.3. Let \((X, M, \ast)\) be a complete fuzzy metric space and let \(A, B, S\) and \(T\) be self mappings of \(X\) satisfying \((i)-(iii)\) of theorem 3.1 and there exists \(q \in (0, 1)\) such that \(M(Ax, By, qt) \geq M(Sx, Ty, t)\) for every \(x, y \in X\) and \(t > 0\).

Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Proof. We have \(M(Ax, By, qt) \geq M(Sx, Ty, t) = M(Sx, Ty, t) \ast 1 \geq M(Sx, Ty, t) \ast M(Ax, Sx, t) \ast M(Sx, By, 2t) \ast M(By, Ty, t) \ast M(Ty, Ax, t)\) and hence, from corollary 3.2, \(A, B, S\) and \(T\) have a unique fixed point in \(X\). \(\square\)

Corollary 3.4. Let \((X, M, \ast)\) be a complete fuzzy metric space and let \(A, B, S\) and \(T\) be self mappings of \(X\) satisfying \((i)-(iii)\) of theorem 3.1 and there exists \(q \in (0, 1)\) such that \(M(Ax, By, qt) \geq M(Sx, Ty, t) \ast M(Ax, Ty, t)\) for every \(x, y \in X\) and \(t > 0\).

Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Proof. We have \(M(Ax, By, qt) \geq M(Sx, Ty, t) \ast M(Ax, Ty, t) = M(Sx, Ty, t) \ast M(Sx, Ax, t) \ast M(Ax, Ty, t) \ast 1 \geq M(Sx, Ty, t) \ast M(Sx, Ax, t) \ast M(Ax, Ty, t) \ast M(Sx, By, 2t) \ast M(By, Ty, t) \ast M(Ty, Ax, t) \ast M(Sx, Ty, t) \ast M(Sx, Ax, t) \ast M(Ax, Ty, t)\) and hence, from corollary 3.2, \(A, B, S\) and \(T\) have a unique fixed point in \(X\). \(\square\)

Theorem 3.5. Let \((X, M, \ast)\) be a complete fuzzy metric space. Then continuous self mappings \(S\) and \(T\) of \(X\) have a common fixed point in \(X\) if and only if there exists a self mapping \(A\) of \(X\) such that the following conditions are satisfied:

(i) \(AX \subset TX \cap SX\),
(ii) the pairs \([A, S]\) and \([A, T]\) are compatible,
(iii) there exists \(q \in (0, 1)\) such that for every \(x, y \in X\) and \(t > 0\),
\(M(Ax, Ay, qt) \geq M(Sx, Ty, t) \ast M(Ax, Sx, t) \ast M(Ay, Ty, t) \ast M(Ax, Ty, t)\).

In fact \(A, S\) and \(T\) have a unique common fixed point in \(X\).

Proof. First, we show that the necessity of the conditions \((i)-(iii)\). Suppose that \(S\) and \(T\) have a common fixed point in \(X\), say \(z\). Then \(Sz = z = Tz\).

Let \(Ax = z\) for all \(x \in X\). Then we have \(AX \subset TX \cap SX\) and we know that \([A, S]\) and \([A, T]\) are compatible, in fact \(A \circ S = S \circ A\) and \(A \circ T = T \circ A\), and hence the conditions \((i)\) and \((ii)\) are satisfied.

For some \(q \in (0, 1)\), we get \(M(Ax, Ay, qt) = 1 \geq M(Sx, Ty, t) \ast M(Ax, Sx, t) \ast M(Ay, Ty, t) \ast M(Ax, Ty, t)\) for every \(x, y \in X\) and \(t > 0\) and hence the condition \((iii)\) is satisfied.

Now, for the sufficiency of the conditions, let \(A = B\) in theorem 3.1. Then \(A, S\) and \(T\) have a unique common fixed point in \(X\). \(\square\)

Corollary 3.6. Let \((X, M, \ast)\) be a complete fuzzy metric space. Then continuous self mappings \(S\) and \(T\) of \(X\) have a common fixed point in \(X\) if and
only if there exists a self mapping $A$ of $X$ satisfying $(i) - (ii)$ of theorem 3.5
and there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$,
$$M(Ax, Ay, qt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Ay, Ty, t) * M(Ax, Sx, 2t) * M(Ax, Ty, t).$$

In fact $A, S$ and $T$ have a unique common fixed point in $X$.

**Corollary 3.7.** Let $(X, M, *)$ be a complete fuzzy metric space. Then continuous self mappings $S$ and $T$ of $X$ have a common fixed point in $X$ if and only if there exists a self mapping $A$ of $X$ satisfying $(i) - (ii)$ of theorem 3.5 and there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$,
$$M(Ax, Ay, qt) \geq M(Sx, Ty, t).$$

In fact $A, S$ and $T$ have a unique common fixed point in $X$.

**Corollary 3.8.** Let $(X, M, *)$ be a complete fuzzy metric space. Then continuous self mappings $S$ and $T$ of $X$ have a common fixed point in $X$ if and only if there exists a self mapping $A$ of $X$ satisfying $(i) - (ii)$ of theorem 3.5 and there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$,
$$M(Ax, Ay, qt) \geq M(Sx, Ty, t) * M(Sx, Ax, t) * M(Ax, Ty, t).$$

In fact $A, S$ and $T$ have a unique common fixed point in $X$.

**Example.** Let $X = [0, 1]$ and $a * b = \min\{a, b\}$. Let $M$ be the standard fuzzy metric induced by $d$, where $d(x, y) = |x - y|$ for $x, y \in X$. Then $(X, M, *)$ is a complete fuzzy metric space. Let $Ax = \frac{x}{16}, Tx = \frac{x}{2}, Bx = \frac{x}{8}$ and $Sx = \frac{x}{4}$. Then the conditions $(i)$ and $(ii)$ of theorem 3.1 are satisfied, and also for $q = 1/2$, condition $(iii)$ of theorem 3.1 is satisfied and zero is the unique common fixed point of $A, B, S$ and $T$.

**References**


[6] Valentin Gregori and Almanzor Sapena, On fixed point theorems in fuzzy


demic Publishers.

[10] O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces,


[12] S. Sessa, On weak commutativity condition of mappings in fixed point

[13] Bijendra Singh and M. S. Chauhan, Common fixed points of compatible


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